



Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control Winter 2018/2019

Lecturer: Sebastian Korczak, PhD, Eng.

Lecture 13

Stability criteria.
Gain margin and phase margin.
System correction.

STABILITY CRITERIA

General stability criterion

Hurwitz criterion

Nyquist stability criterion

General stability criterion

LTI SISO system is asymptotically stable if real part of every pole of the system's transfer function is less than zero.

$$G(s) = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

$$\operatorname{Re} p_1 < 0 \wedge \operatorname{Re} p_2 < 0 \wedge \dots \wedge \operatorname{Re} p_n < 0$$

Hurwitz criterion

mathematics

a necessary and sufficient
condition whether all
the roots of the polynomial
are in the left half of
the complex plane

Hurwitz criterion

mathematics

a necessary and sufficient condition whether all the roots of the polynomial are in the left half of the complex plane

control theory

a necessary and sufficient condition whether all the poles of transfer function of a linear system have negative real parts

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

is stable if:

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

is stable if:

$$\textcircled{1} \quad a_n > 0, \quad a_{n-1} > 0, \quad \dots, \quad a_1 > 0, \quad a_0 > 0$$

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

is stable if:

① $a_n > 0, a_{n-1} > 0, \dots, a_1 > 0, a_0 > 0$

②

$$M_n = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

is stable if:

① $a_n > 0, a_{n-1} > 0, \dots, a_1 > 0, a_0 > 0$

② $\det \Delta_2 > 0$

$\det \Delta_3 > 0$

...

$\det \Delta_{n-1} > 0$

Δ_i - leading principal minor
of order i

$$M_n = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Δ_2 (blue arrow) points to the top-left 2x2 submatrix.
 Δ_3 (green arrow) points to the top-left 3x3 submatrix.
 Δ_{n-1} (red arrow) points to the top-left (n-1)x(n-1) submatrix.

Hurwitz criterion

Hurwitz criterion \neq Routh criterion
(1895) (1876)

Liénard–Chipart criterion – modification of Hurwitz criterion

Hurwitz criterion

Example 1

$$G(s) = \frac{5s + 3}{10s^2 + 3s + 1}$$

Hurwitz criterion

Example 2

$$G(s) = \frac{2s}{2s^3 + s + 20}$$

Hurwitz criterion

Example 3

$$G(s) = \frac{3s - 5}{s^3 + 4s^2 + 3s + 10}$$

Hurwitz criterion

Example 4

$$G(s) = \frac{1}{3s^4 + 4s^3 + 6s^2 + 4s + 5}$$

Hurwitz criterion

Example 5

Choose k parameter to satisfy Hurwitz criterion

$$\frac{k s}{4 s^3 + 3 s^2 + k s + 1}$$

Hurwitz criterion

Example 6

Choose k parameter to satisfy Hurwitz criterion

2

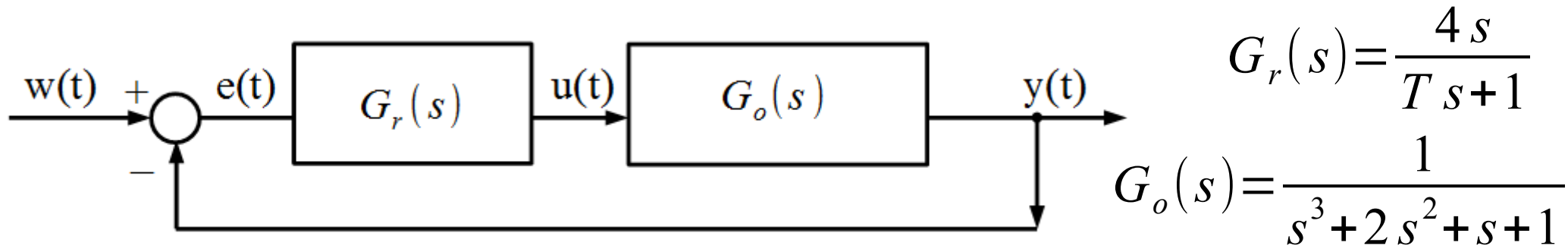
$$\frac{2s^3 + ks^2 + (1+k)s + 3}{2}$$

Homework

Hurwitz criterion

Example 7

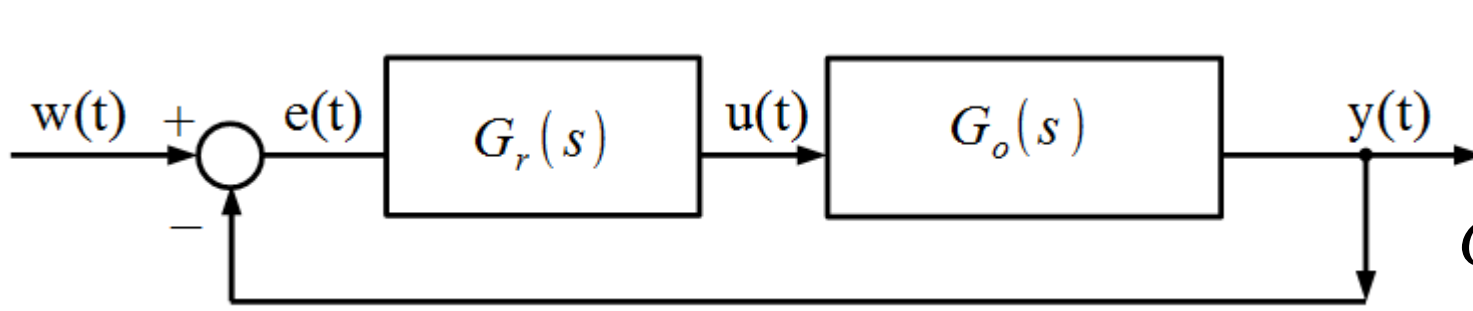
Choose T parameter to satisfy Hurwitz criterion



Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s) = \frac{4s}{Ts+1}$$

$$G_o(s) = \frac{1}{s^3 + 2s^2 + s + 1}$$

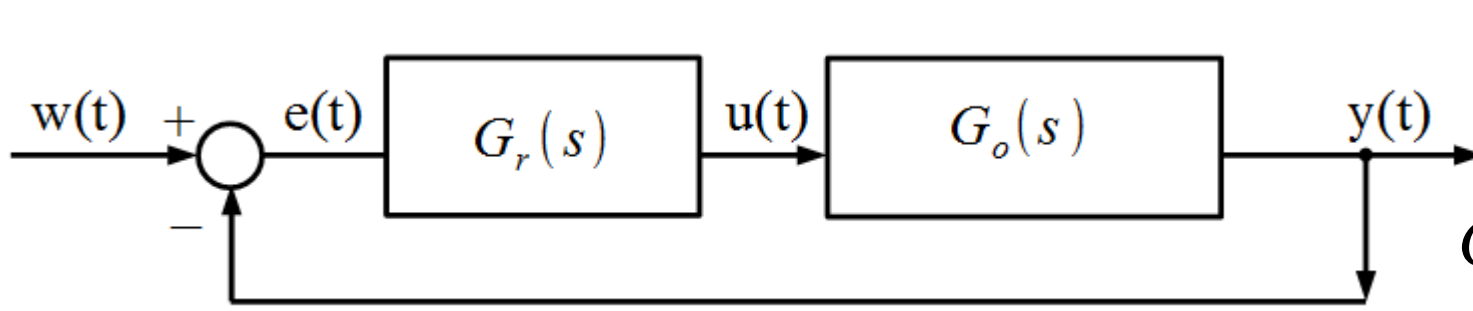
$$G_z(s) = \frac{G_r G_o}{1 + G_r G_o G_p} = \frac{4s}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

$$a_4 = T, \quad a_3 = 2T + 1, \\ a_2 = T + 2, \quad a_1 = T + 5, \quad a_0 = 1$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s) = \frac{4s}{Ts+1}$$

$$G_o(s) = \frac{1}{s^3 + 2s^2 + s + 1}$$

$$G_z(s) = \frac{G_r G_o}{1 + G_r G_o G_p} = \frac{4s}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

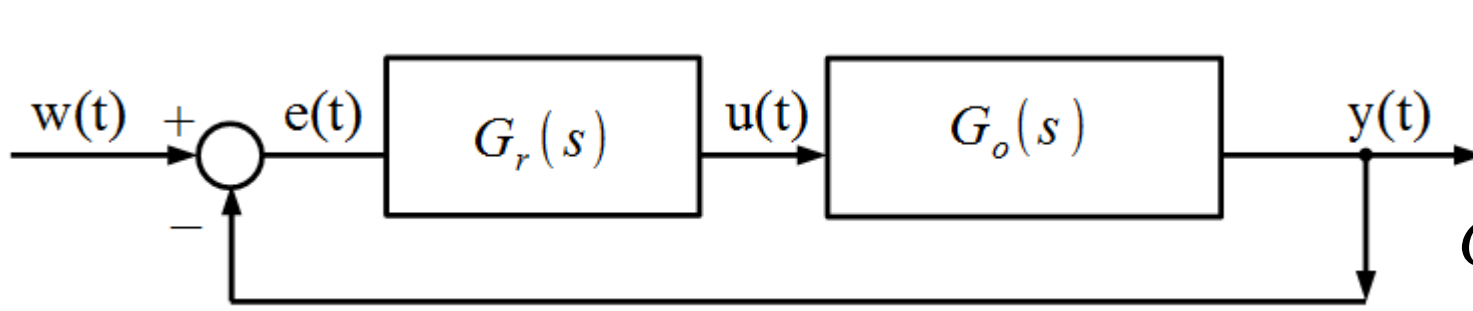
$$a_4 = T, \quad a_3 = 2T + 1, \\ a_2 = T + 2, \quad a_1 = T + 5, \quad a_0 = 1$$

$$a_4 > 0, \quad a_3 > 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 > 0 \rightarrow T > 0$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s) = \frac{4s}{Ts+1}$$

$$G_o(s) = \frac{1}{s^3 + 2s^2 + s + 1}$$

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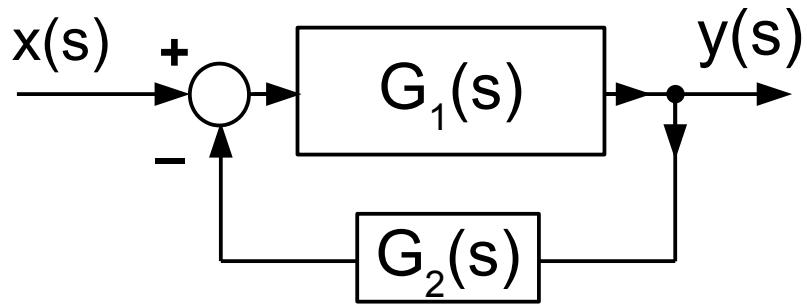
$$a_4 = T, \quad a_3 = 2T + 1, \\ a_2 = T + 2, \quad a_1 = T + 5, \quad a_0 = 1$$

$$a_4 > 0, \quad a_3 > 0, \quad a_2 > 0, \quad a_1 > 0, \quad a_0 > 0 \rightarrow T > 0$$

$$\Delta_2 = \begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \end{bmatrix} = T^2 + 2 > 0 \quad T \in \mathbb{R}$$

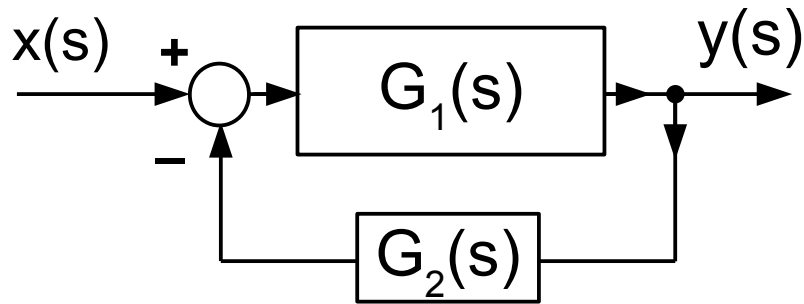
$$\Delta_3 = \begin{bmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{bmatrix} = T^3 + T^2 - 2T + 9 > 0 \rightarrow T > 2.83$$

Nyquist stability criterion



$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

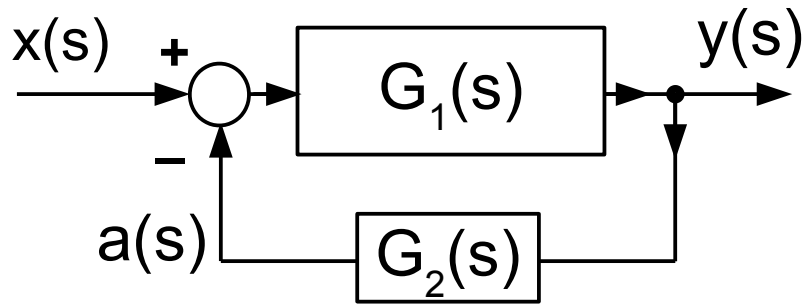
Nyquist stability criterion



$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

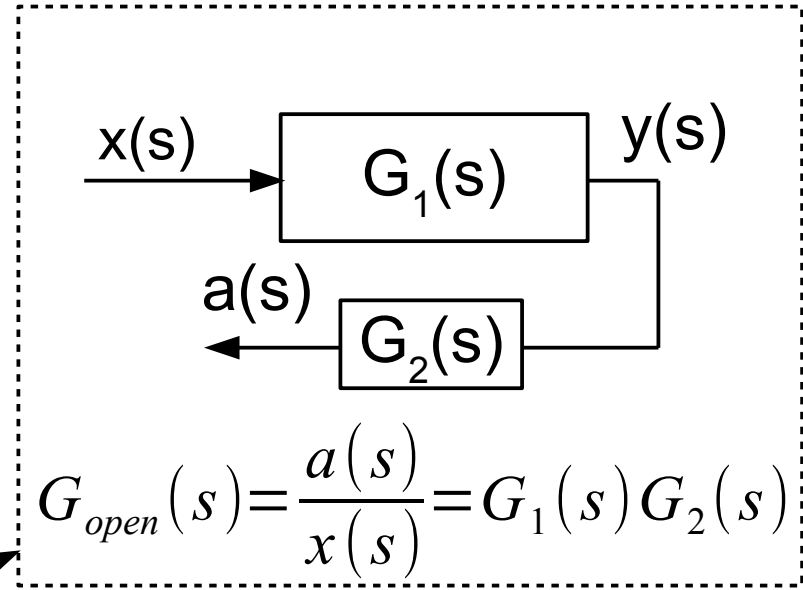
Unstable if: $G_1 G_2 = -1$

Nyquist stability criterion



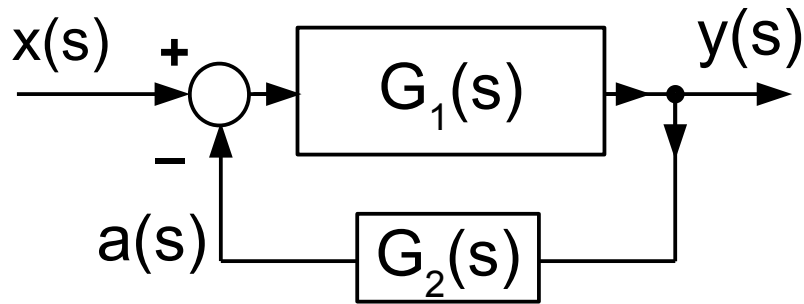
$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

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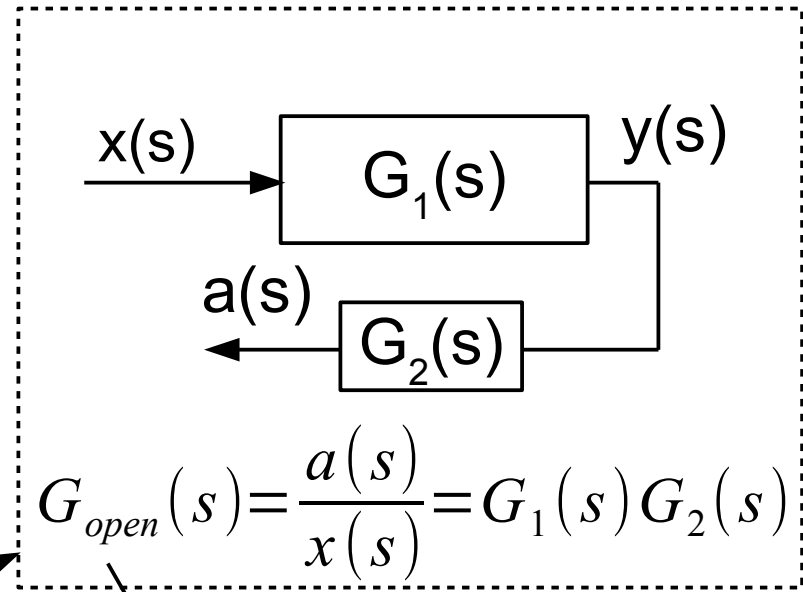
$$G_{open}(s) = \frac{a(s)}{x(s)} = G_1(s)G_2(s)$$

