Faculty of Automotive and Construction Machinery Engineering
WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control
Winter 2019/2020

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Lecture 8

Laplace transform.
Transfer function.
Inputs and outputs in time domain.

Laplace transform

Assumption: $x(t)$ - signal such that for $t<0 \quad x(t)=0$

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where: $s \in \mathbb{C}, \quad s=\sigma+j\omega, \quad j=\sqrt{-1}$

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Inverse Laplace transform of x(t): $$x(t)=L^{-1}\{X(s)\}=\frac{1}{2\pi j}\lim_{\omega\to\infty}\int_{\gamma-j\omega}^{\gamma+j\omega} X(s)e^{st}ds$$

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A necessary condition for existence of the integral is that x(t) must be locally integrable on t in <0, ∞).

Laplace transform

Example 1

Calculate Laplace transform of x(t) function from definition.

$$x(t)=e^{-2t}$$

Laplace transform

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Calculate Laplace transform of x(t) function from definition.

$$x(t)=e^{-2t}$$

$$X(s)=L\{e^{-2t}\}=\int_0^{\infty} e^{-2t}e^{-st}\,dt=\int_0^{\infty} e^{-(2+s)t}\,dt=\left[\frac{e^{-(2+s)t}}{-(2+s)}\right]_0^{\infty}=\frac{1}{s+2}$$

Laplace transform

$f(t),\ t\geqslant 0$	$F(s)$
$\delta(t)$ unit impulse	1
$1(t)$ unit step	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-bt}	$\frac{1}{s+b}$
$1-e^{-bt}$	$\frac{b}{s(s+b)}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega}$
$a\cdot f(t)$	$a\cdot F(s)$
$x(t)+y(t)$	$X(s)+Y(s)$
$x(t)*y(t)$ convolution	$X(s)\cdot Y(s)$
$\frac{dy(t)}{dt}$	$sY(s)-y(0)$
$\frac{d^2y(t)}{dt^2}$	$s^2Y(s)-sy(0)-\frac{dy(0)}{dt}$
$\frac{d^ny(t)}{dt^n}$	$s^nY(s)-\frac{d^{n-1}y(0)}{dt^{n-1}}-s\frac{d^{n-2}y(0)}{dt^{n-2}}-\ldots-s^{n-1}y(0)$

table on the website

Laplace transform pairs

$f(t),\ t \geqslant 0$	$F(s)$
$\delta(t)$ unit impulse	1
$1(t)$ unit step	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
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$\cosh(\omega t)$	$\frac{s}{s^2-\omega}$

Properties of Laplace transform

$f(t),\ t \geqslant 0$	$F(s)$
$a \cdot f(t)$	$a \cdot F(s)$
$x(t)+y(t)$	$X(s)+Y(s)$
$x(t)*y(t)$ convolution	$X(s) \cdot Y(s)$

Properties of Laplace transform cont.

$\frac{dy(t)}{dt}$	$sY(s)-y(0)$
$\frac{d^2y(t)}{dt^2}$	$s^2Y(s)-sy(0)-\frac{dy(0)}{dt}$
$\frac{d^ny(t)}{dt^n}$	$s^nY(s)-\frac{d^{n-1}y(0)}{dt^{n-1}}-s\frac{d^{n-2}y(0)}{dt^{n-2}}-...-s^{n-1}y(0)$

Properties of Laplace transform cont.

$\int\limits_{t=0}^{\infty} f(t)dt$	$\dfrac{F(s)}{s}$
$\underbrace{\int \int}_{n} \ldots \int f(t)dt$	$\dfrac{F(s)}{s^n}$
$f(t-\tau)$	$e^{-\tau s}F(s)$

Properties of Laplace transform

$f(t),\ t \geqslant 0$	$F(s)$
$a \cdot f(t)$	$a \cdot F(s)$
$x(t) + y(t)$	$X(s) + Y(s)$
$x(t) * y(t)$ convolution	$X(s) \cdot Y(s)$
$\frac{dy(t)}{dt}$	$sY(s) - y(0)$
$\frac{d^2 y(t)}{dt^2}$	$s^2 Y(s) - s\,y(0) - \frac{dy(0)}{dt}$
$\frac{d^n y(t)}{dt^n}$	$s^n Y(s) - \frac{d^{n-1} y(0)}{dt^{n-1}} - s\frac{d^{n-2} y(0)}{dt^{n-2}} - \ldots - s^{n-1} y(0)$
$\int_{t=0}^{\infty} f(t)\,dt$	$\frac{F(s)}{s}$
$\underbrace{\int \int \ldots \int}_{n} f(t)\,dt$	$\frac{F(s)}{s^n}$
$f(t-\tau)$	$e^{-\tau s} F(s)$

Laplace transform

<u>Example 2</u>

Solve equation for a given initial conditions using Laplace transform.

$$\frac{d^2 y(t)}{dt^2} - 3\frac{dy(t)}{dt} + 2y(t) = 1(t)\ ,\ \frac{dy(0)}{dt} = 2,\ y(0) = 3,\ t \geqslant 0$$

Laplace transform

Example 2

Solve equation for a given initial conditions using Laplace transform.

$$\frac{d^2y(t)}{dt^2}-3\frac{dy(t)}{dt}+2y(t)=1(t)\ ,\ \frac{dy(0)}{dt}=2,\ y(0)=3,\ t\geqslant 0$$

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after Laplace transformation

$$Y(s) = \frac{1 - 7s + 3s^2}{s(s-1)(s-2)}$$

after partial fraction decomposition/expansion

$$Y(s) = \frac{1}{2}\frac{1}{s} + 3\frac{1}{s-1} - \frac{1}{2}\frac{1}{s-2}$$

after inverse Laplace transformation

$$y(t) = \frac{1}{2}1(t) + 3e^{t} - \frac{1}{2}e^{2t}$$

Transfer function – definition

For LTI SISO system with continuous input x(t) and output y(t) for zero initial conditions, transfer function is a ratio of the output of a system to the input of a system described in complex domain by the Laplace transformation.

$$H(s)=\frac{L\{y(t)\}}{L\{x(t)\}}=\frac{Y(s)}{X(s)}$$

Transfer function form

Standard form: $$H(s)=\frac{b^m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

Factored form: $$H(s)=\frac{(s-z_1)(s-z_2)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

$z_{1,}\ z_2,\ \dots,\ z_m$ - zeroes

$p_{1,}\ p_2,\ \dots,\ p_n$ - poles

Drawing of a transfer function

$$H(s)$$

$$s = \sigma + j\omega$$

for every $s \in \mathbb{C}$
there is $H(s) \in \mathbb{C}$

Drawing of a transfer function

$$H(s) = |H(s)| e^{j \arg H(s)}$$

$$s = \sigma + j\omega$$

for every $s \in \mathbb{C}$
there is $H(s) \in \mathbb{C}$

for every $s \in \mathbb{C}$
there is $|H(s)| \in \mathbb{R}$

for every $s \in \mathbb{C}$
there is $\operatorname{Arg} H(s) \in \mathbb{R}$

Drawing of a transfer function

Example

$$H(s)=\frac{2-s}{s^3+s^2-2}=\frac{s-2}{(s-1)(s+j+1)(s-j+1)}$$

Drawing of a transfer function

Example

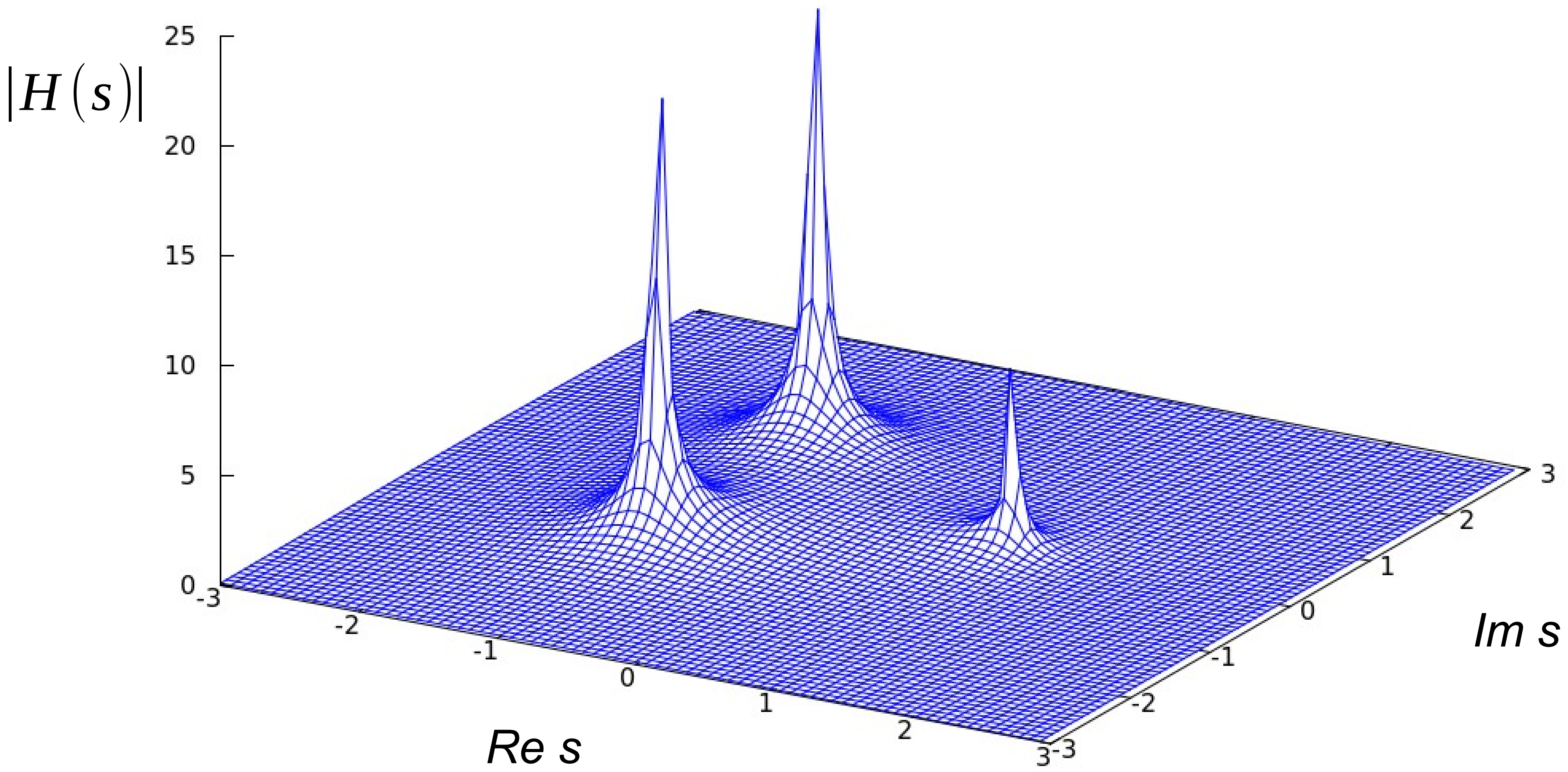
$$H(s)=\frac{2-s}{s^3+s^2-2}=\frac{s-2}{(s-1)(s+j+1)(s-j+1)}$$