Nyquist stability criterion

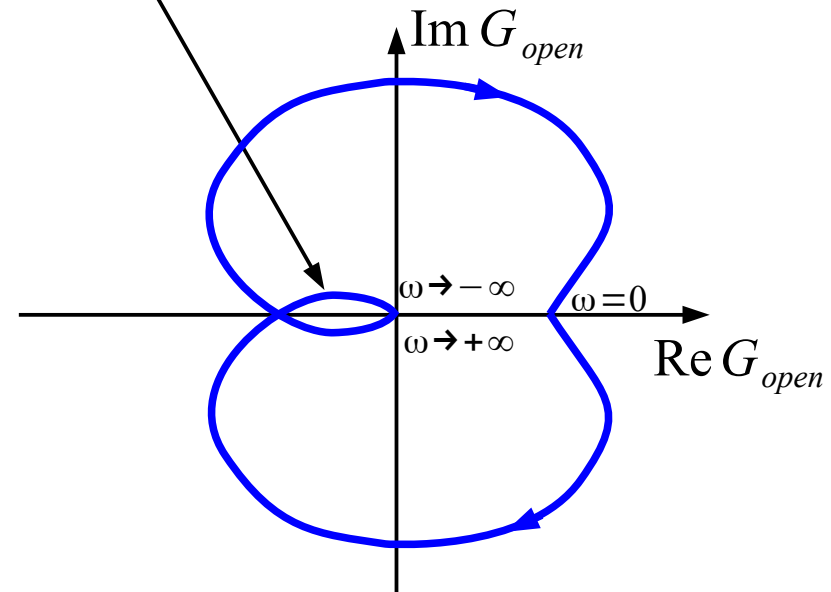


$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

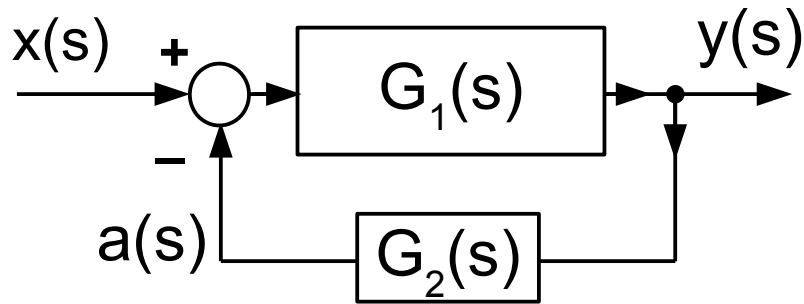
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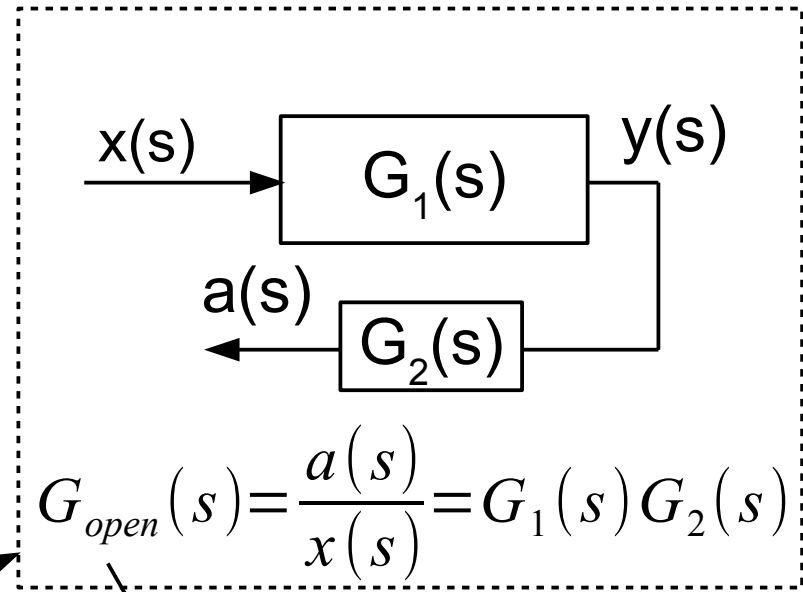
$$G_{open}(s) = \frac{a(s)}{x(s)} = G_1(s)G_2(s)$$



Nyquist stability criterion



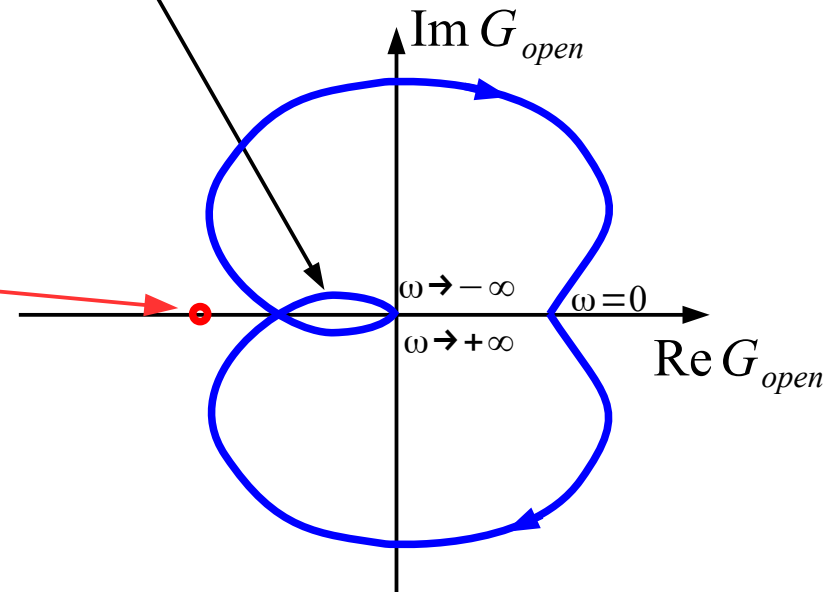
$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$



$$G_{open}(s) = \frac{a(s)}{x(s)} = G_1(s)G_2(s)$$

Unstable if:

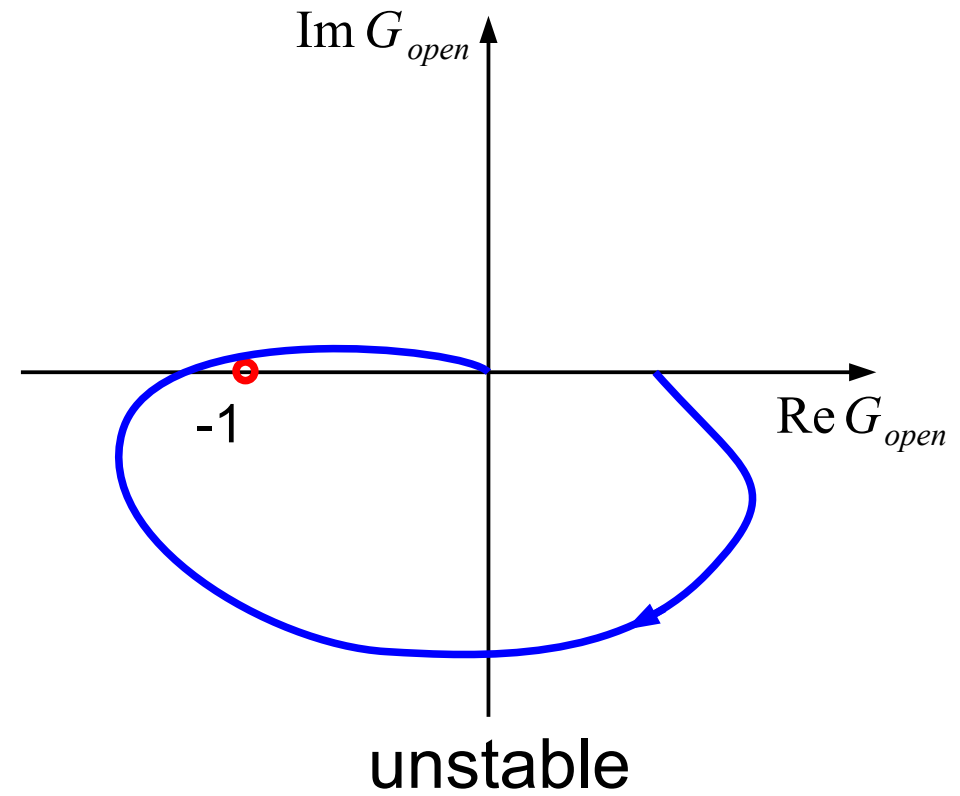
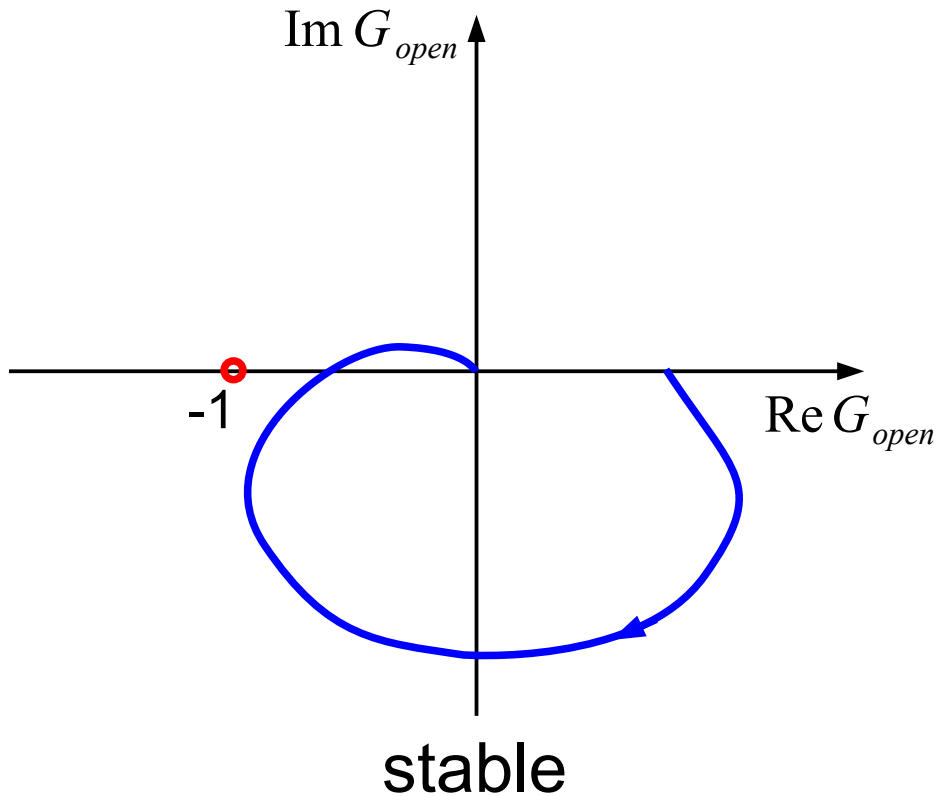
$$G_1 G_2 = -1$$



Nyquist stability criterion (particular)

The closed-loop system is stable if:

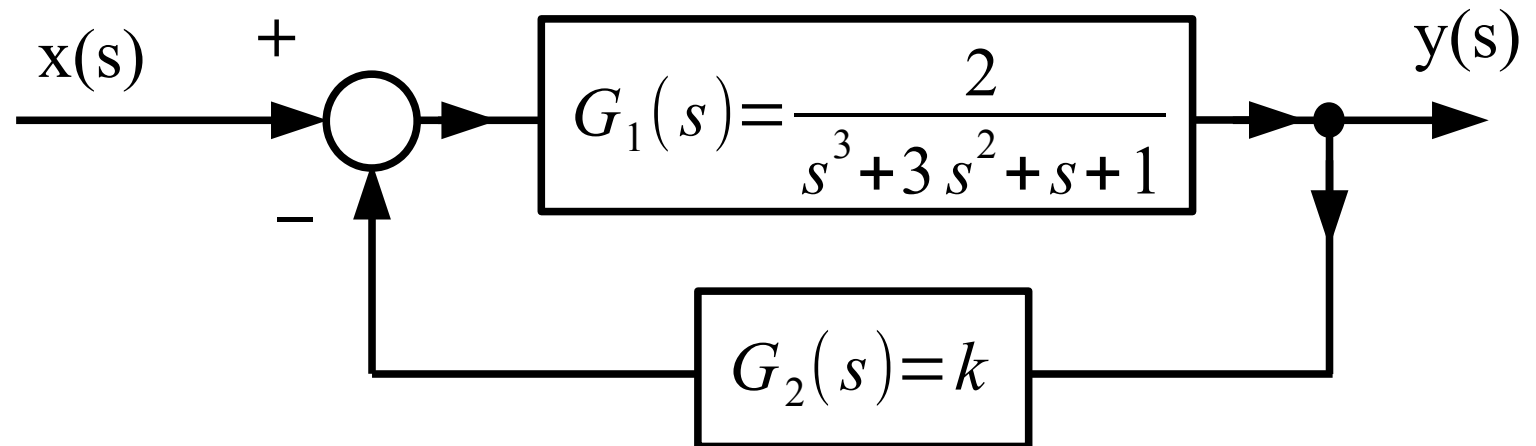
- 1) open-loop transfer function is stable AND
- 2) open-loop transfer function not enclosing the point $(-1, j0)$.



Nyquist criterion

Example 8

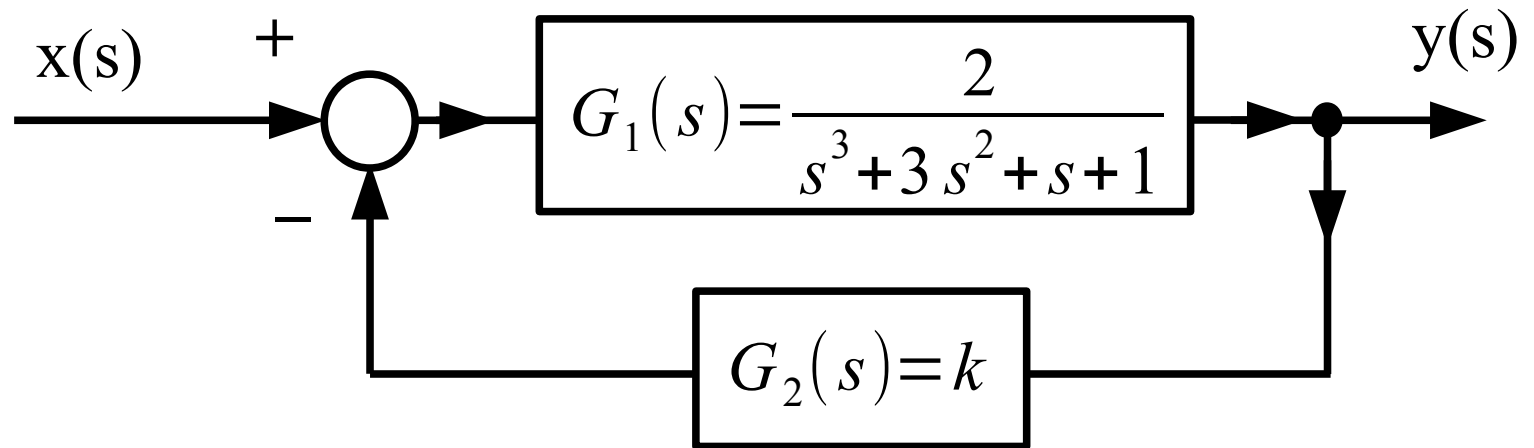
Choose k parameter to satisfy Nyquist criterion



Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion

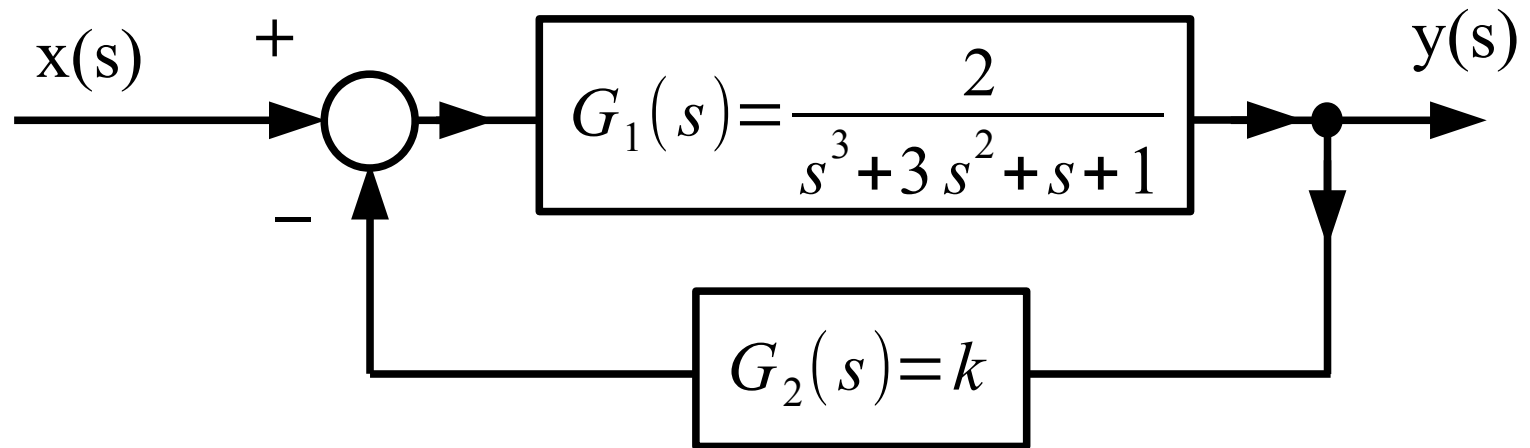


$$G_{open}(s) = G_1 G_2 = \frac{2k}{s^3 + 3s^2 + s + 1}$$

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



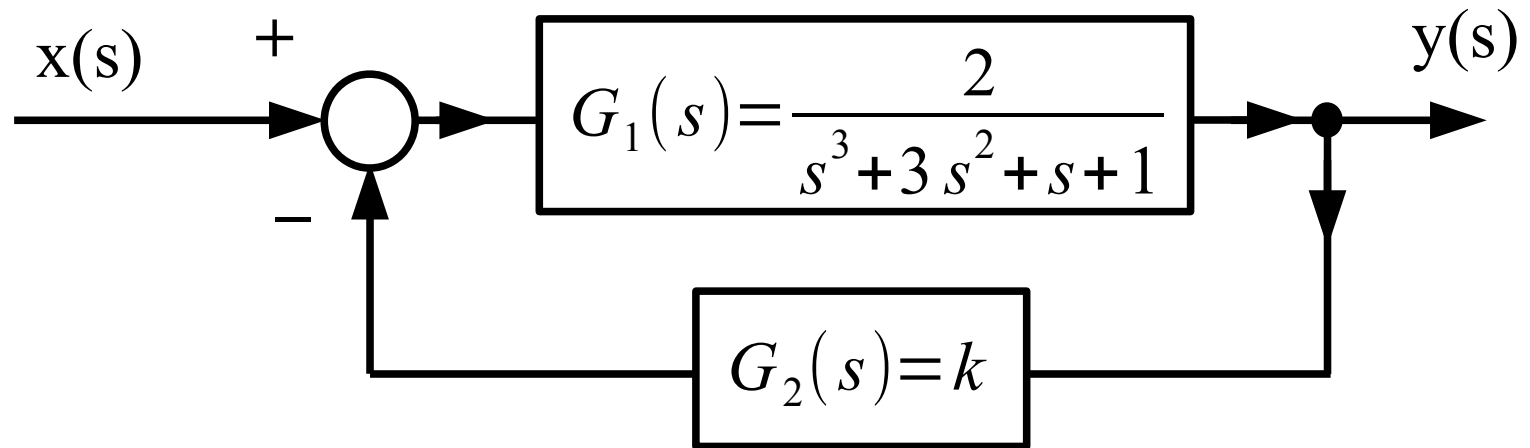
$$G_{open}(s) = G_1 G_2 = \frac{2k}{s^3 + 3s^2 + s + 1}$$

- stable from Hurwitz

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



$$G_{open}(s) = G_1 G_2 = \frac{2k}{s^3 + 3s^2 + s + 1} \quad \text{- stable from Hurwitz}$$

$$P(\omega) = \frac{2k - 6k\omega^2}{(1 - 3\omega^2)^2 + (\omega - \omega^3)^2}, \quad Q(\omega) = \frac{2k\omega^3 - 2k\omega}{(1 - 3\omega^2)^2 + (\omega - \omega^3)^2}$$

Nyquist criterion

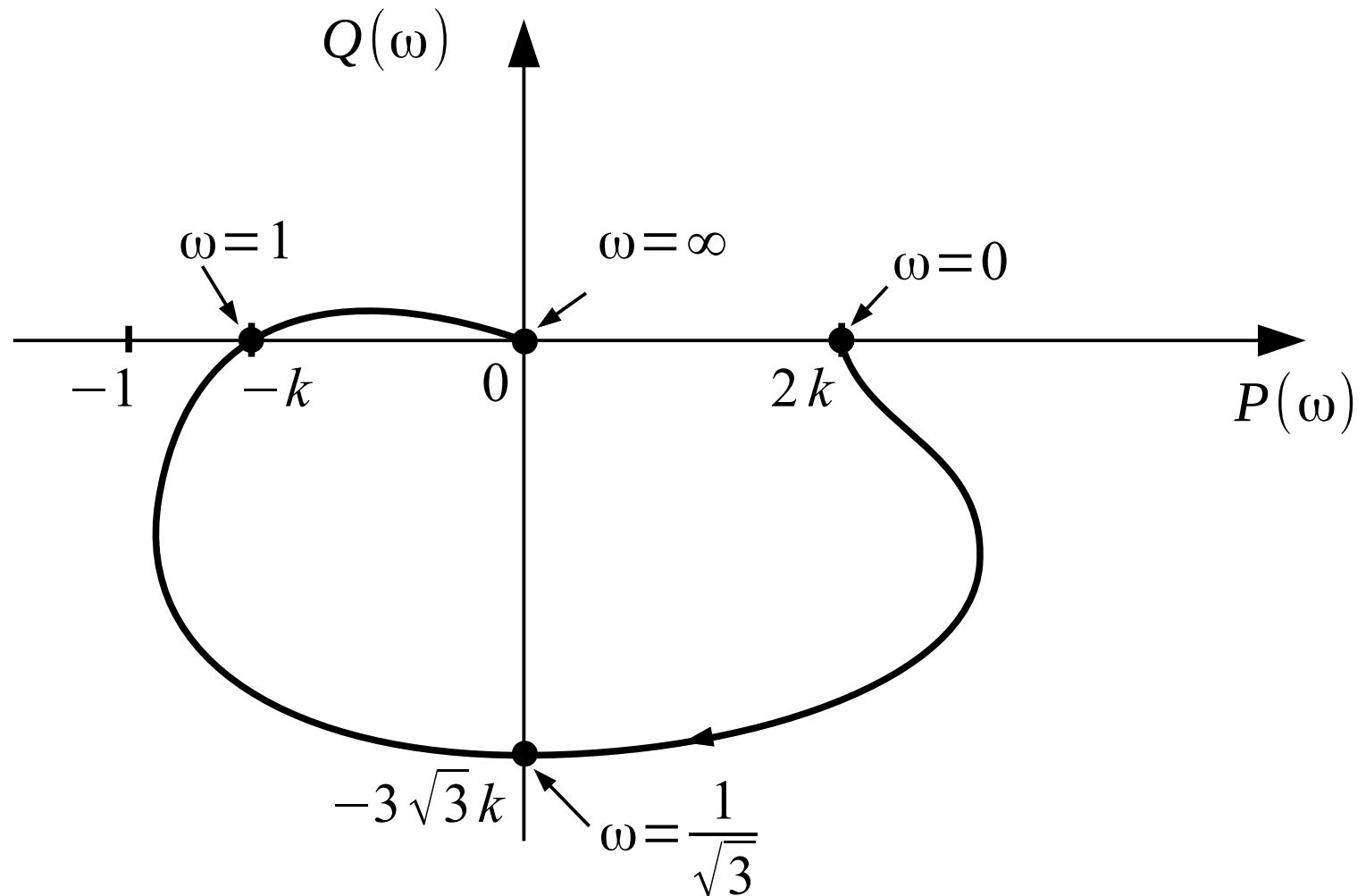
Example 8

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Nyquist criterion

Example 8

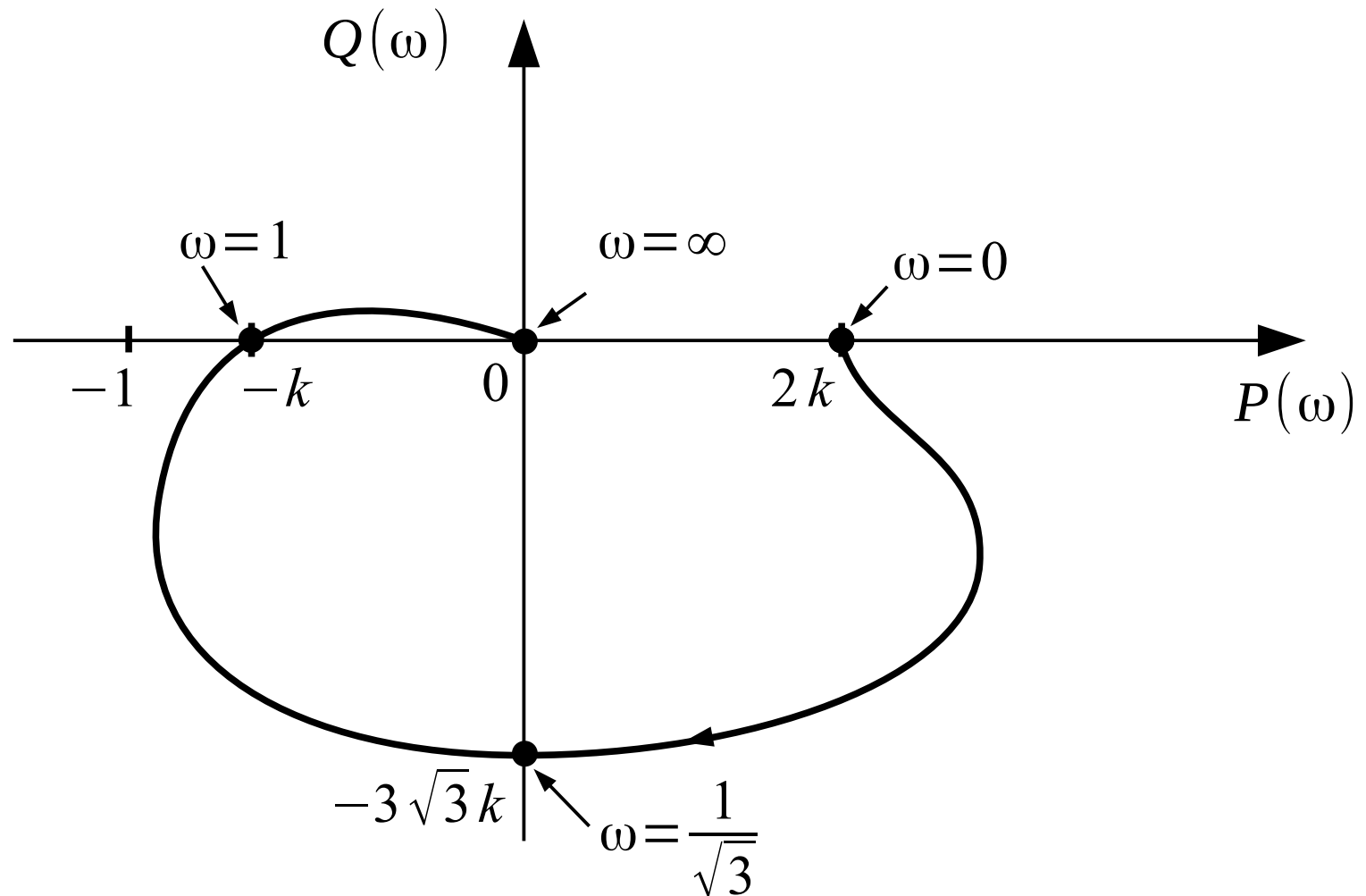
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Nyquist criterion

Example 8

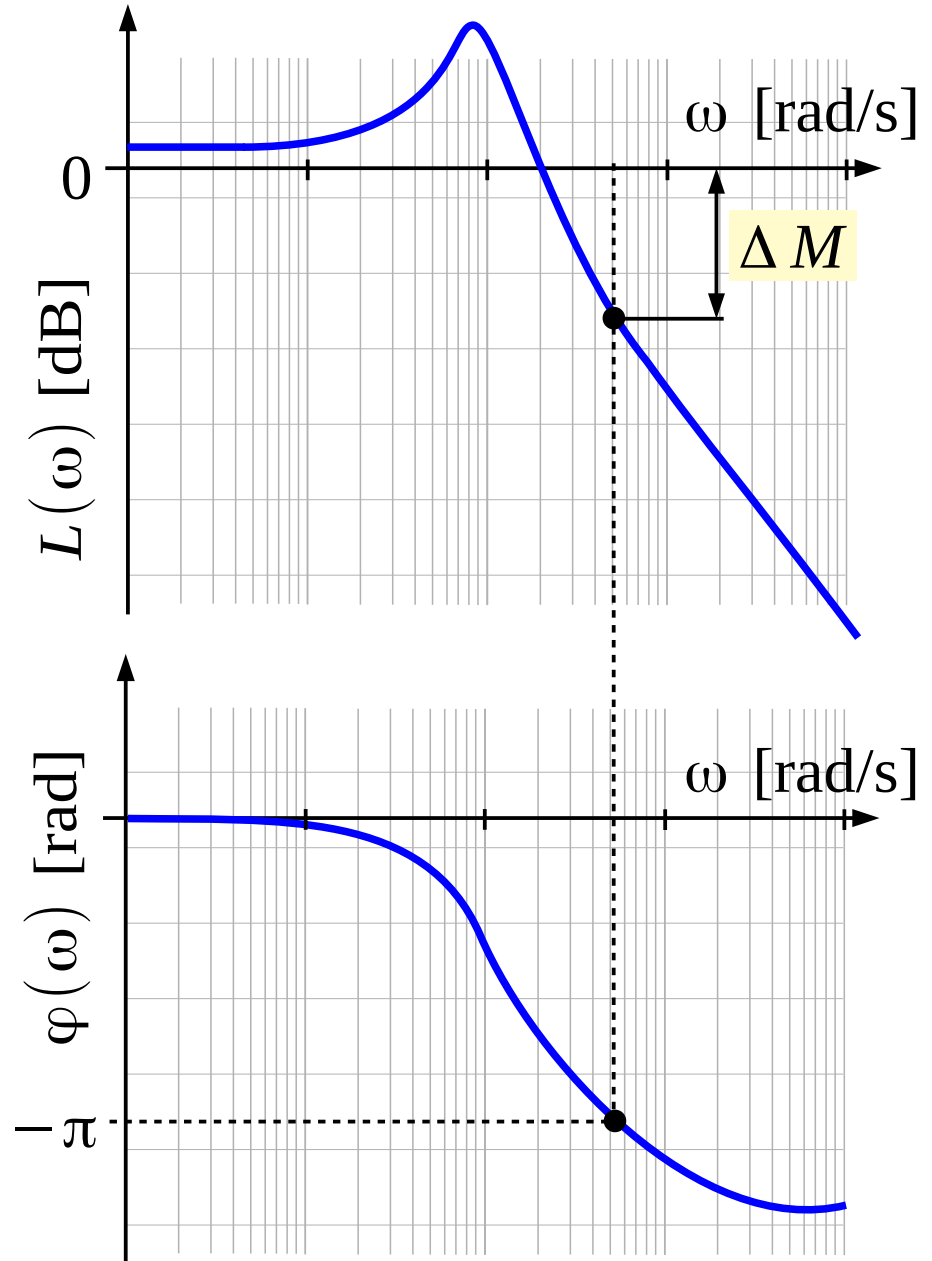
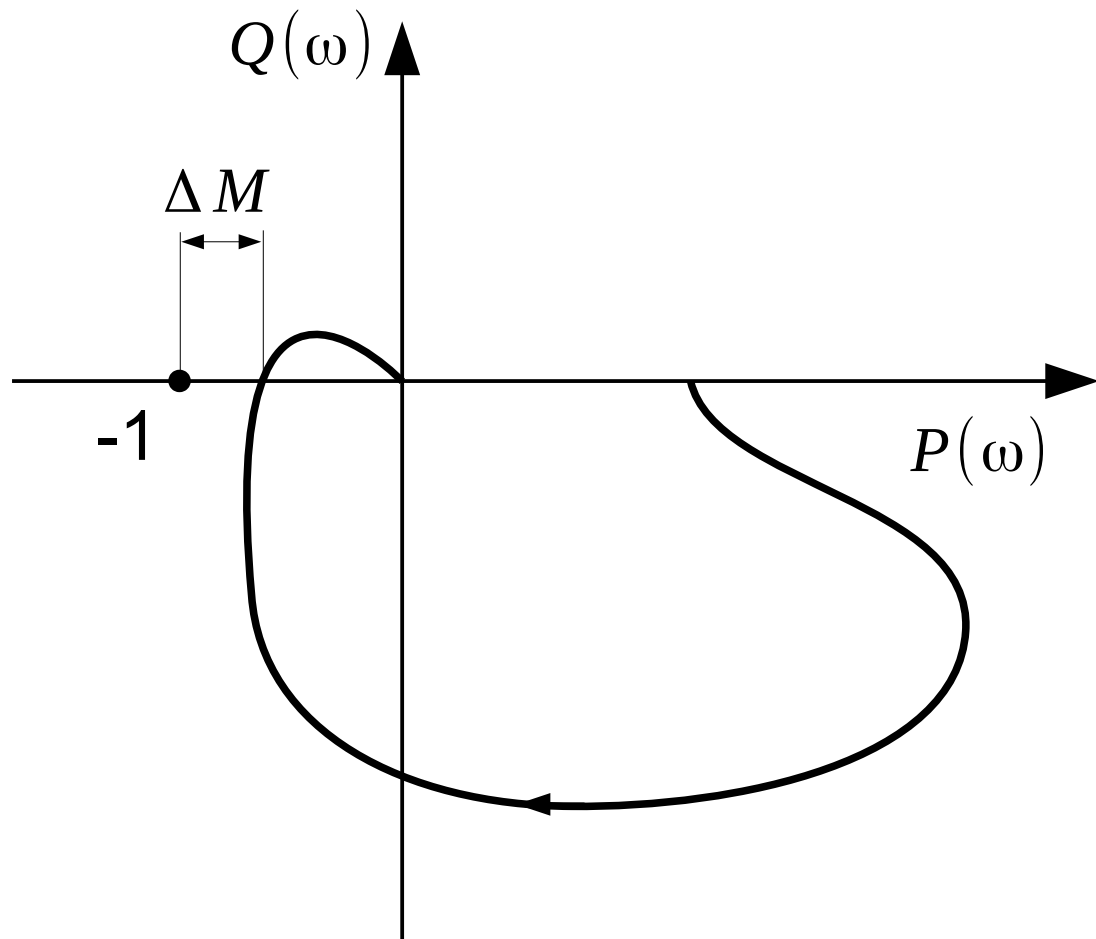
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closed-loop
system
stable for
 $0 < k < 1$

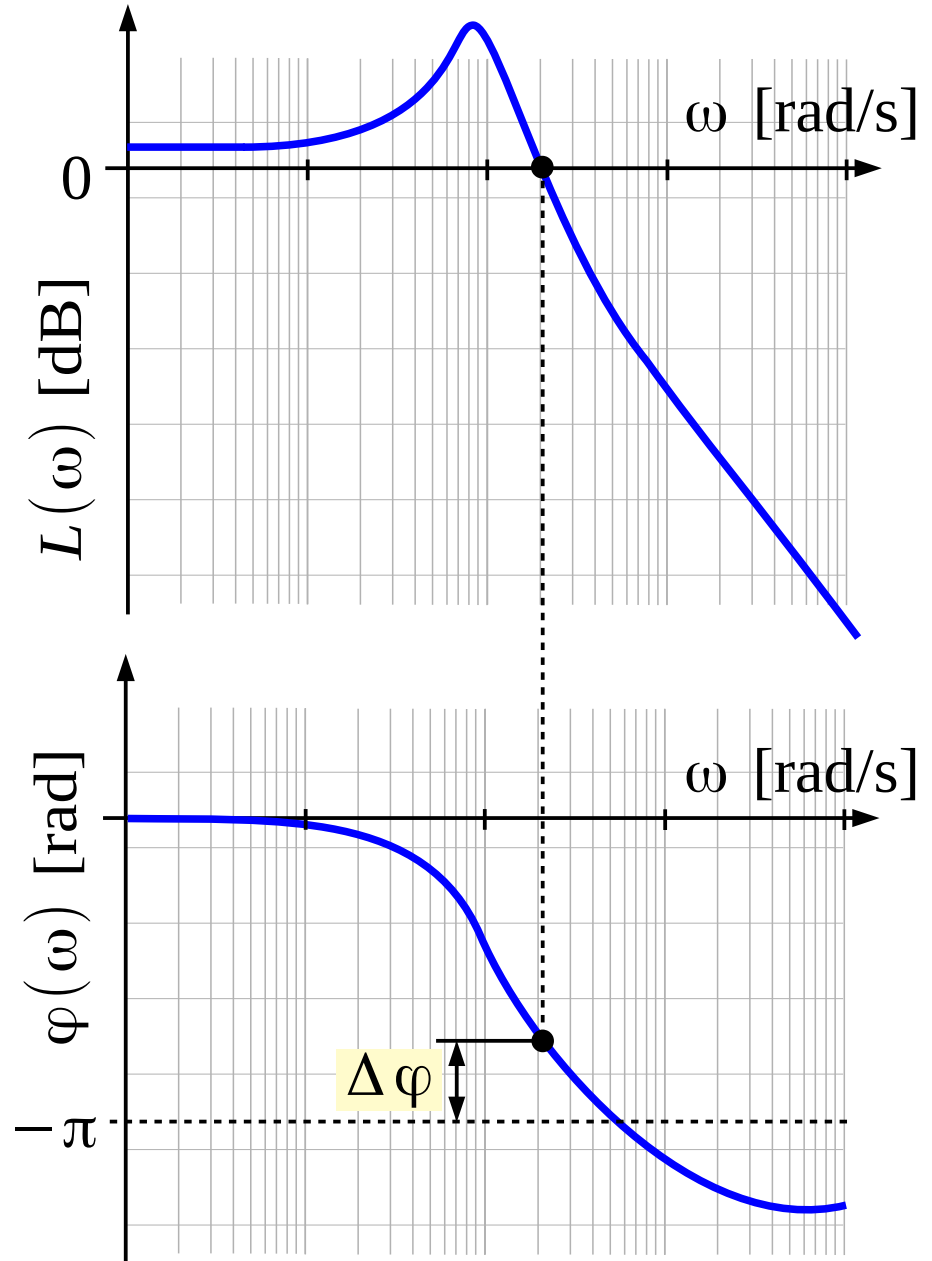
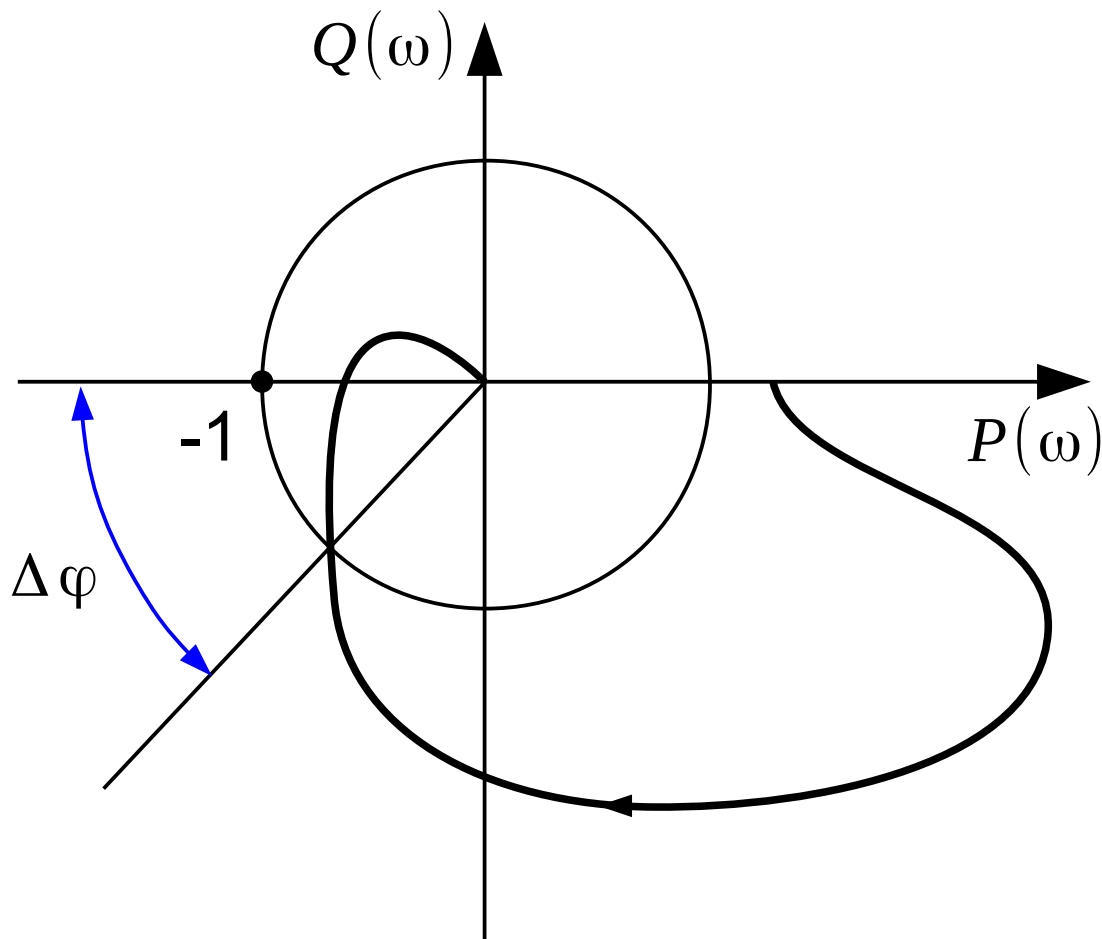
Gain margin

Closed-loop system will lose its stability if we add additional gain (in serial) greater or equals to gain margin.



Phase margin

Closed-loop system will lose its stability if we add additional delay (in serial) greater or equals to phase margin.

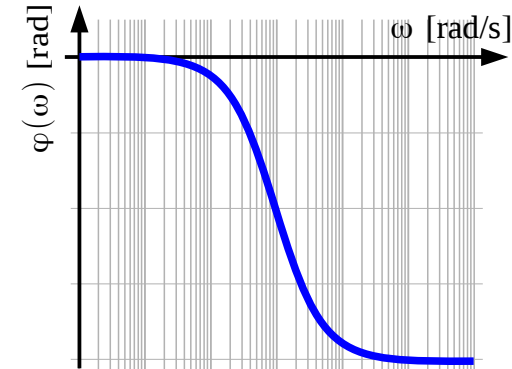
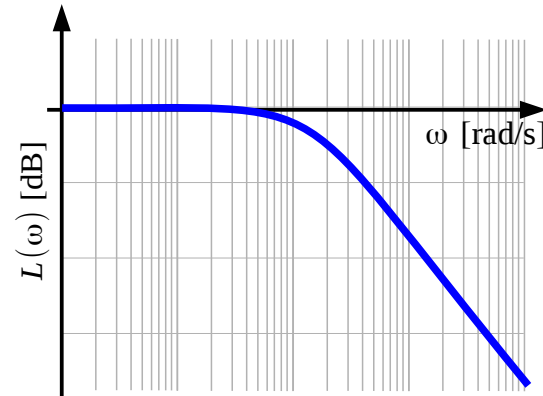
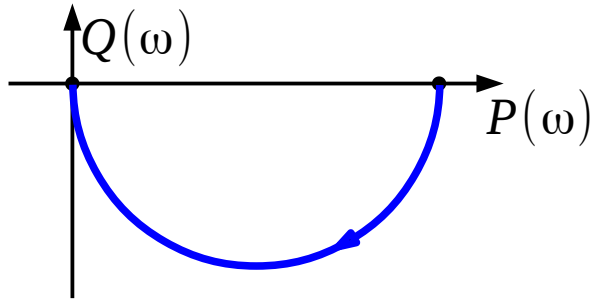


Stability vs Bode plot

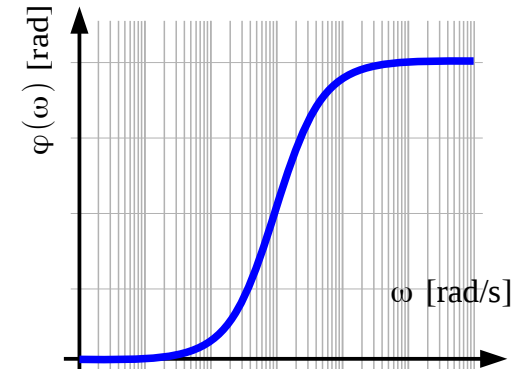
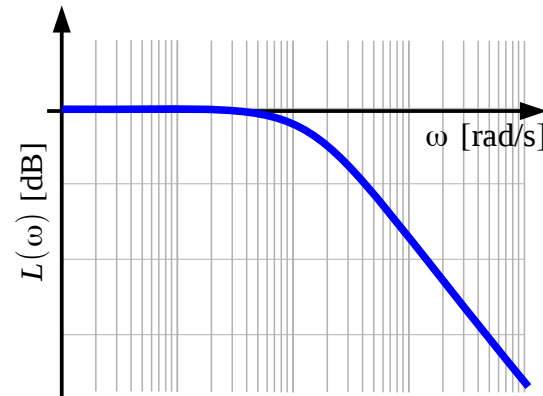
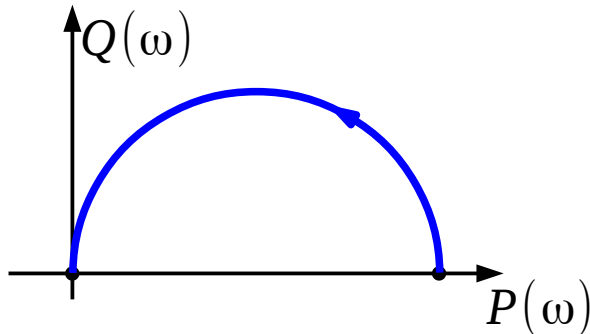
Bodego plot (gain + delay) has no physical meaning if the system is unstable!

Example:

$$G(s) = \frac{1}{s+1}$$



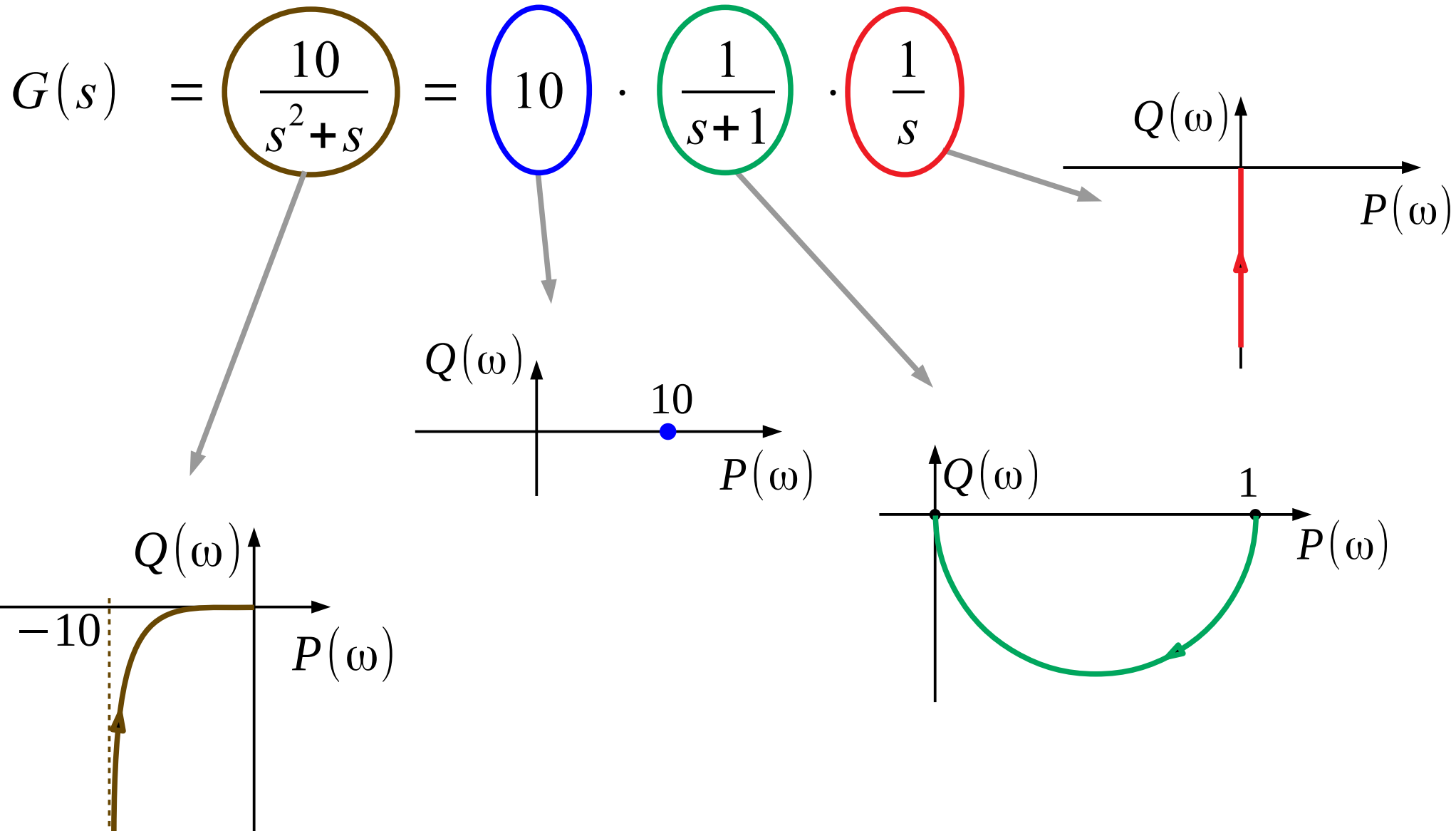
$$G(s) = \frac{1}{s-1}$$



Summing of Bode plots – example

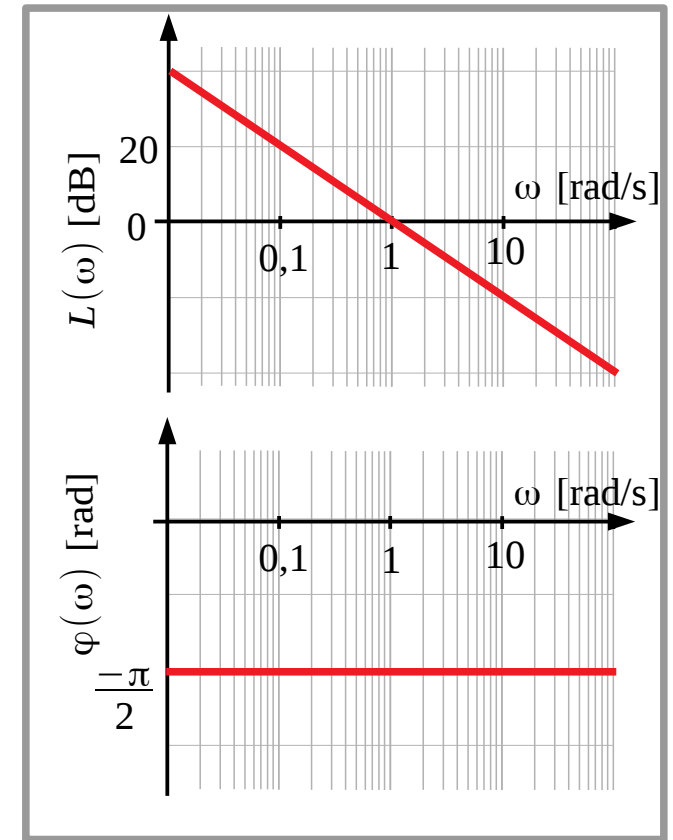
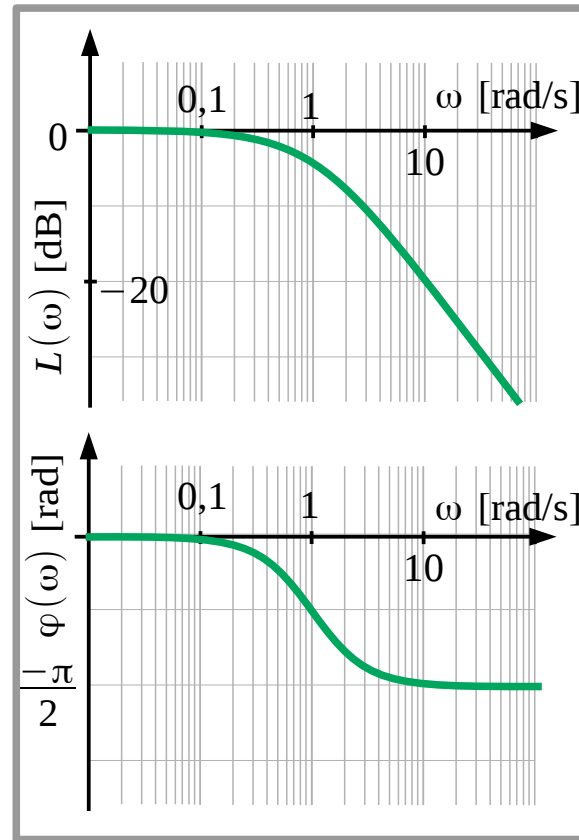
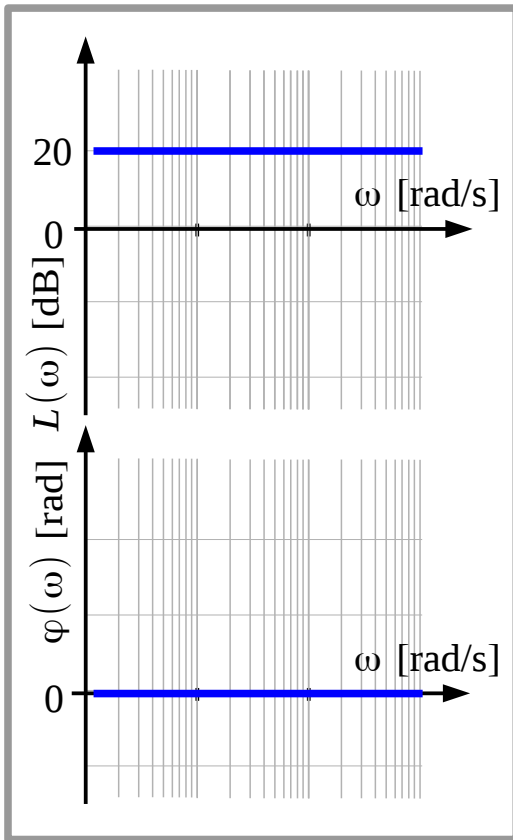
$$G(s) = \frac{10}{s^2 + s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$

Summing of Bode plots – example



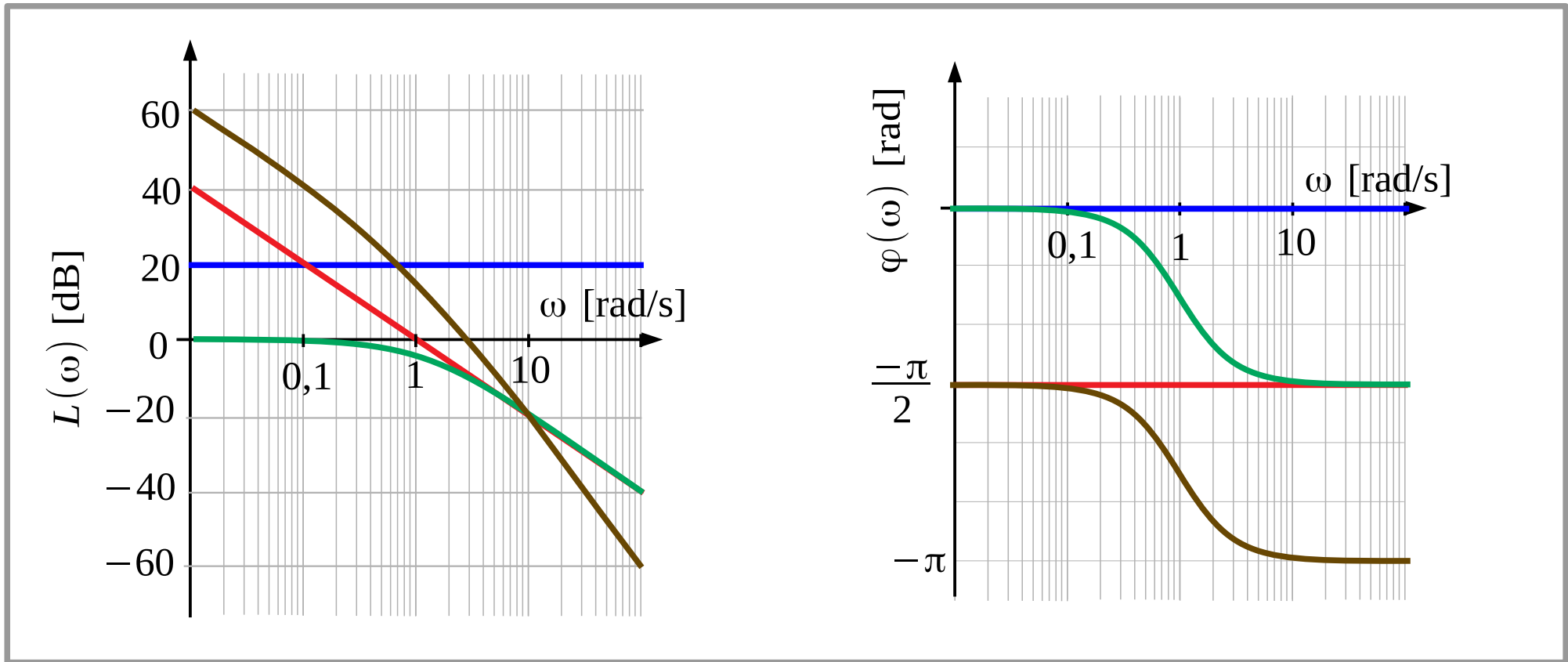
Summing of Bode plots – example

$$G(s) = \frac{10}{s^2 + s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$



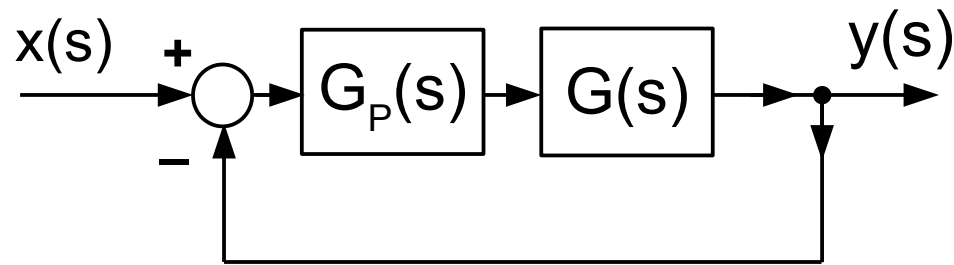
Summing of Bode plots – example

$$G(s) = \frac{10}{s^2 + s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$



Nyquist stability criterion

control loop with P controller



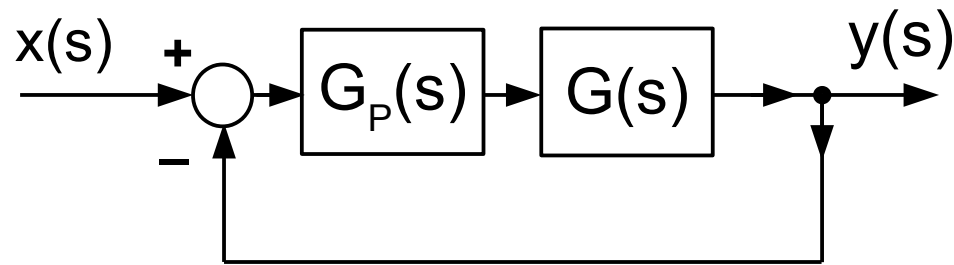
$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

Nyquist stability criterion

control loop with P controller

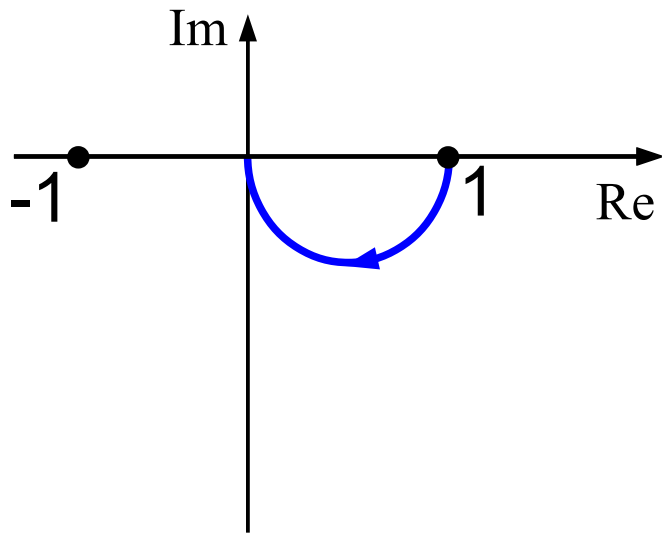


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

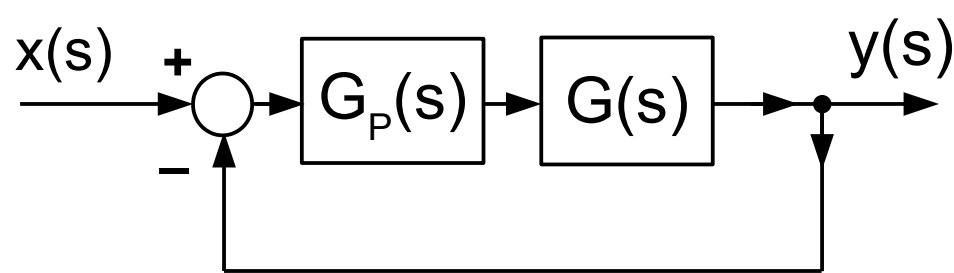
$$G_P(s) = k_P$$

$$G(s) = \frac{1}{Ts + 1}$$



Nyquist stability criterion

control loop with P controller

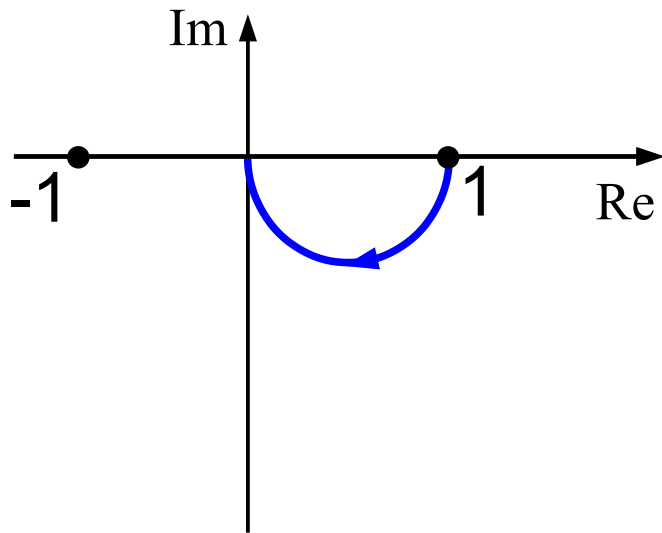


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

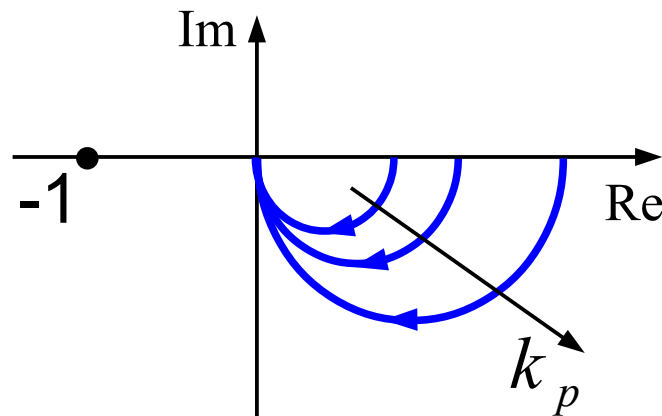
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_p$$

$$G(s) = \frac{1}{Ts + 1}$$

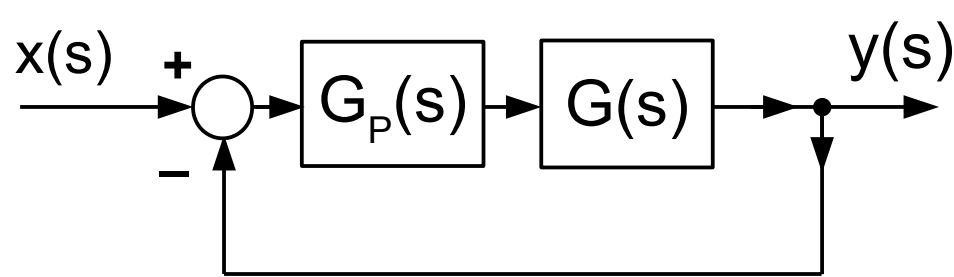


$$G_{opened}(s) = k_p \frac{1}{Ts + 1}$$



Nyquist stability criterion

control loop with P controller

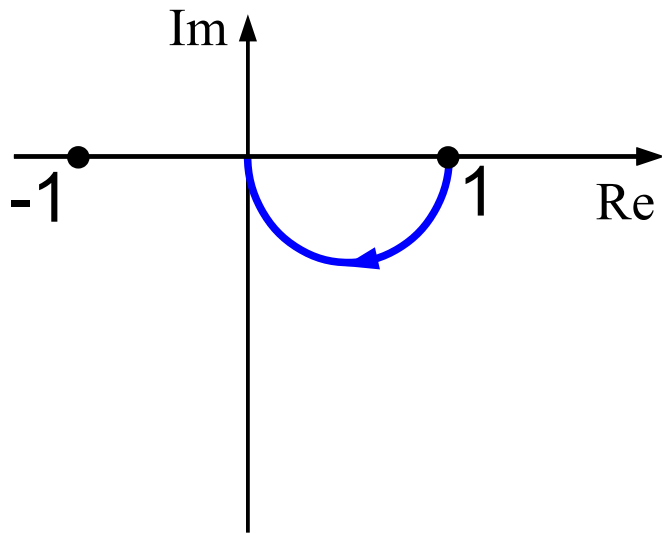


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

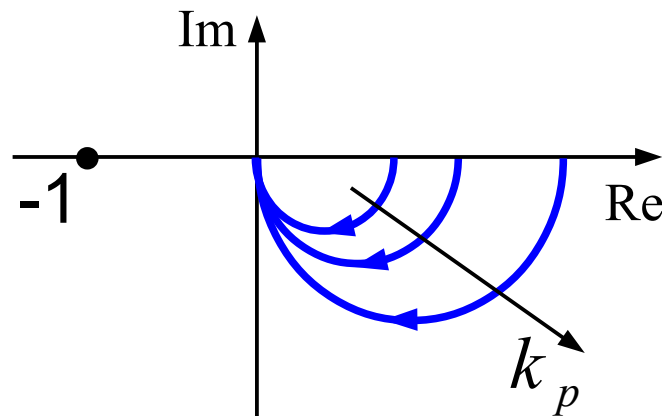
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