Poles: $p_1=1,\ p_2=-1-j,\ p_3=-1+j$ Zeroes: $z_1=2$

Drawing of a transfer function

Example

Poles: $p_1 = 1$, $p_2 = -1 - j$, $p_3 = -1 + j$ Zeroes: $z_1 = 2$



Drawing of a transfer function

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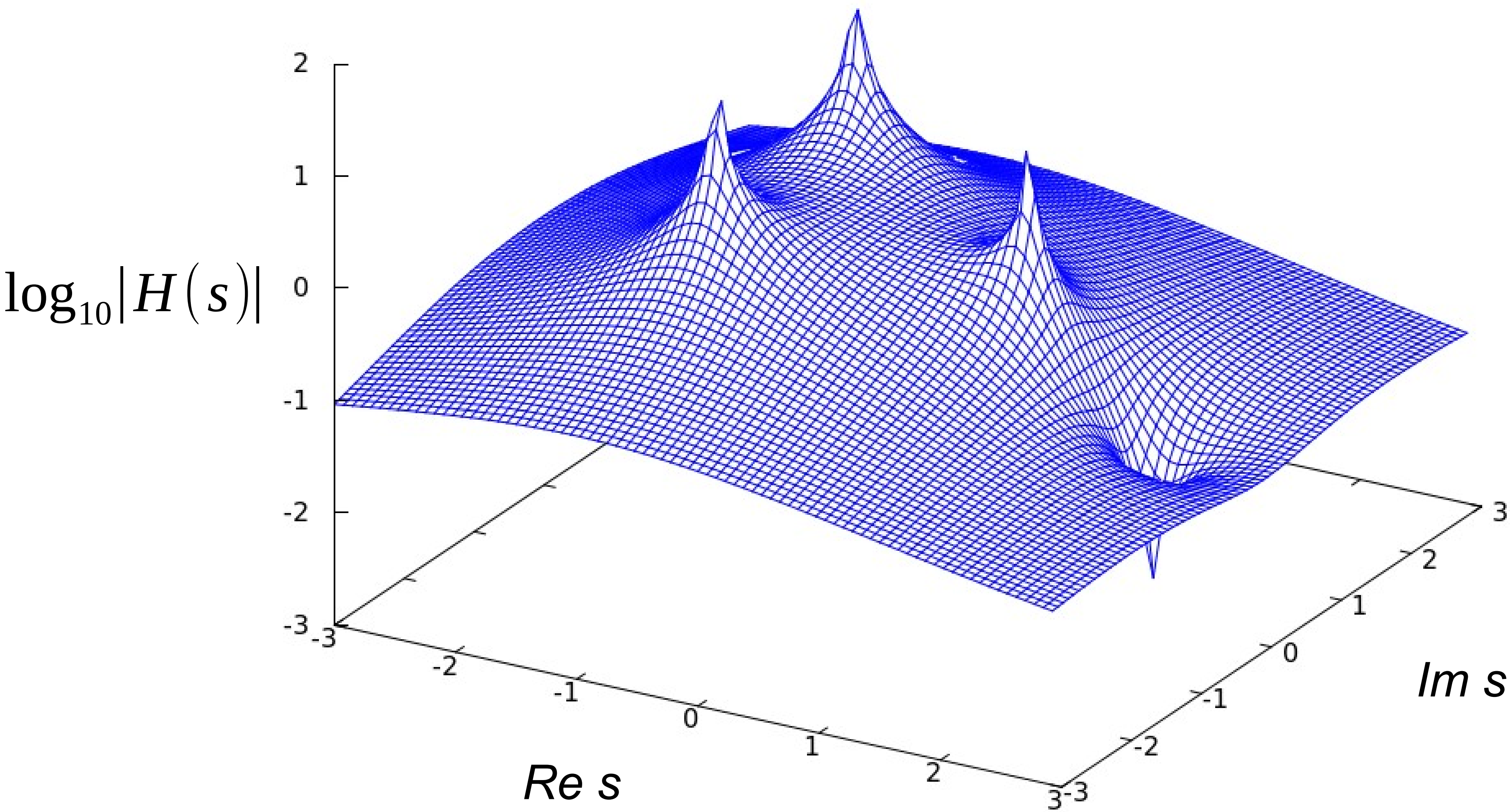

Drawing of a transfer function

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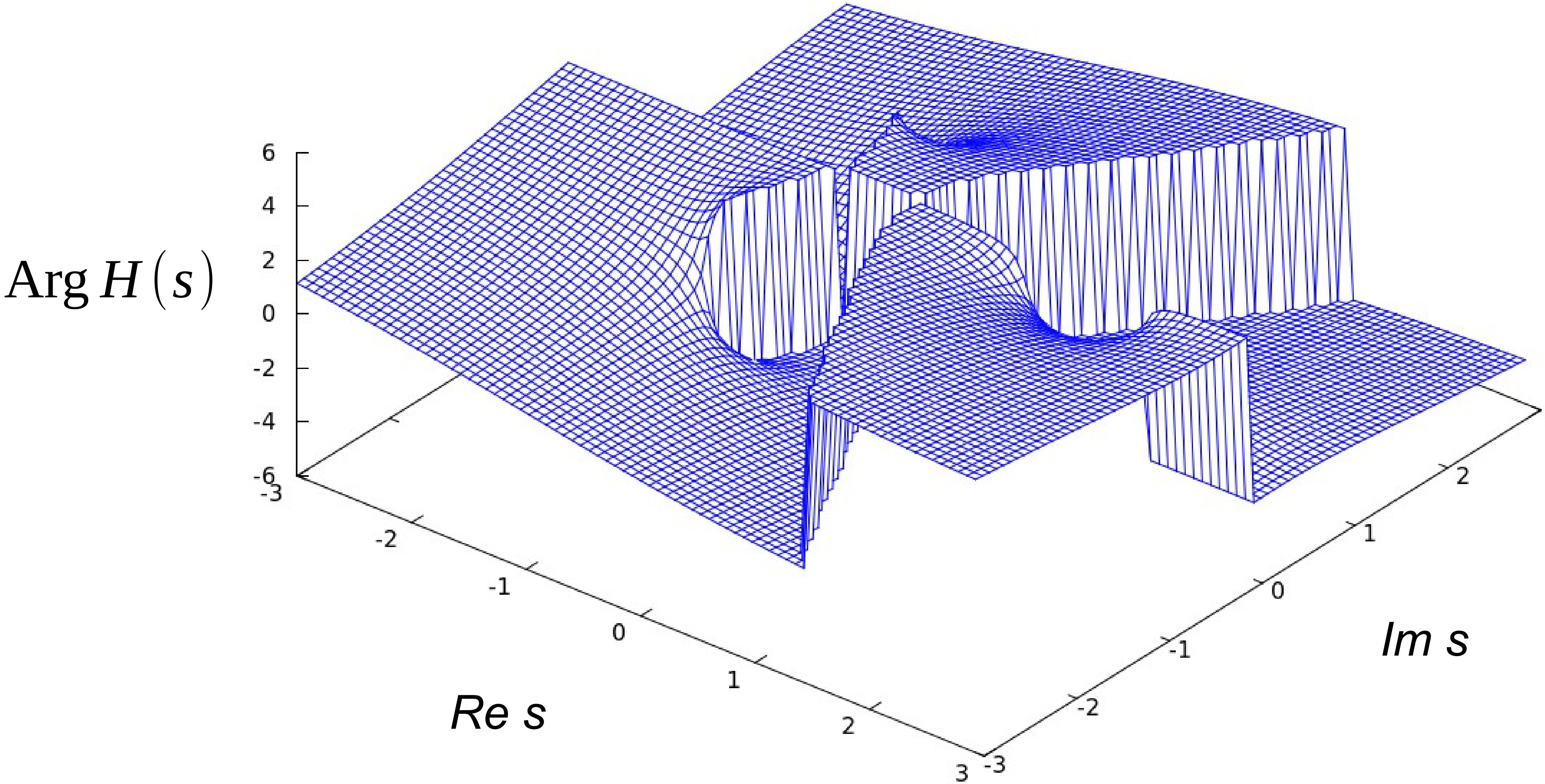

Input and output

Transfer function: $$H(s)=\frac{Y(s)}{X(s)}$$

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Laplace transform of output: $$Y(s)=H(s)X(s)$$

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Output in time domain: $$y(t)=L^{-1}\{Y(s)\}$$

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Transfer function: $H(s)=\frac{Y(s)}{X(s)}$