$$G(s) = \frac{1}{Ts + 1}$$



$$G_{opened}(s) = k_P \frac{1}{Ts + 1}$$



G_{opened} is
always stable

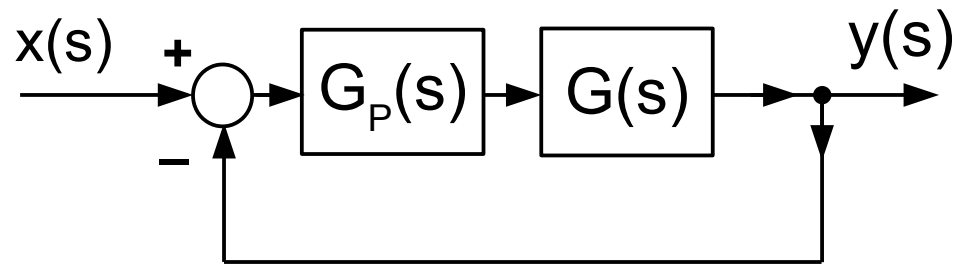
G_{closed} is
always stable

steady state
error ratio:

$$\frac{k_P}{k_P + 1}$$

Nyquist stability criterion

control loop with P controller



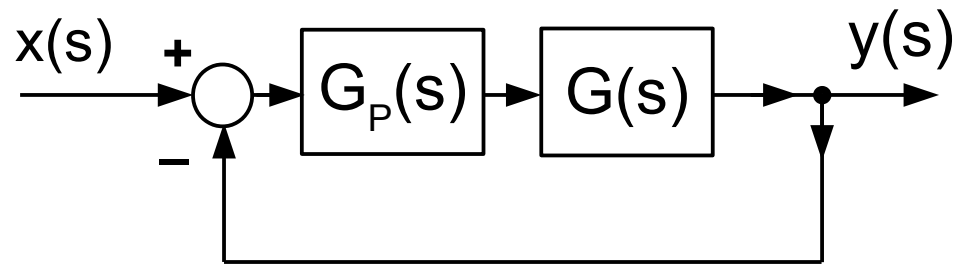
$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

Nyquist stability criterion

control loop with P controller

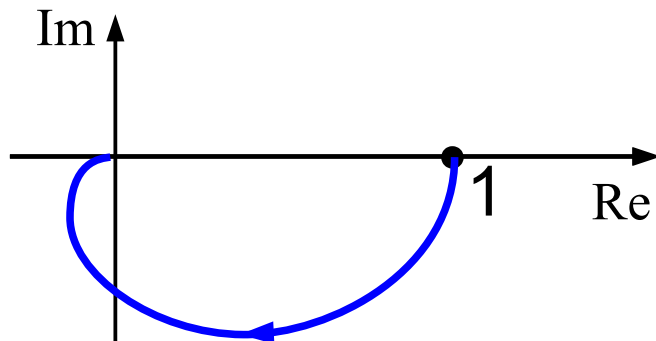


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

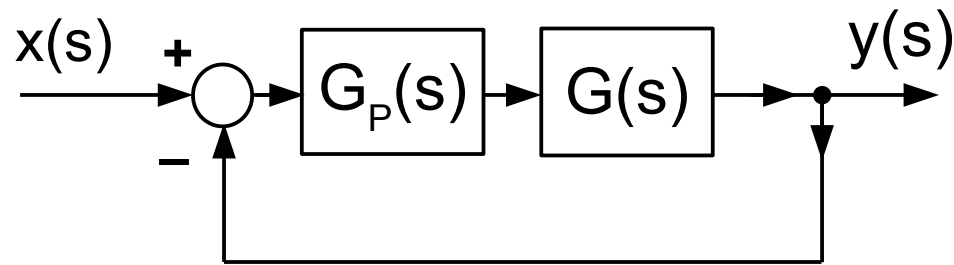
$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_1^2 s^2 + T_2 s + 1}$$



Nyquist stability criterion

control loop with P controller

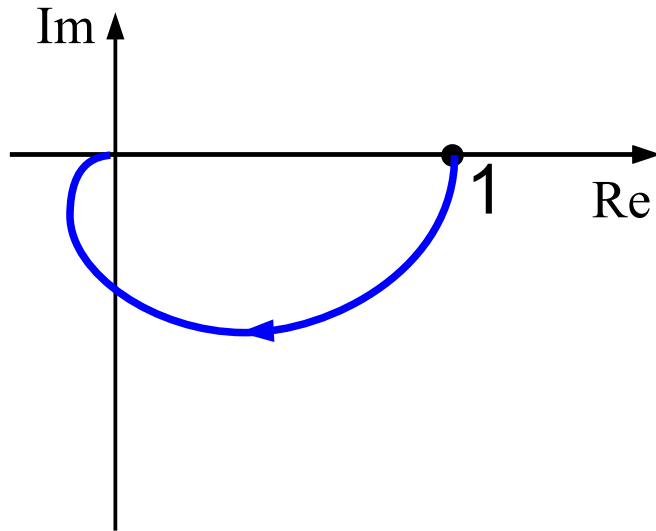


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

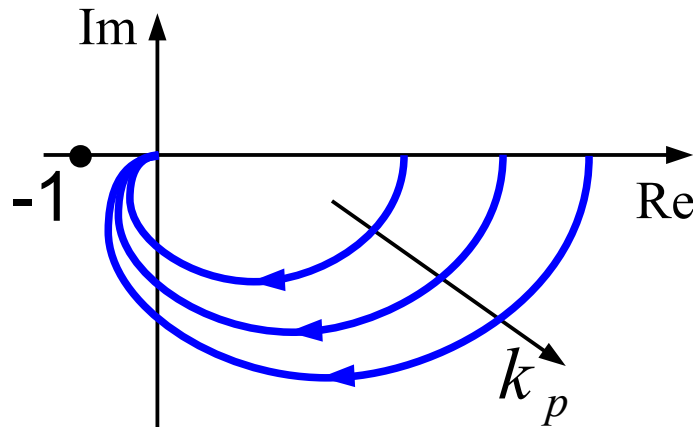
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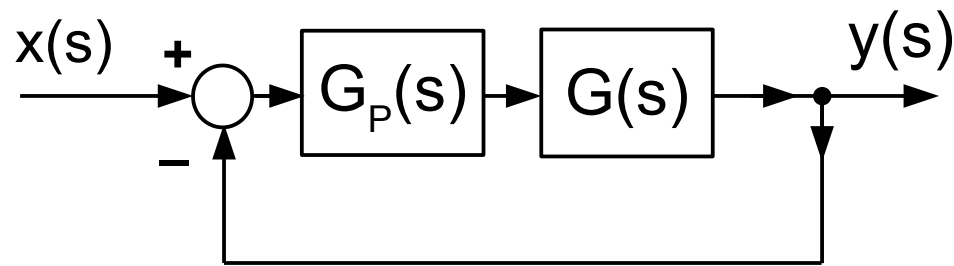


$$G_{opened}(s) = \frac{k_P}{T_1^2 s^2 + T_2 s + 1}$$



Nyquist stability criterion

control loop with P controller

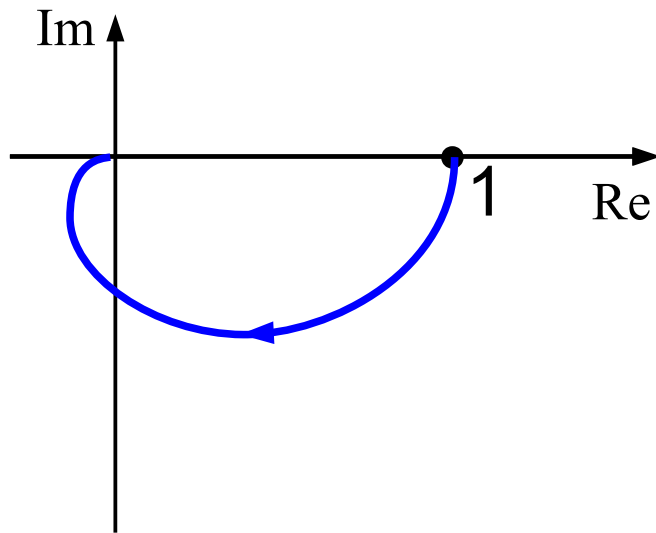


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

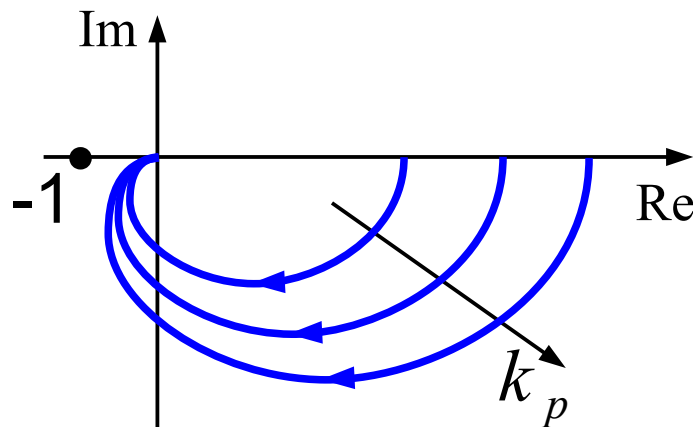
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_1^2 s^2 + T_2 s + 1}$$



$$G_{opened}(s) = \frac{k_P}{T_1^2 s^2 + T_2 s + 1}$$



G_{opened} is
always stable

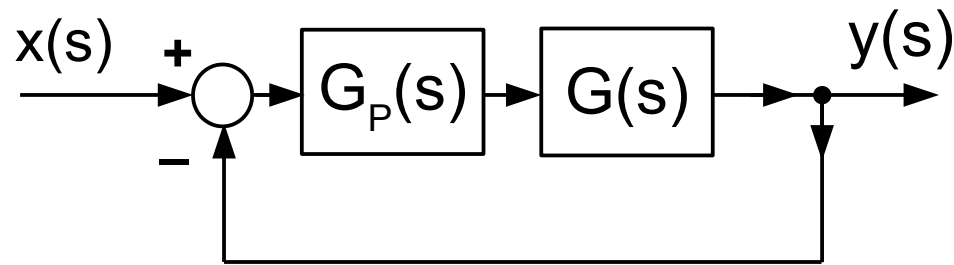
G_{closed} is
always stable

steady state
error ratio:

$$\frac{k_P}{k_P + 1}$$

Nyquist stability criterion

control loop with P controller



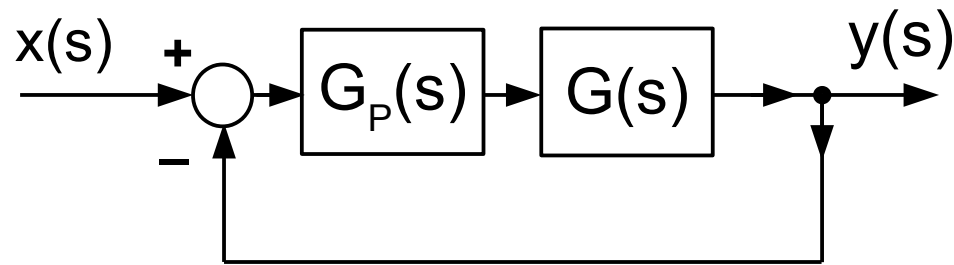
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$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

Nyquist stability criterion

control loop with P controller

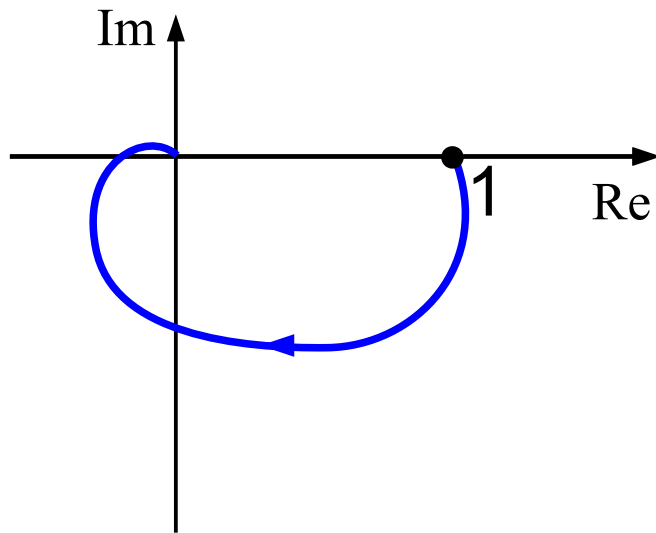


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

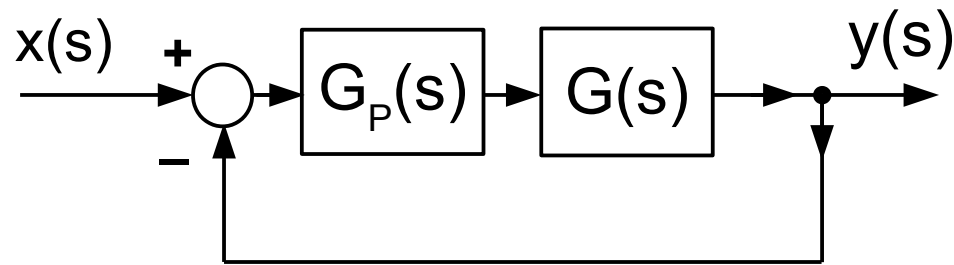
$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$



Nyquist stability criterion

control loop with P controller

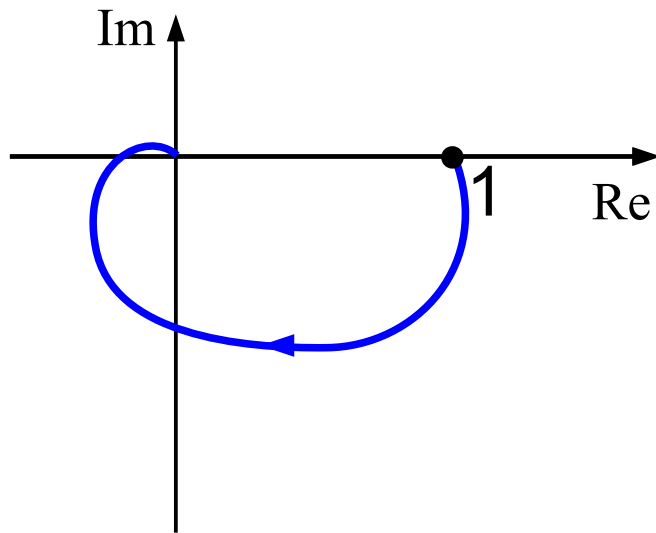


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

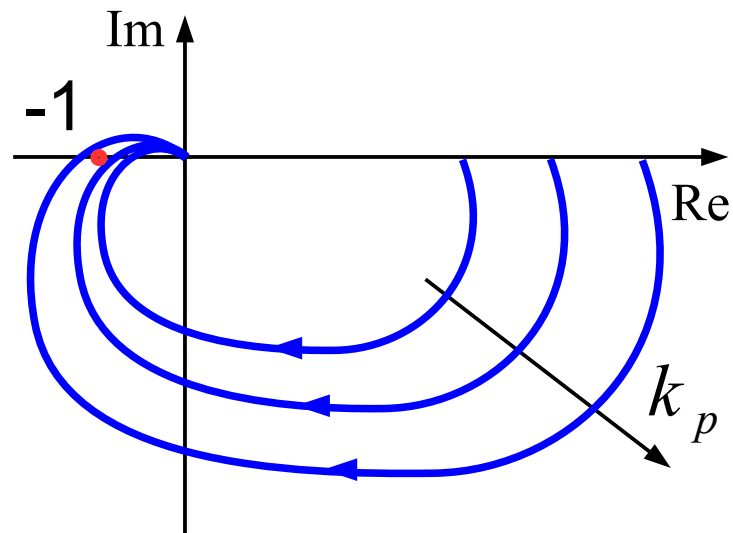
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$

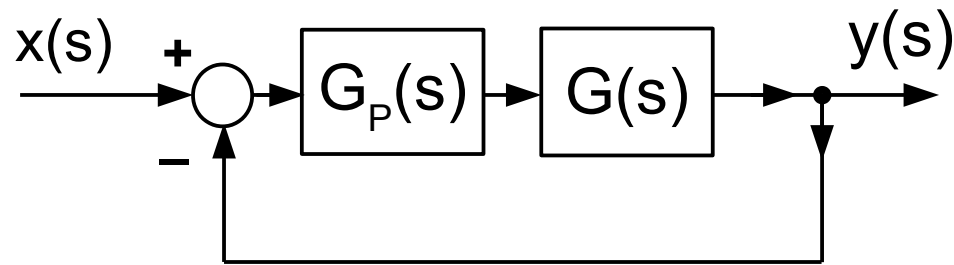


$$G_{opened}(s) = \frac{k_P}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$



Nyquist stability criterion

control loop with P controller

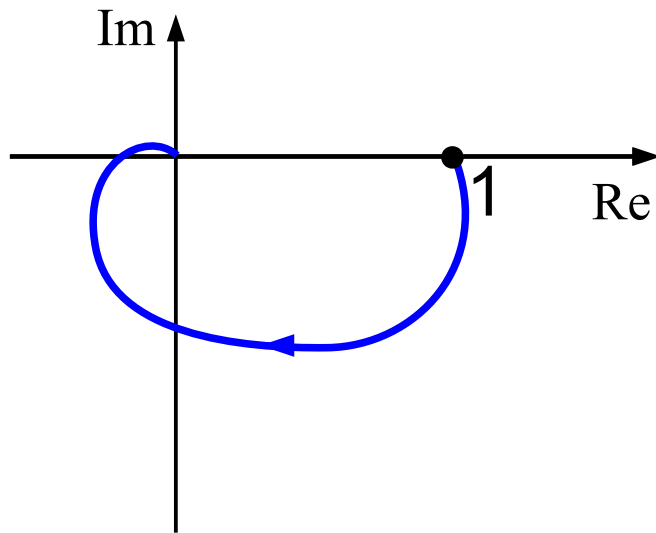


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

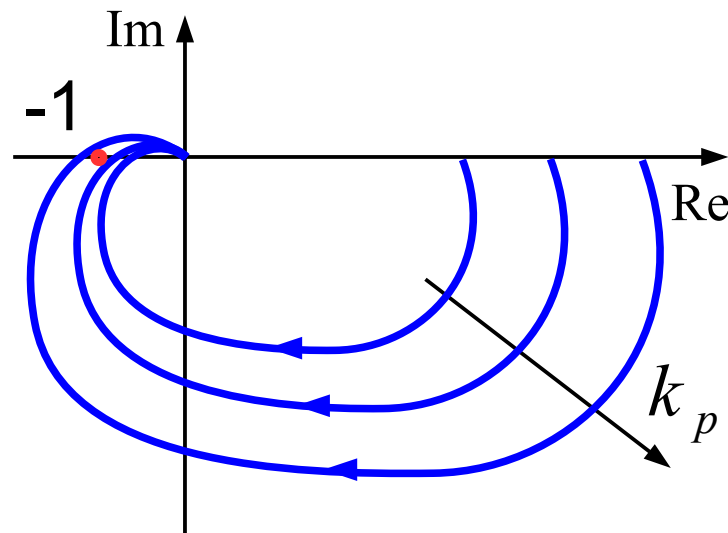
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$



$$G_{opened}(s) = \frac{k_P}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$



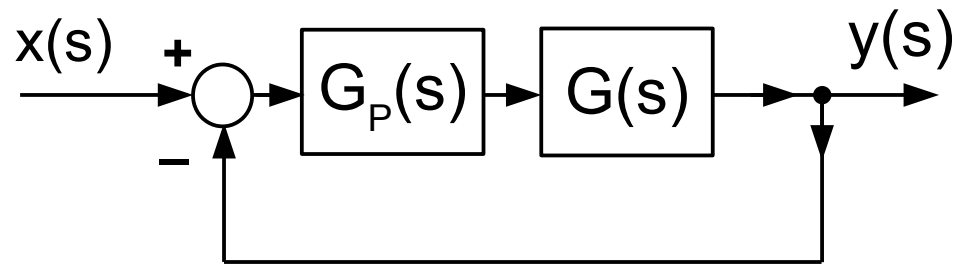
G_{closed} is not
always stable

steady state
error ratio:

$$\frac{k_P}{k_P + 1}$$

Nyquist stability criterion

control loop with P controller



$$G(s) = \frac{1}{T_3^2 s^3 + T_2^2 s^2 + T_1 s + 1}$$

$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

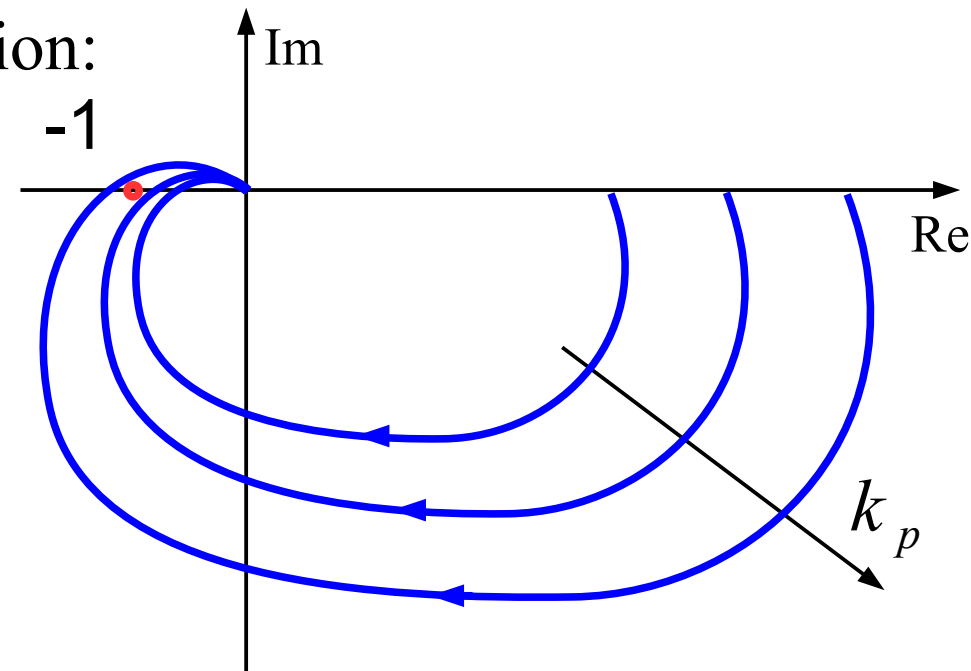
$$G_P(s) = k_P$$

conclusion for open-loop transfer function:

higher $k_p \rightarrow$ lower steady state error

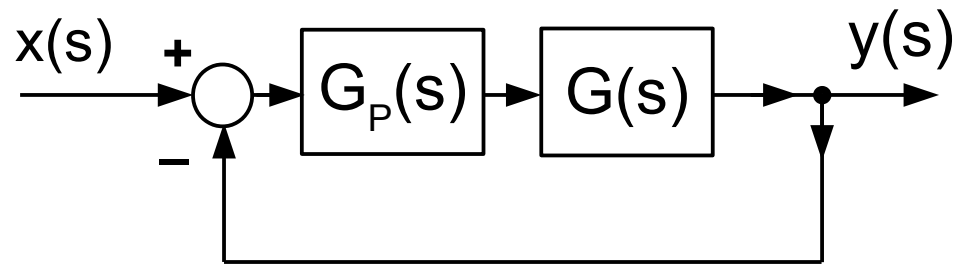
lower $k_p \rightarrow$ better stability

(higher gain margin)



Nyquist stability criterion

control loop with PI controller

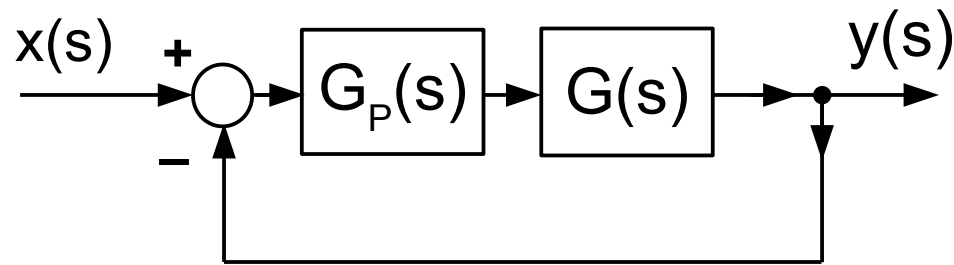


$$G_P(s) = k_P \left(1 + \frac{1}{T_i s} \right)$$

$$G_{opened}(s) = G_P(s) G(s)$$

Nyquist stability criterion

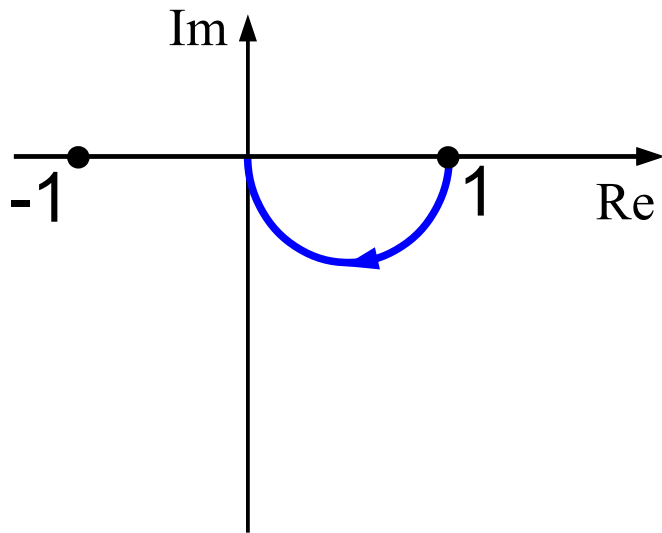
control loop with PI controller



$$G_P(s) = k_P \left(1 + \frac{1}{T_i s} \right)$$

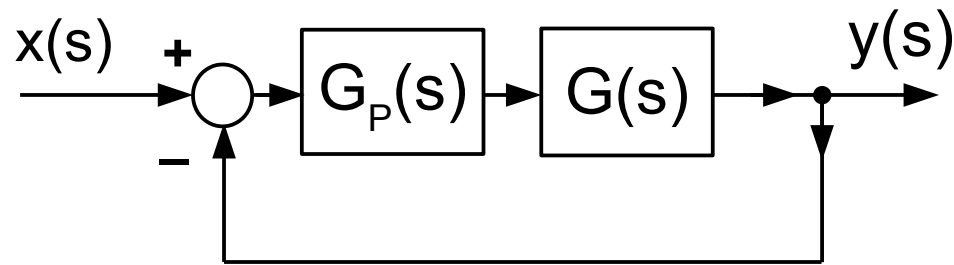
$$G_{opened}(s) = G_P(s) G(s)$$

$$G(s) = \frac{1}{Ts + 1}$$



Nyquist stability criterion

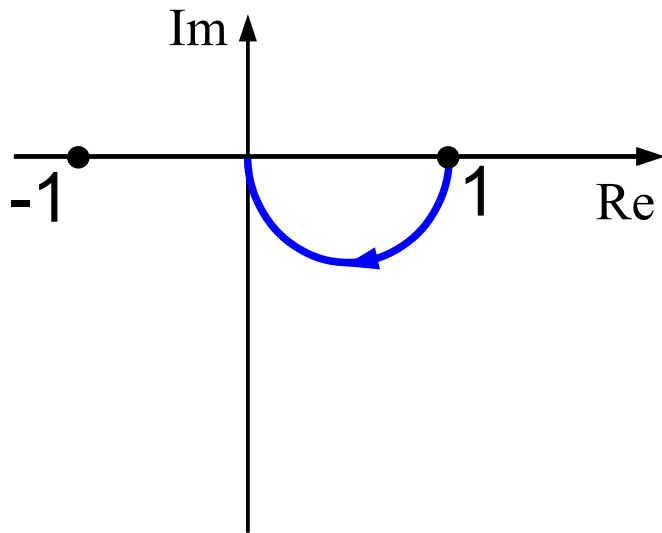
control loop with PI controller



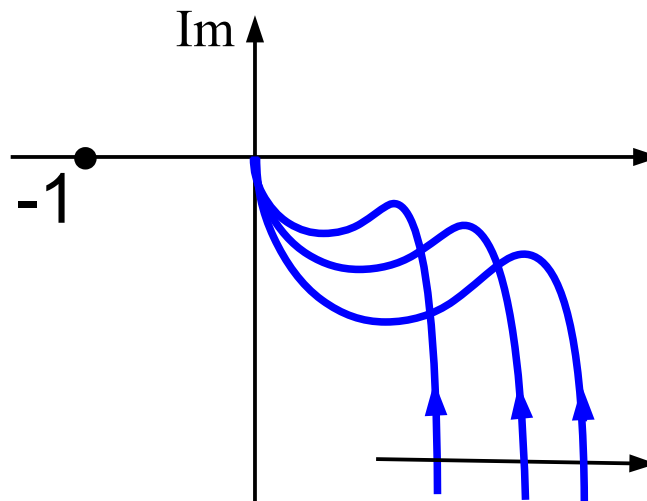
$$G_P(s) = k_P \left(1 + \frac{1}{T_i s} \right)$$

$$G_{opened}(s) = G_P(s) G(s)$$

$$G(s) = \frac{1}{T s + 1}$$

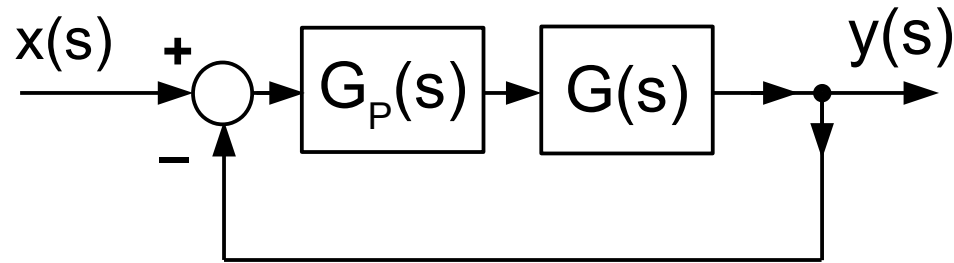


$$G_{opened}(s) = k_P^2 \frac{s T_i^2 + 2 T_i}{T_i^3 T s^2 + T_i^2 s}$$



Nyquist stability criterion

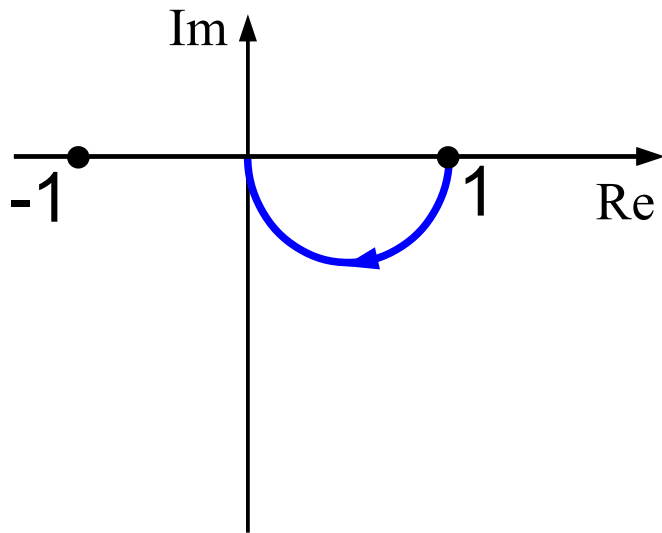
control loop with PI controller



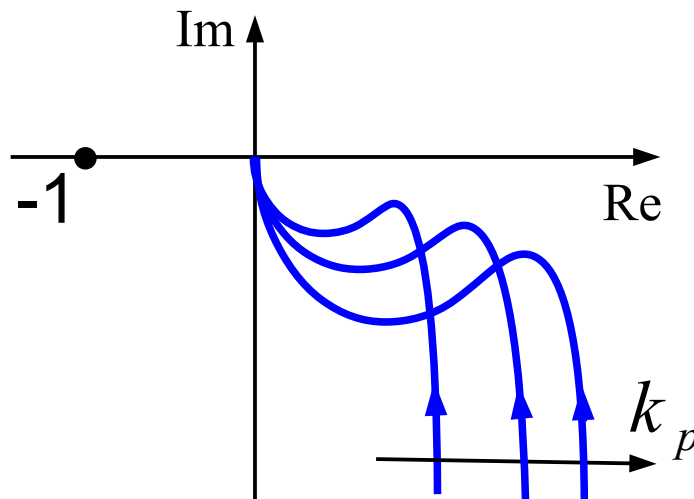
$$G_P(s) = k_P \left(1 + \frac{1}{T_i s} \right)$$