Laplace transform of output: $Y(s)=H(s)X(s)$

Output in time domain: $y(t)=L^{-1}\{Y(s)\}$

$$y(t)=L^{-1}\{H(s)X(s)\}=L^{-1}\{H(s)\}*L^{-1}\{X(s)\}=h(t)*x(t)$$

Input and output

Transfer function: $H(s)=\dfrac{Y(s)}{X(s)}$

Laplace transform of output: $Y(s)=H(s)X(s)$

Output in time domain: $y(t)=L^{-1}\{Y(s)\}$

$$y(t)=L^{-1}\{H(s)X(s)\}=L^{-1}\{H(s)\}*L^{-1}\{X(s)\}=h(t)*x(t)$$

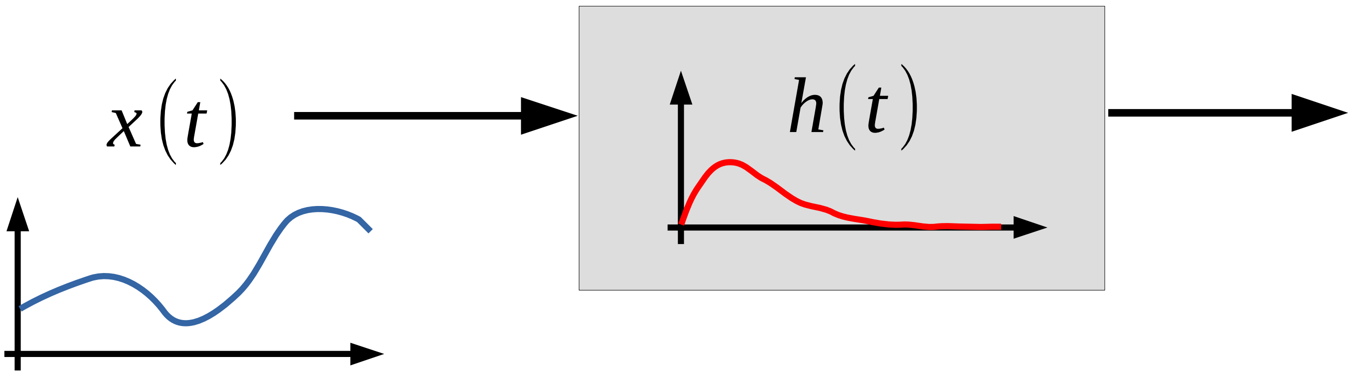
$h(t)$ - system impulse response ($y(t)$ when $x(t)=\delta(t)$)

Input and output

Convolution $h(t)*x(t)=\int_0^\infty x(\tau)h(t-\tau)d\tau$

Convolution

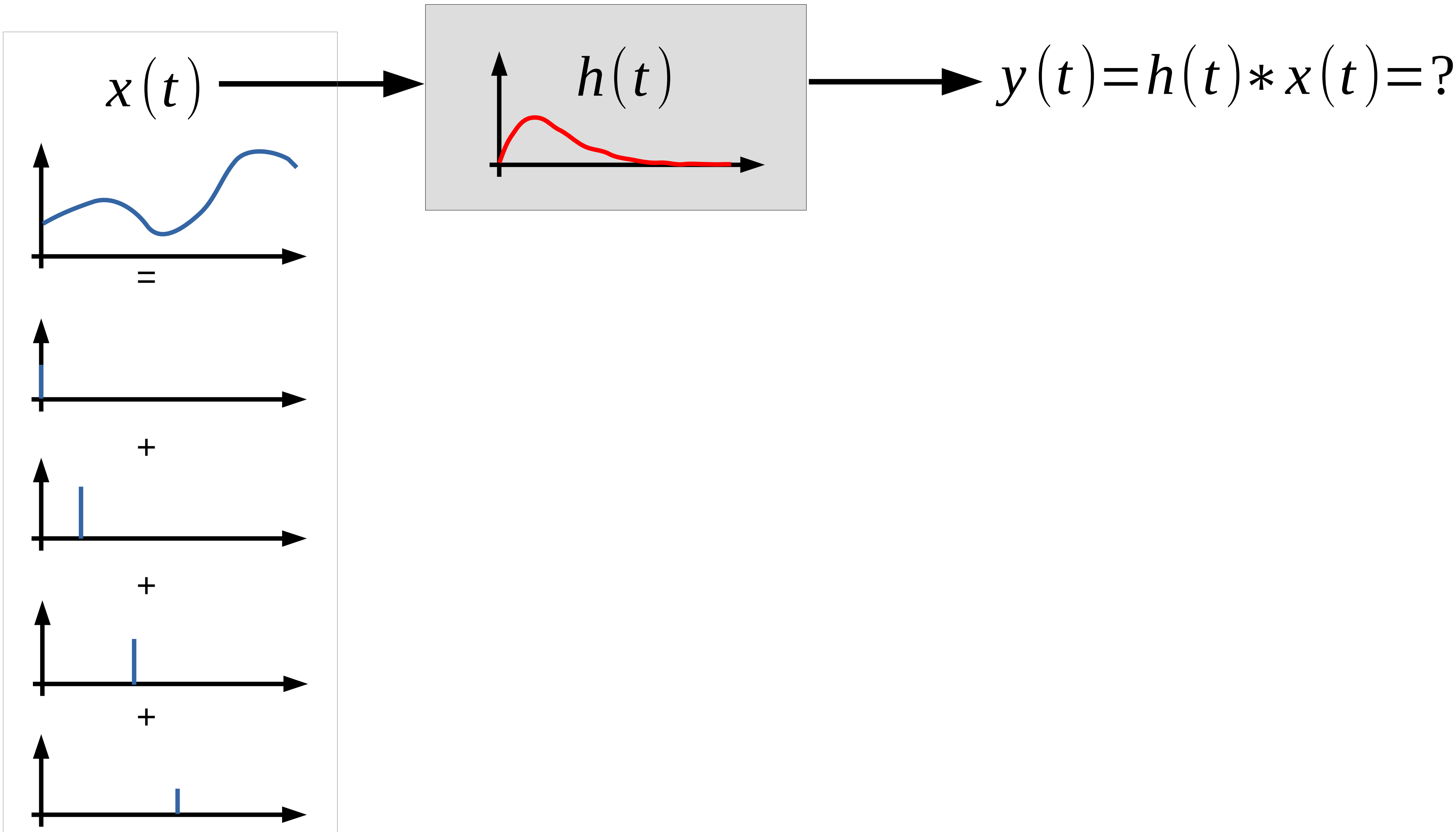
$$h(t)*x(t)=\int_{0}^{\infty} x(\tau)h(t-\tau)\,d\tau$$



$$y(t)=h(t)*x(t)=?$$

Convolution

$$h(t)*x(t)=\int_{0}^{\infty} x(\tau)h(t-\tau)d\tau$$



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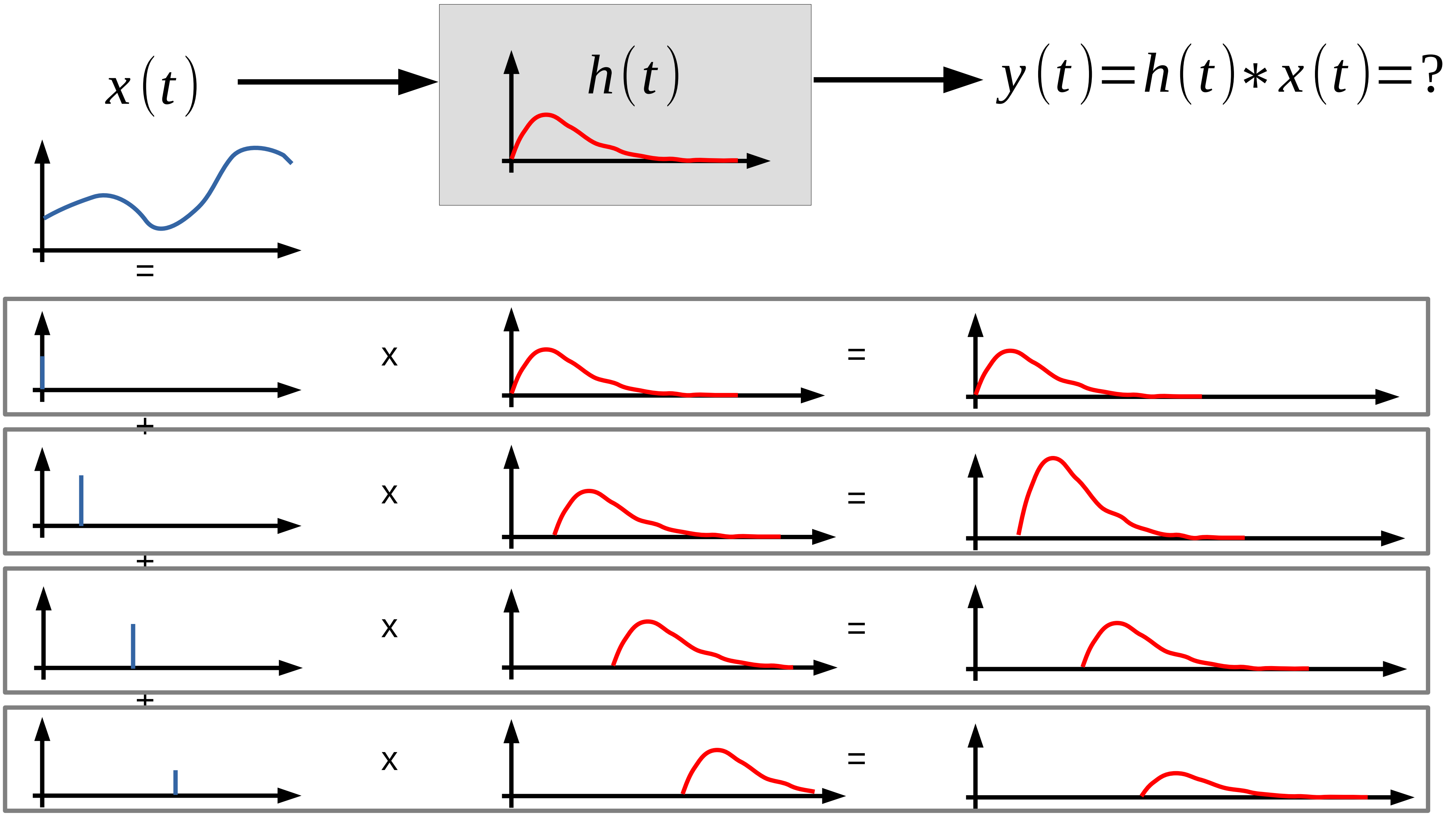

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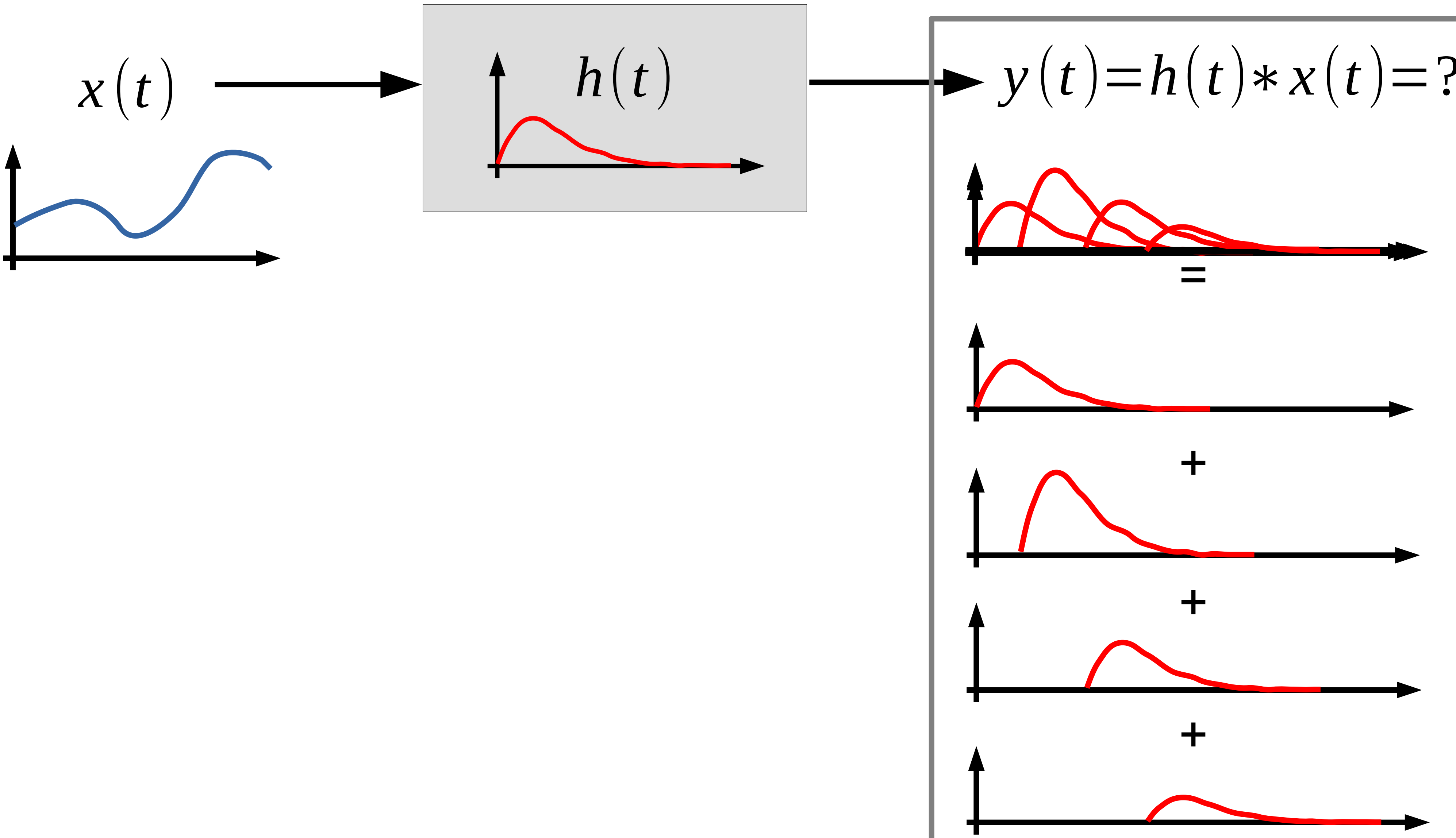

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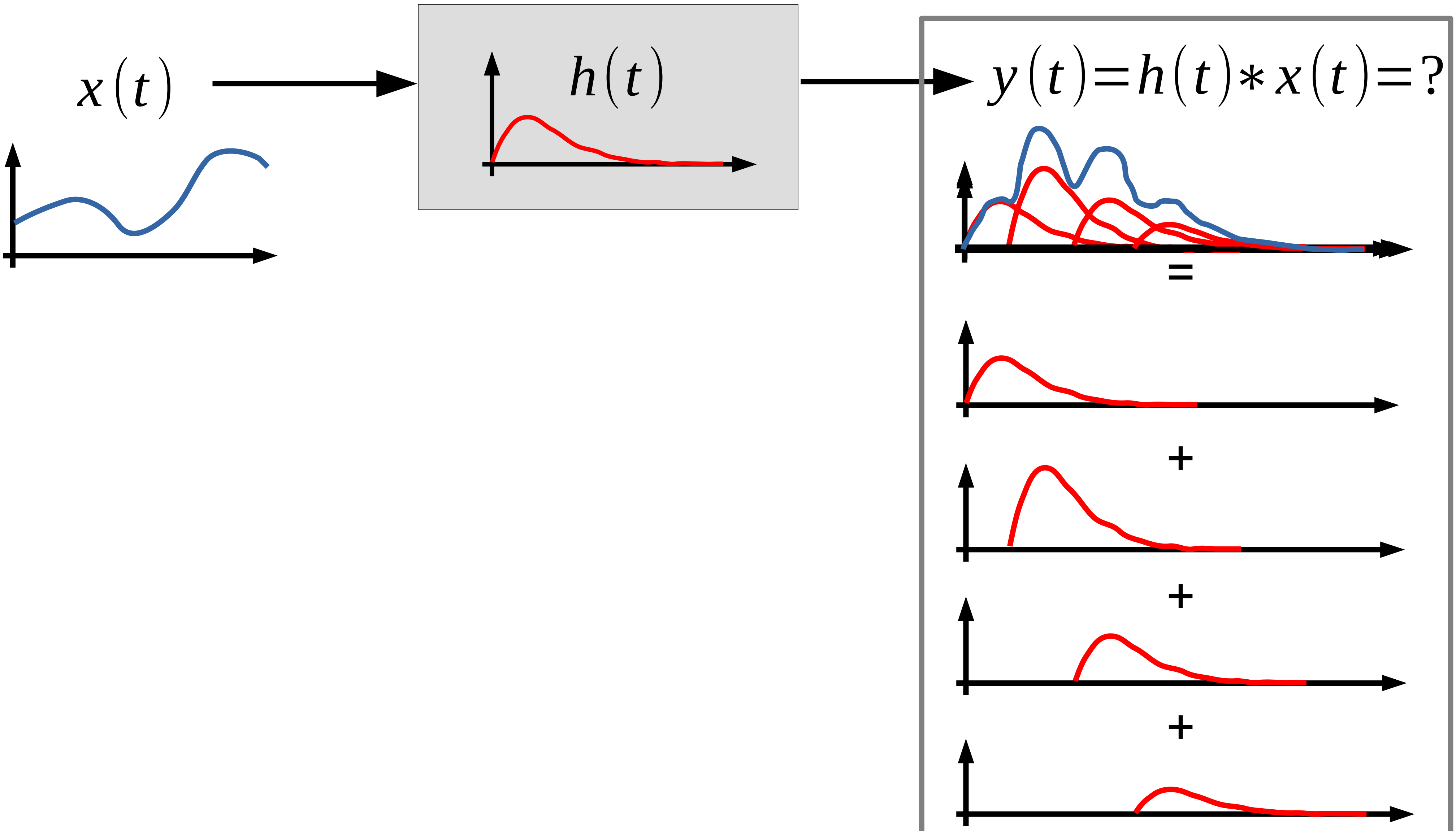