$$G_{opened}(s) = G_P(s) G(s)$$

$$G(s) = \frac{1}{Ts + 1}$$



$$G_{opened}(s) = k_P^2 \frac{s T_i^2 + 2 T_i}{T_i^3 T s^2 + T_i^2 s}$$



G_{opened} is stable,

so G_{closed} is stable

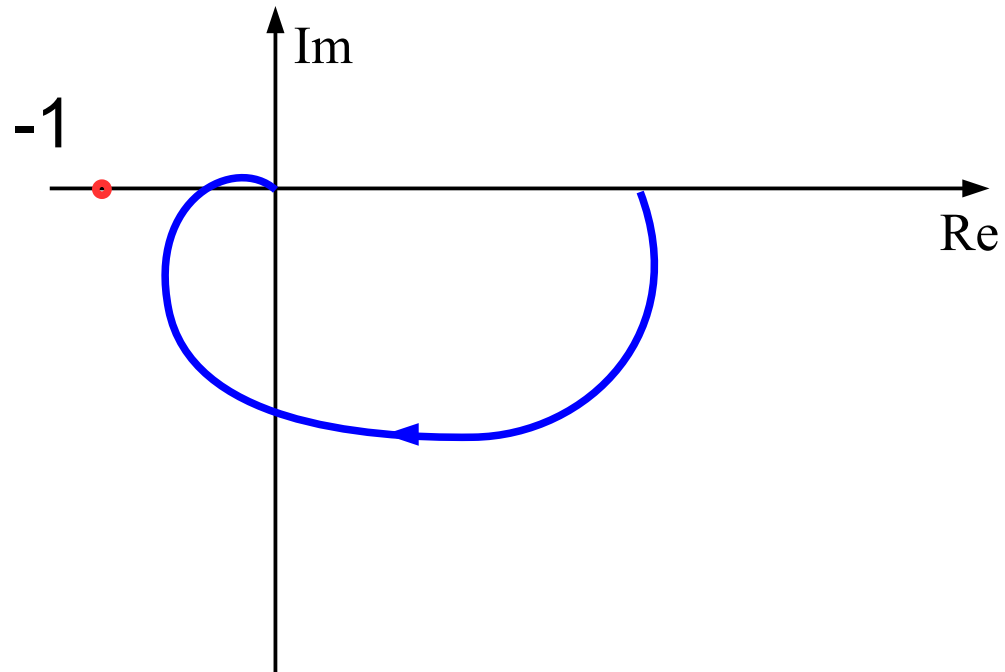
$G_{opened}(\omega=0) \rightarrow \infty$

so steady state error $\rightarrow 0$

Correction of the system

Correction by proportional term

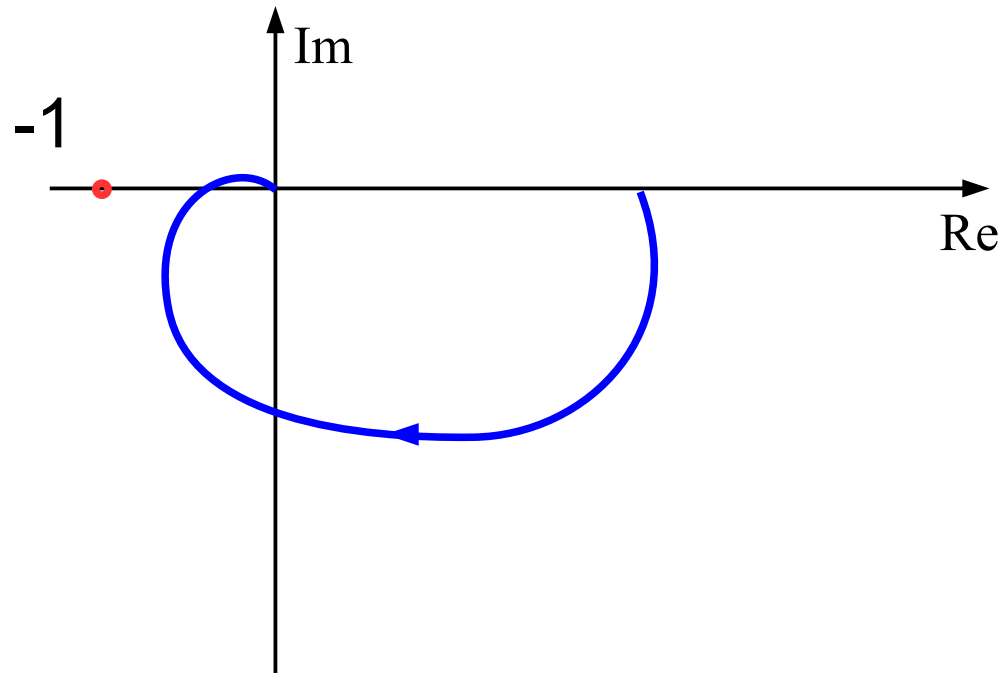
$G(s)$



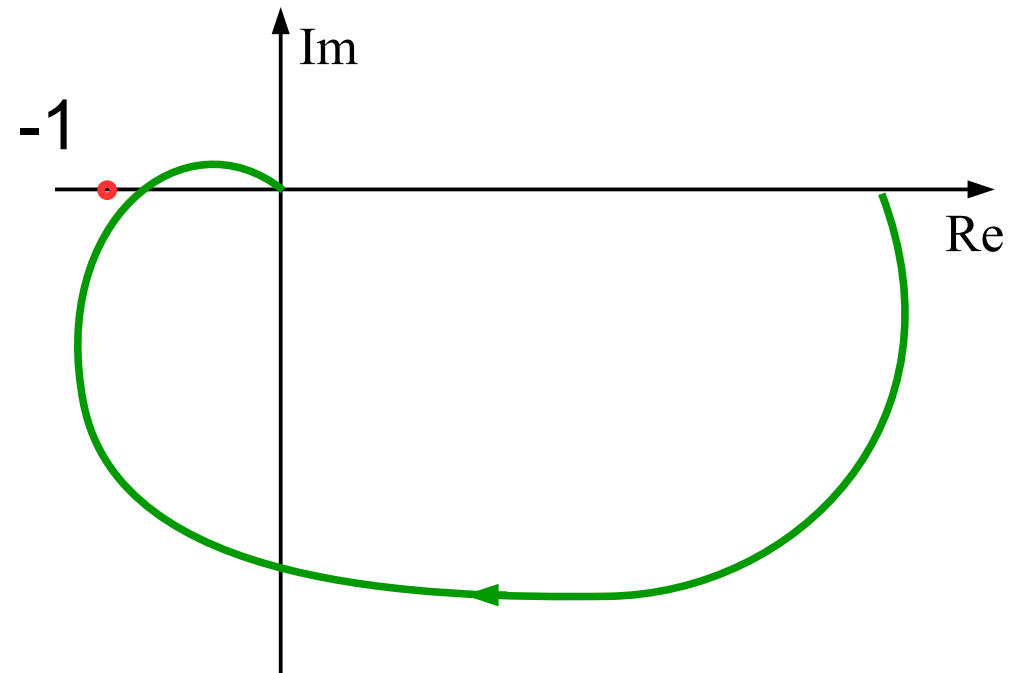
Correction of the system

Correction by proportional term

$$G(s)$$



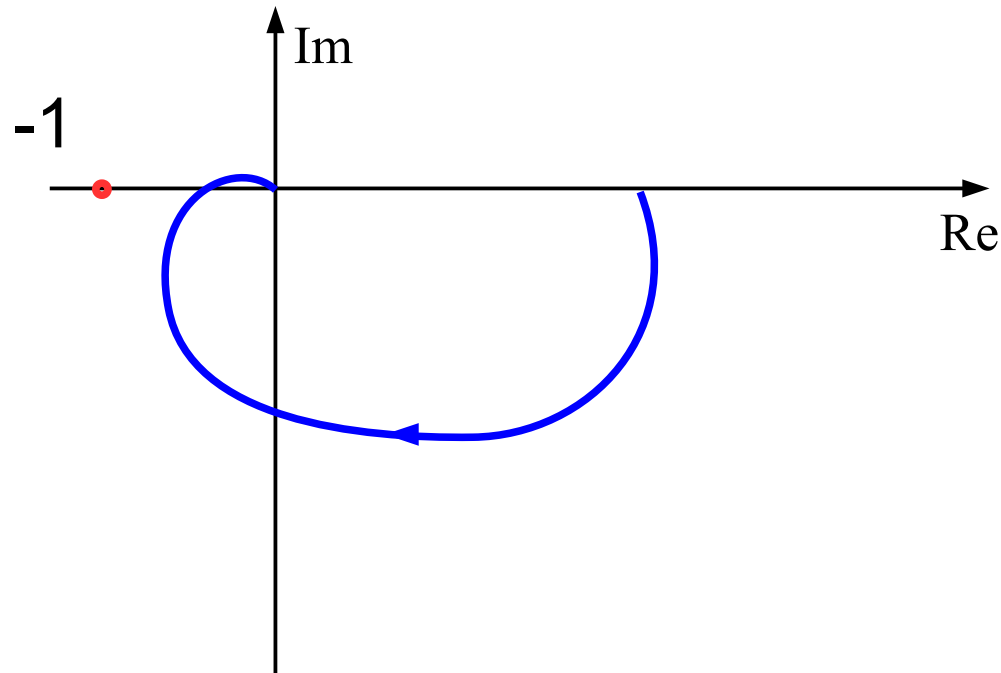
$$k \cdot G(s)$$



Correction of the system

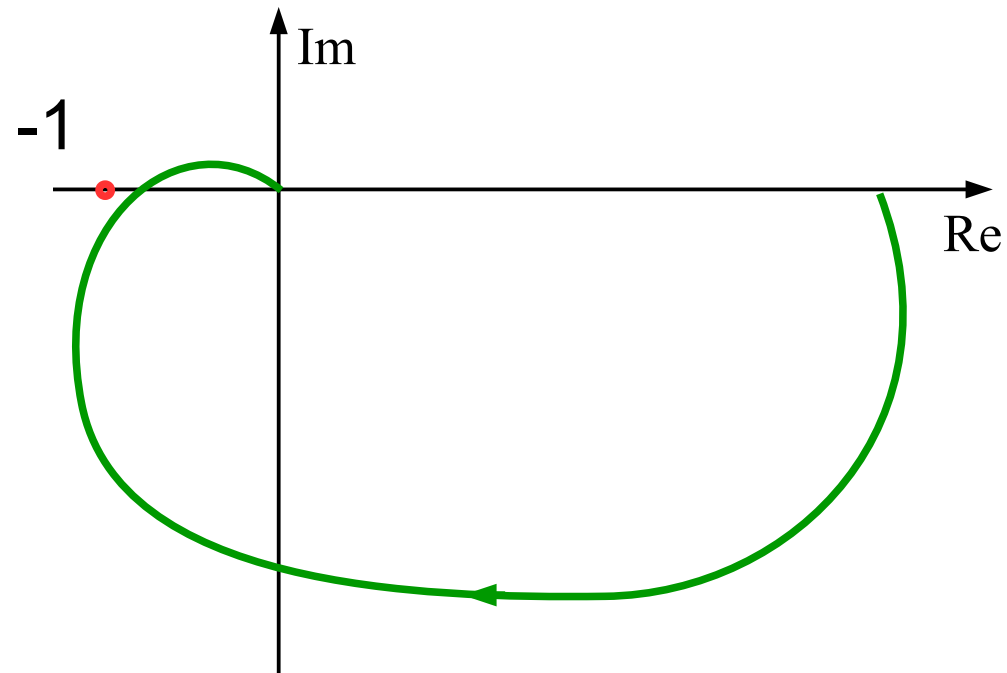
Correction by proportional term

$$G(s)$$



Higher gain margin,
higher phase margin,
higher steady state error

$$k \cdot G(s)$$

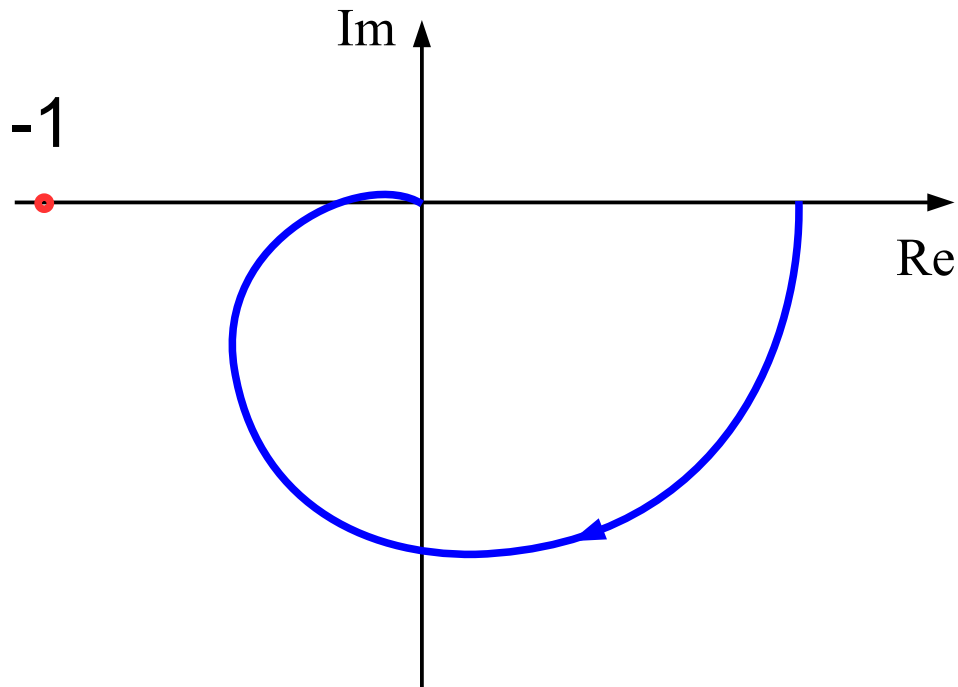


Lower gain margin,
lower phase margin,
lower steady state error

Correction of the system

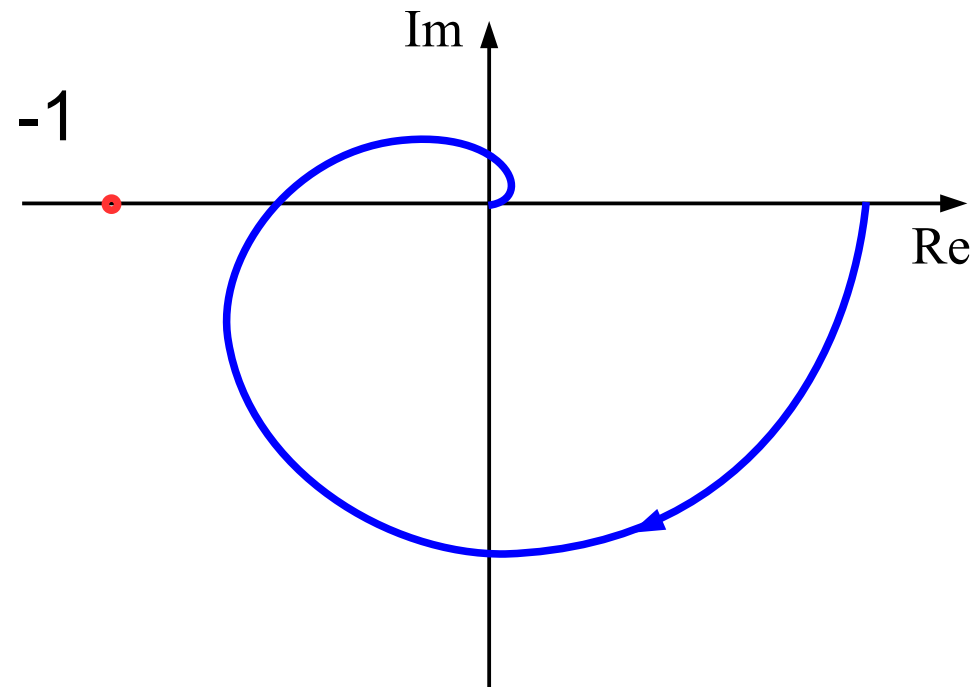
Correction by delay

$$G(s)$$



Higher gain and phase margins

$$G(s) \cdot e^{-\tau s}$$



Lower gain and phase margins

Correction of the system

Derivative

$$K(s) = \frac{1 + T s}{1 + a s + b s^2}$$

Proportional-derivative

$$K(s) = k_P \frac{T s + 1}{\alpha T s + 1}, \quad \alpha < 1$$

Integral

$$K(s) = 1 + \frac{k}{1 + T s}$$

Proportional-integral

$$K(s) = \alpha \frac{T s + 1}{\alpha T s + 1}, \quad \alpha > 1$$

Proportional-integral-derivative

$$K(s) = k (T_d s + 1) \left(1 + \frac{1}{T_i s} \right)$$

Materials for exam – lectures from 1 to 13 (>1100 slides...)

**Lecture 14 – material repeat, supplementary info,
informations about the exam,
WUT questionnaires,
consultations**

**Lecture 15 – modern control theory overview,
experiment with control system,
consultations**

**Exam: Wednesday, 30th January, 10:30-11:30
Wednesday, 6th February, 10:30-11:30**