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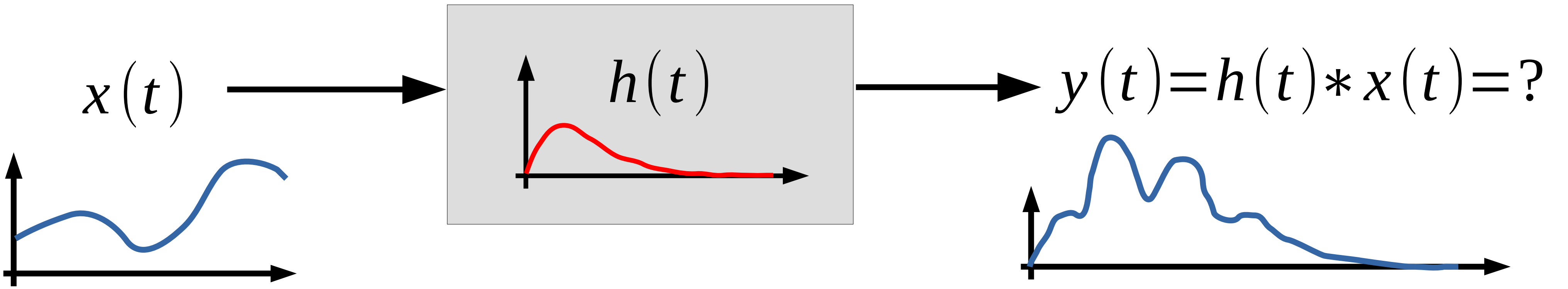
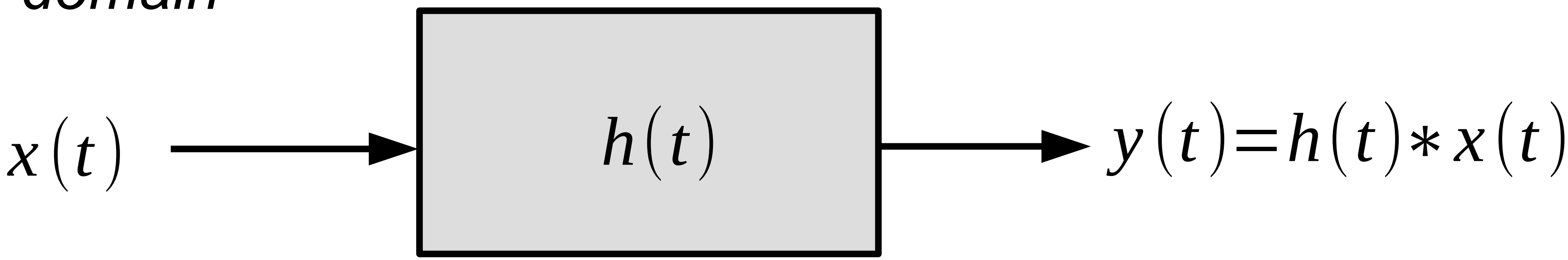
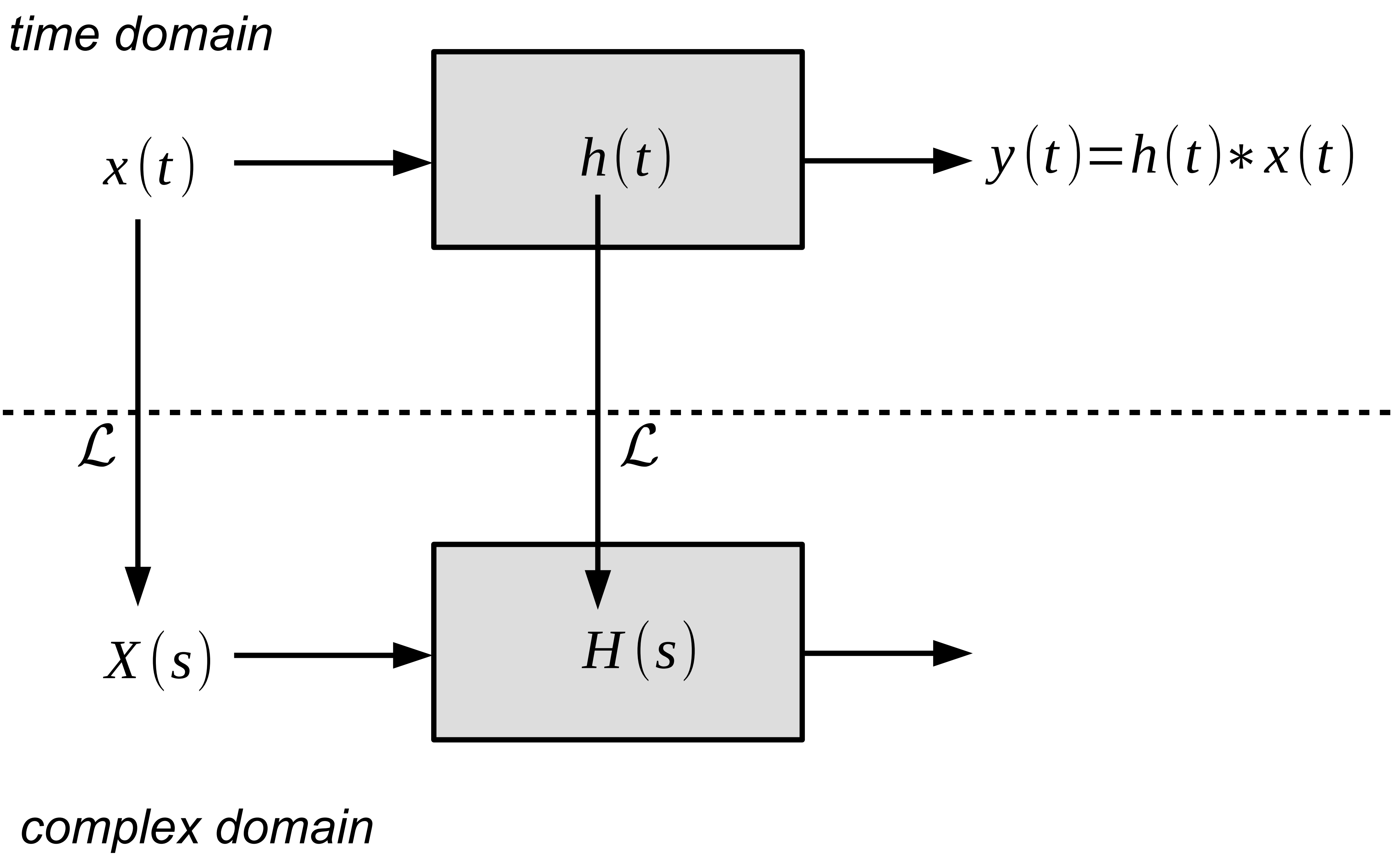


Input and output

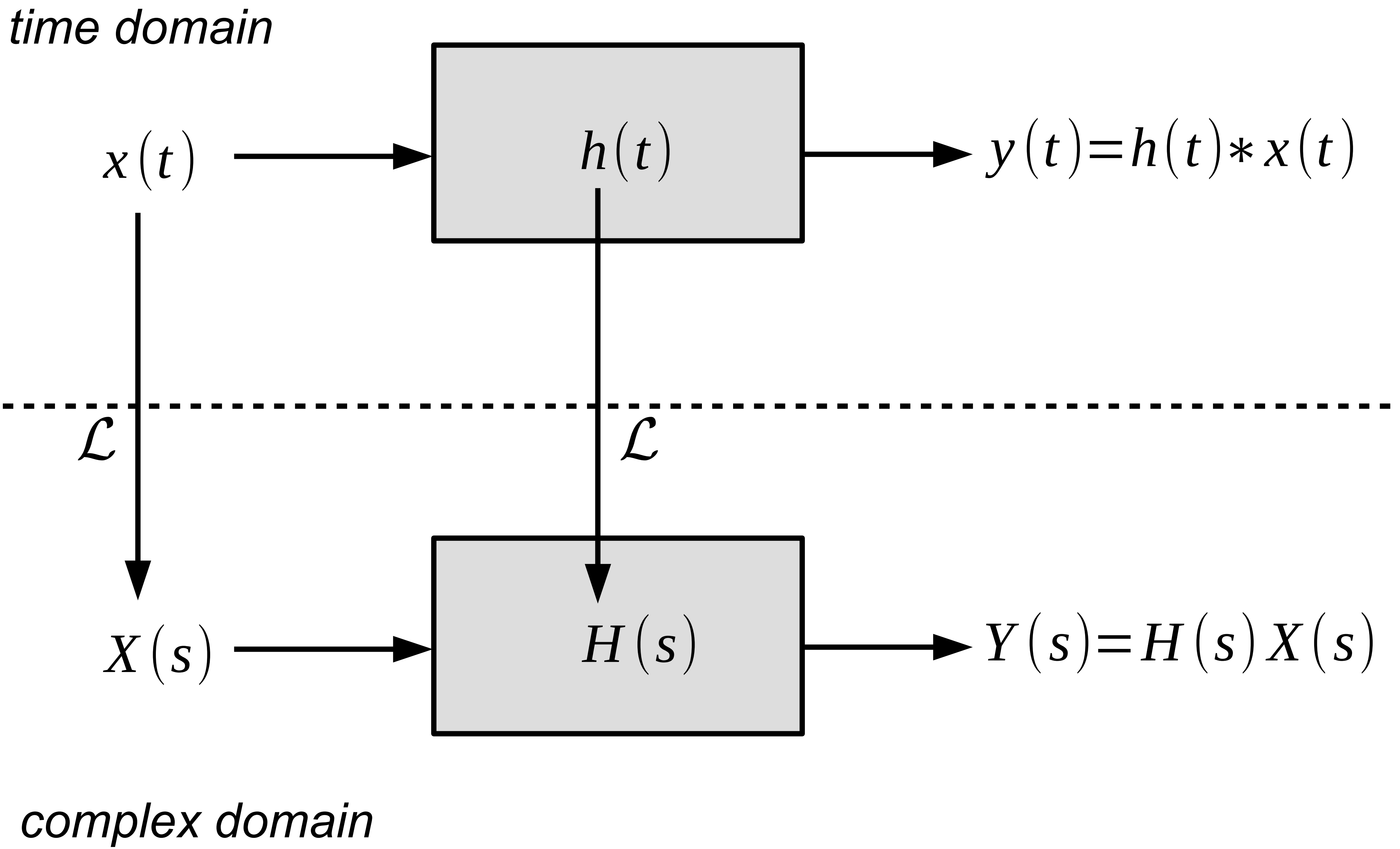
time domain



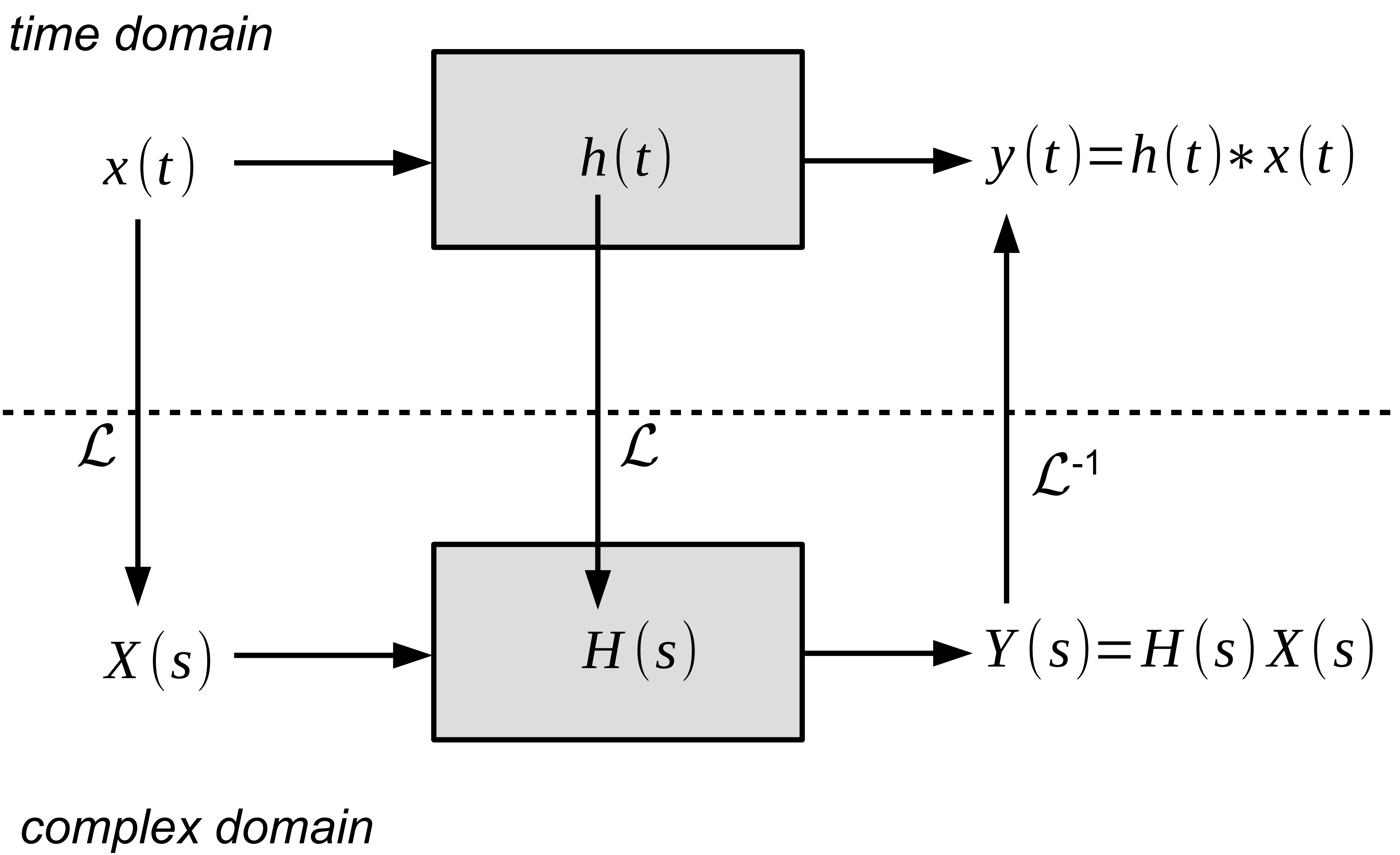
Input and output



Input and output



Input and output

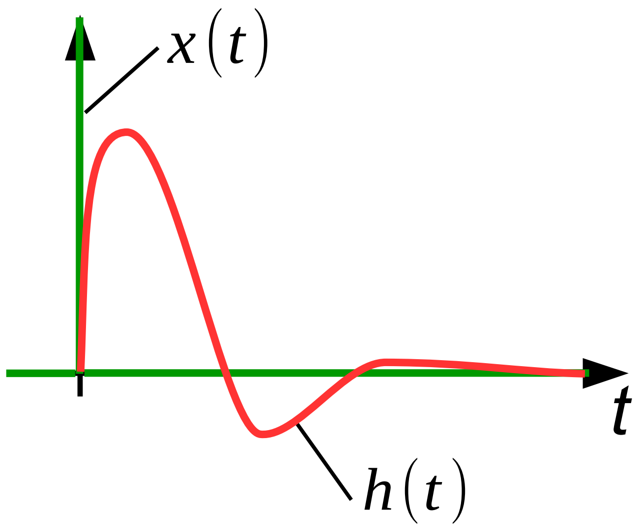


Input and output

$h(t)$

impulse response

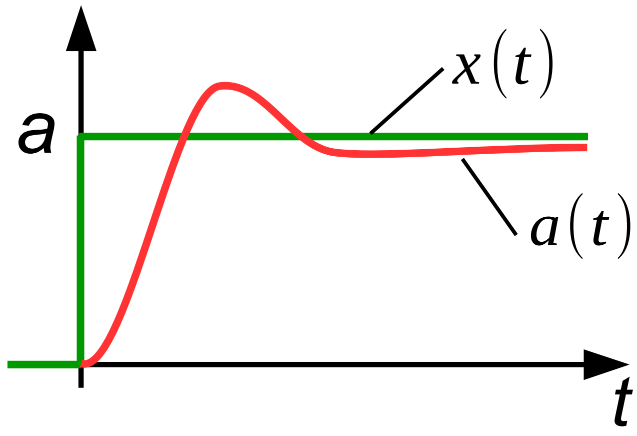
$y(t)$ for $x(t)=\delta(t)$



$a(t)$

step response

$y(t)$ for $x(t)=1(t)$



$$\frac{d\,a(t)}{dt}=h(t)$$

Exemplary input signals

No input: $x(t)=0$

Unit impulse (Dirac delta pseudofunction): $\delta(t)=\begin{cases} 0, & t<0 \\ \infty, & t=0 \\ 0, & t>0 \end{cases}$

Unit step function (Heviside step function): $1(t)=\begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$

$H(t)$ or $1_{+}(t)$

Ramp function: $x(t)=\begin{cases} 0, & t<0 \\ t, & t>0 \end{cases}$

Harmonic function: $x(t)=a\sin(\omega t)$

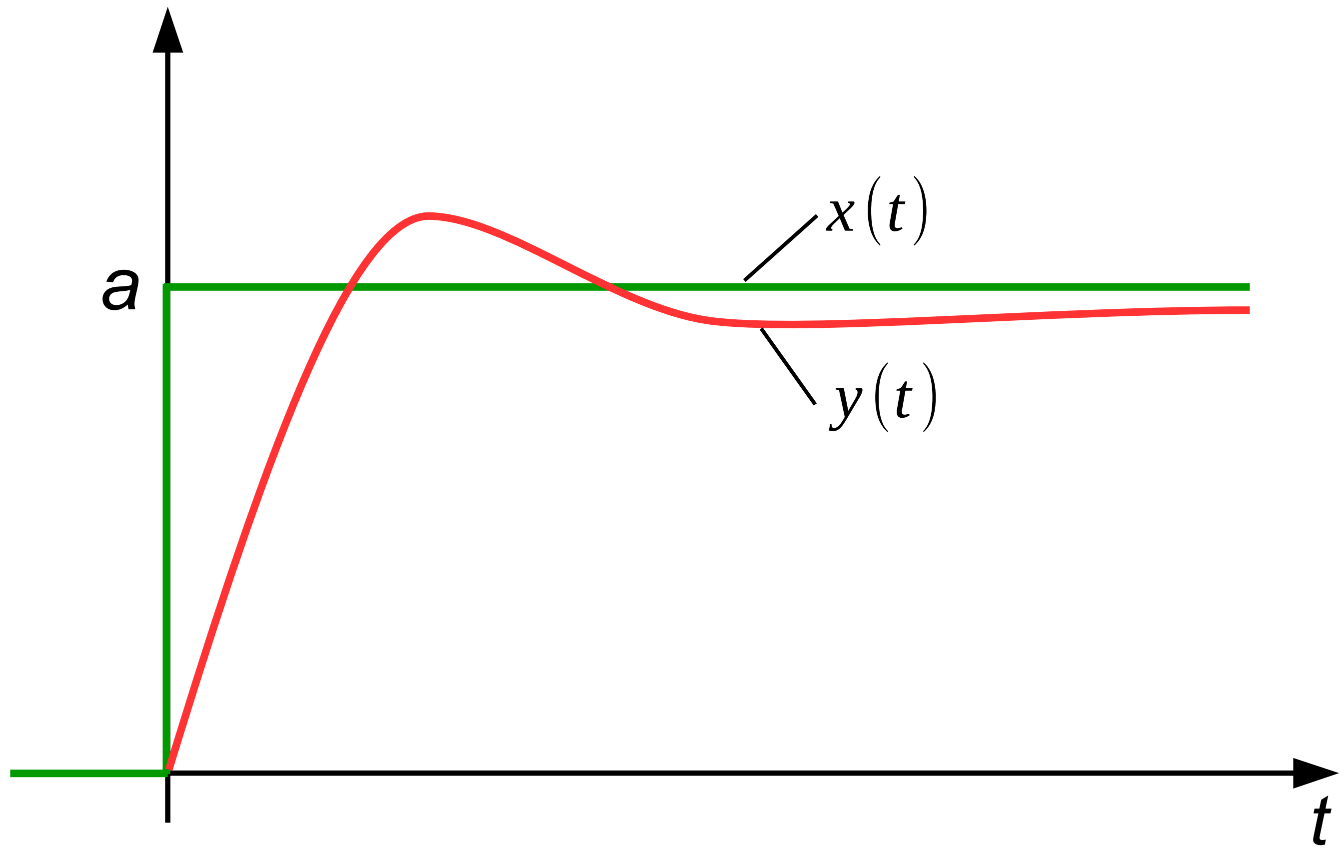
System step response

input: $x(t)=a\cdot 1(t)$ transfer function: $H(s)$ output: $y(t)=?$

$$X(s)=L\{x(t)\}=a\cdot\frac{1}{s}$$

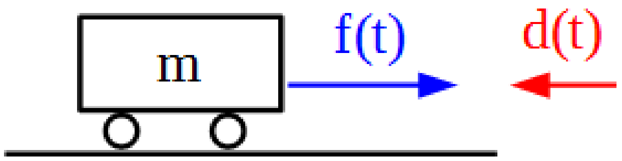
$$Y(s)=X(s)\cdot H(s)$$

$$y(t)=L^{-1}\{Y(s)\}$$



Step response – example 1

Car on a flat surface
m – mass,
f(t) – driving force,
d(t)=c*v(t) – air ressistance,
v(t) – velocity



$$m\frac{dv(t)}{dt}=f(t)-d(t)$$

Step response – example 1

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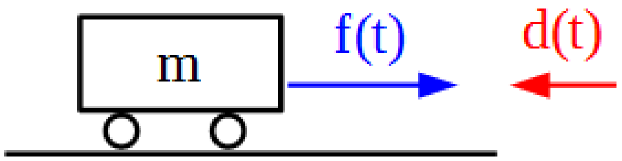

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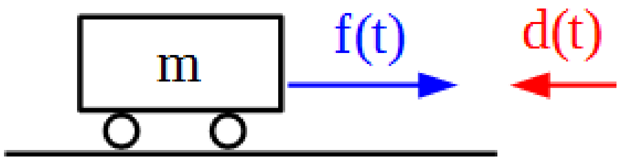

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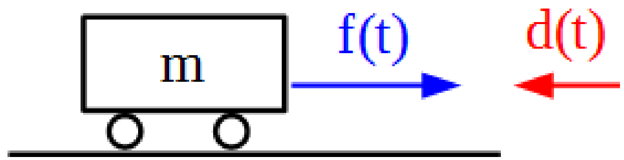
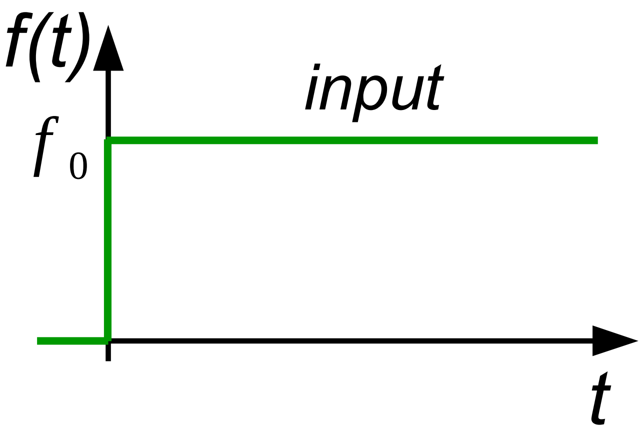


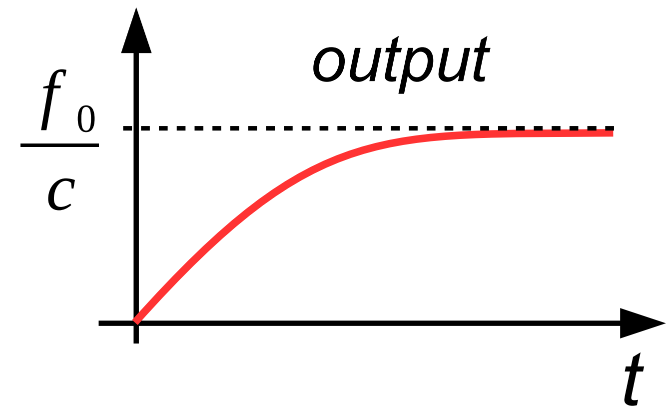
$$m\frac{dv(t)}{dt}=f(t)-d(t)$$



$$m\frac{dv(t)}{dt}=f(t)-cv(t)$$

$$msV(s)=F(s)-cV(s)$$

$$H(s)=\frac{V(s)}{F(s)}=\frac{1}{ms+c}$$



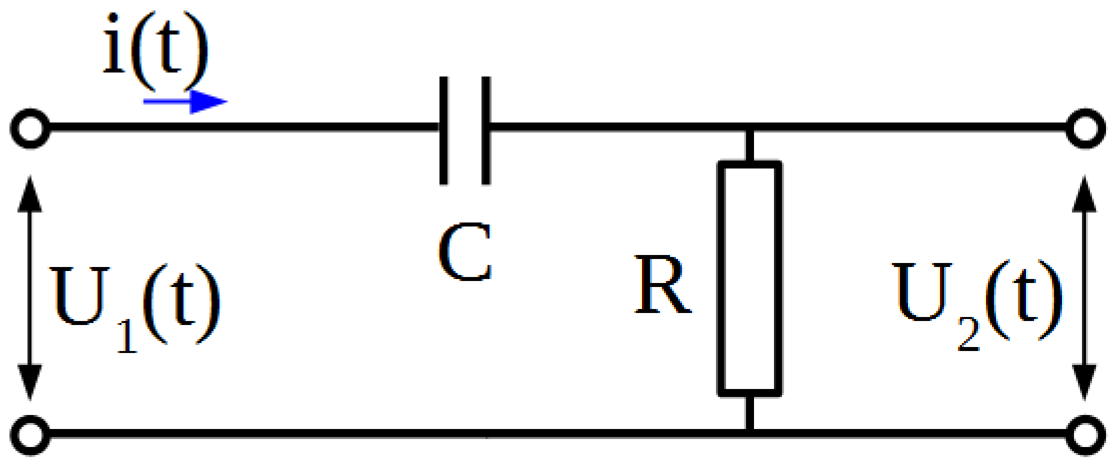
$$f(t)=f_0 1(t)$$

$$F(s)=f_0\frac{1}{s}$$

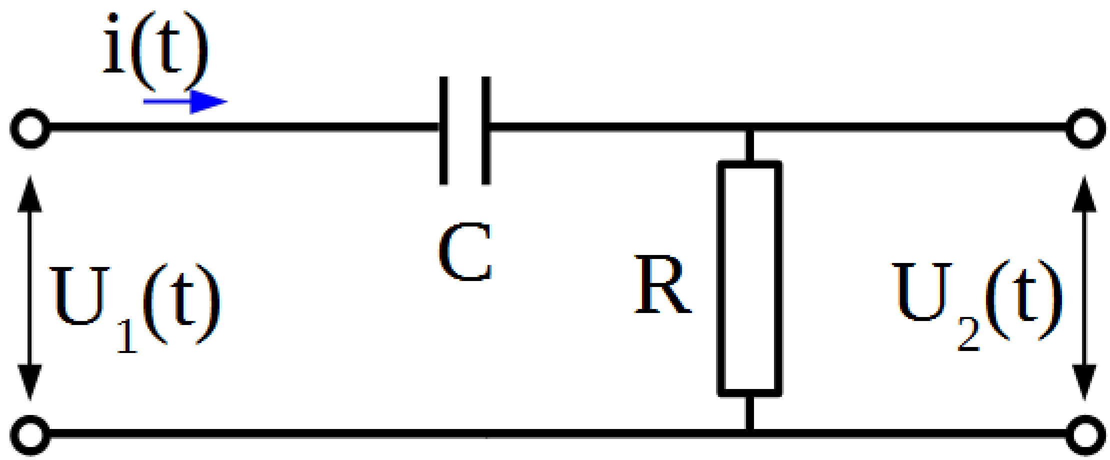
$$V(s)=H(s)F(s)=\frac{1}{ms+c}f_0\frac{1}{s}=\frac{f_0}{s(ms+c)}$$

$$v(t)=L^{-1}\left\{\frac{f_0}{s(ms+c)}\right\}=L^{-1}\left\{\frac{f_0}{c}\frac{c/m}{s(s+c/m)}\right\}=\frac{f_0}{c}\left(1-e^{-\frac{c}{m}t}\right)$$

Step response – example 2



Step response – example 2

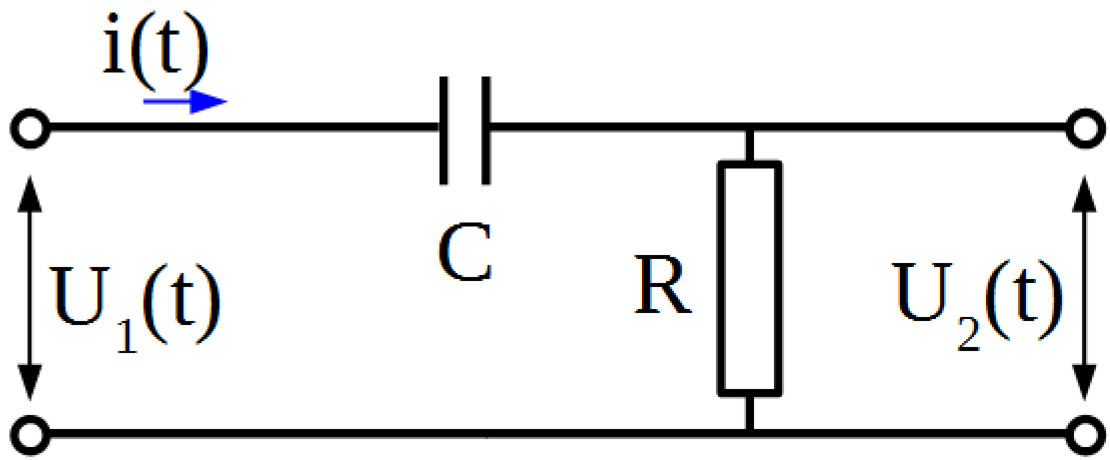


$$u_1(t) = u_C(t) + u_R(t)$$

$$u_C(t) = \frac{q(t)}{C}\ , \quad u_R(t) = i(t)R\ , \quad i = \frac{dq}{dt} \qquad u_2(t) = u_R(t)$$

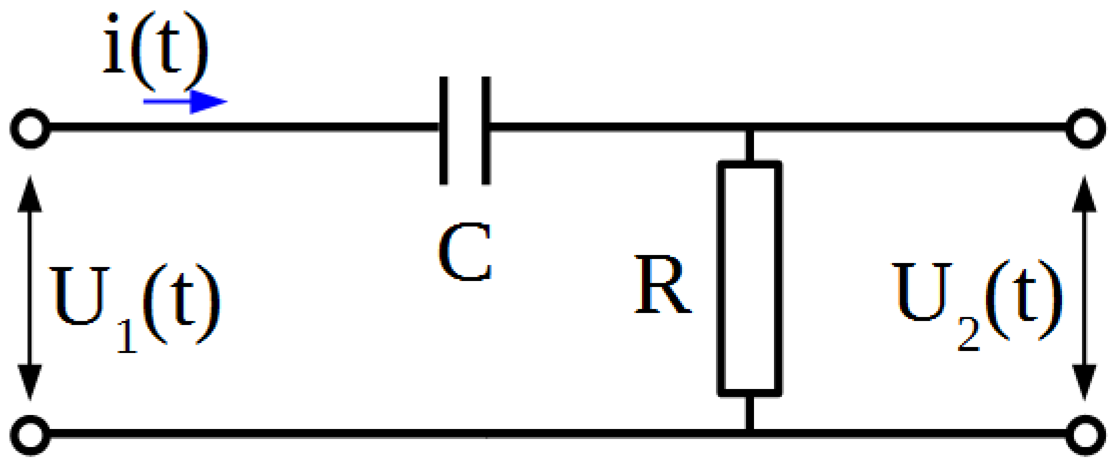
$$u_C(t) = \frac{q(t)}{C} = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \frac{u_R}{R}dt = \frac{1}{CR}\int u_2\,dt$$

Step response – example 2



$$\frac{1}{CR}\int u_2(t)\,dt + u_2(t) = u_1(t)$$

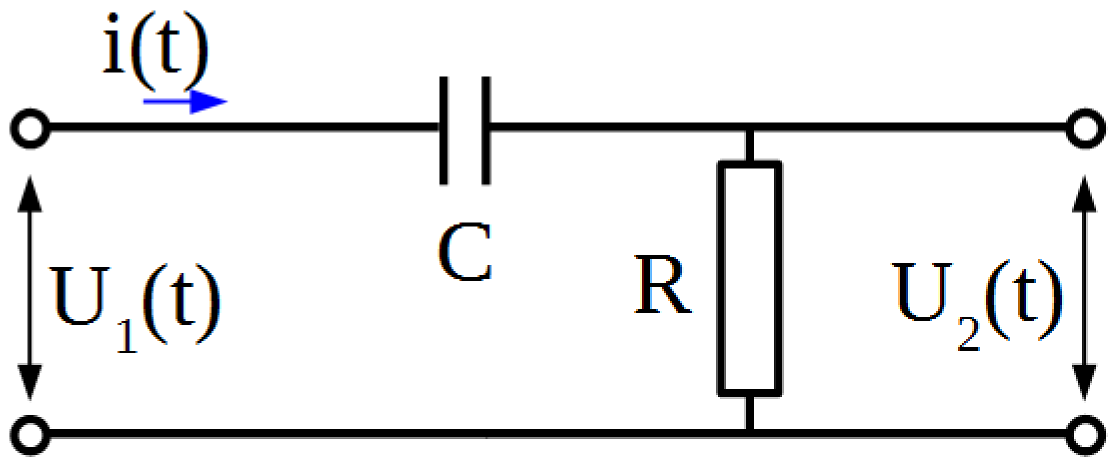
Step response – example 2



$$\frac{1}{CR}\int u_2(t)\,dt + u_2(t) = u_1(t)$$

$$\frac{1}{CR}u_2(t) + \frac{du_2(t)}{dt} = \frac{du_1(t)}{dt}$$

Step response – example 2

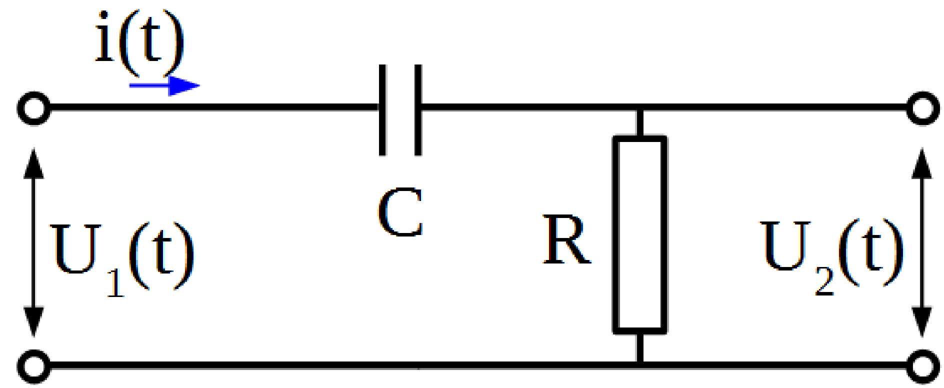


$$\frac{1}{T}U_2(s)+sU_2(s)=sU_1(s) \qquad T=CR$$

$$G(s)=\frac{U_2(s)}{U_1(s)}=\boxed{\frac{Ts}{1+Ts}}$$

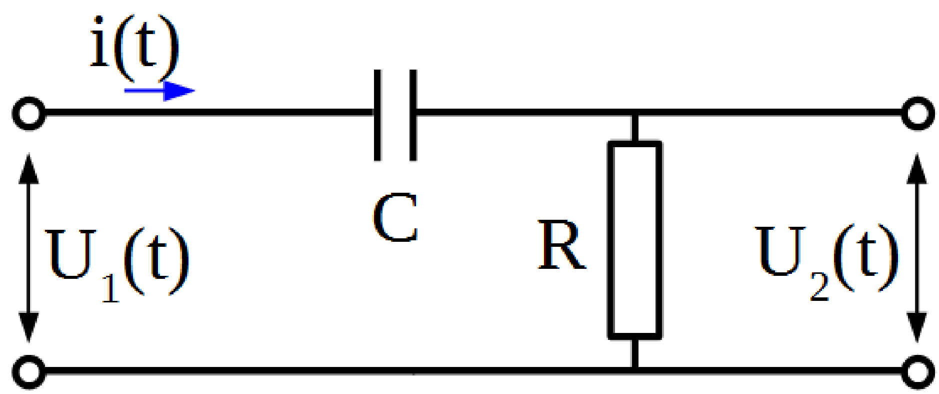
Step response – example 2

$$G(s) = \frac{Ts}{1+Ts}$$



Step response – example 2

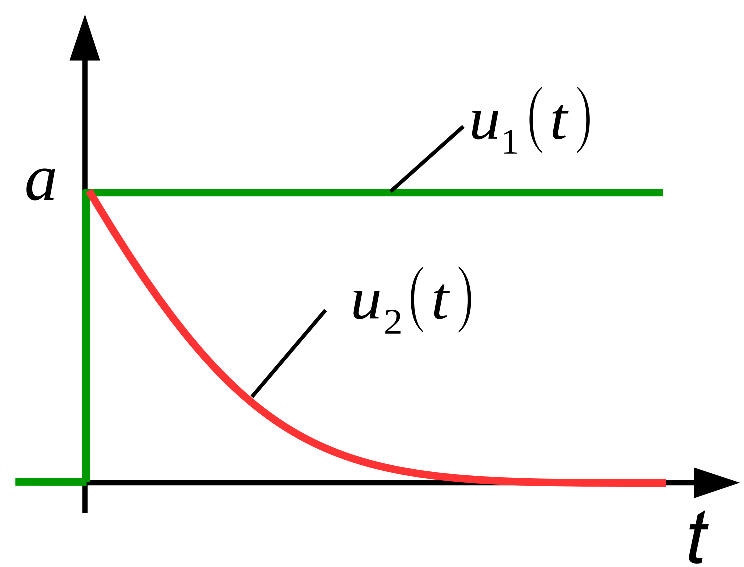
$$G(s)=\frac{Ts}{1+Ts}$$



$$u_1(t)=a\cdot 1(t),$$

$$U_2(s)=U_1(s)\cdot G(s)=a\frac{1}{s+\frac{1}{T}}$$

$$u_2(t)=L^{-1}\left[U_s(s)\right]=ae^{\frac{-t}{T}}$$



Computer methods
for transfer function analysis

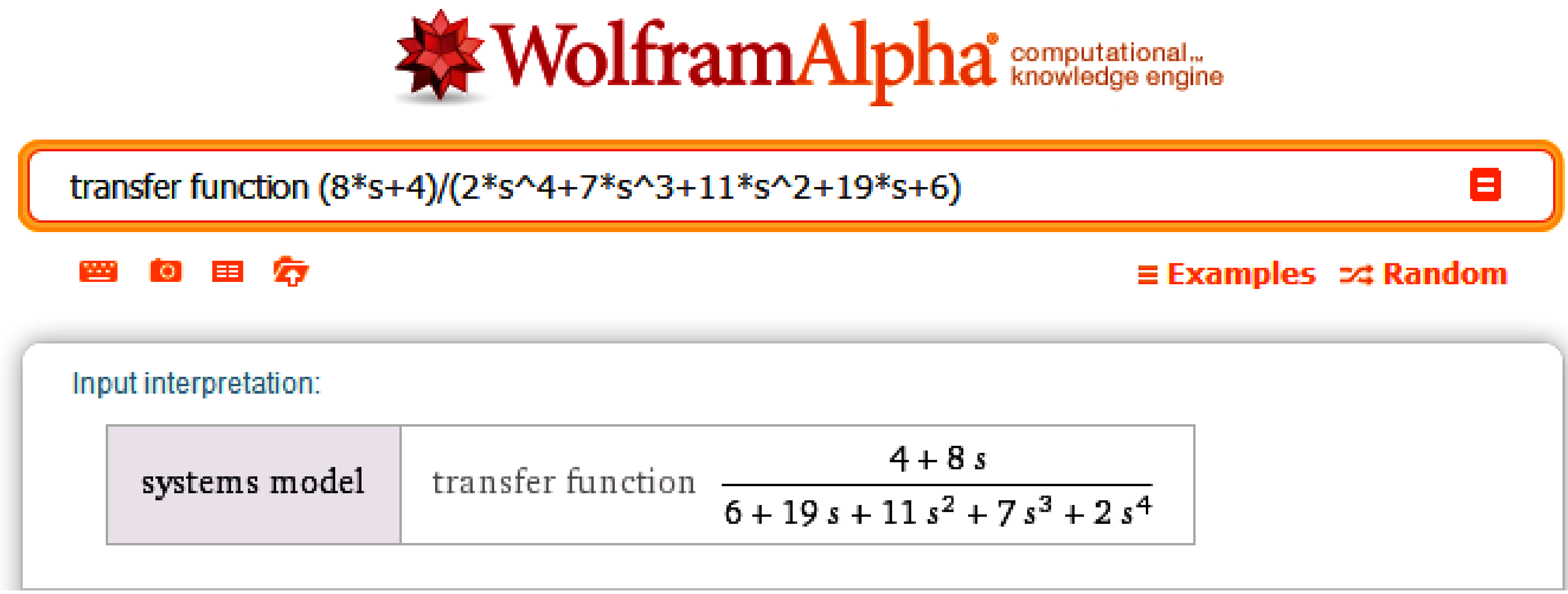
Exemplary computer algebra systems:

- Maxima/wxMaxima (free and open source)
- Wolfram Mathematica (http://www.wolfram.com/mathematica/)
- Mathcad
- Website: www.wolframalpha.com

(en.wikipedia.org/wiki/List_of_computer_algebra_systems)

Spreadsheet for graphs (Excel, LibreOffice Calc)

WolframAlpha



WolframAlpha