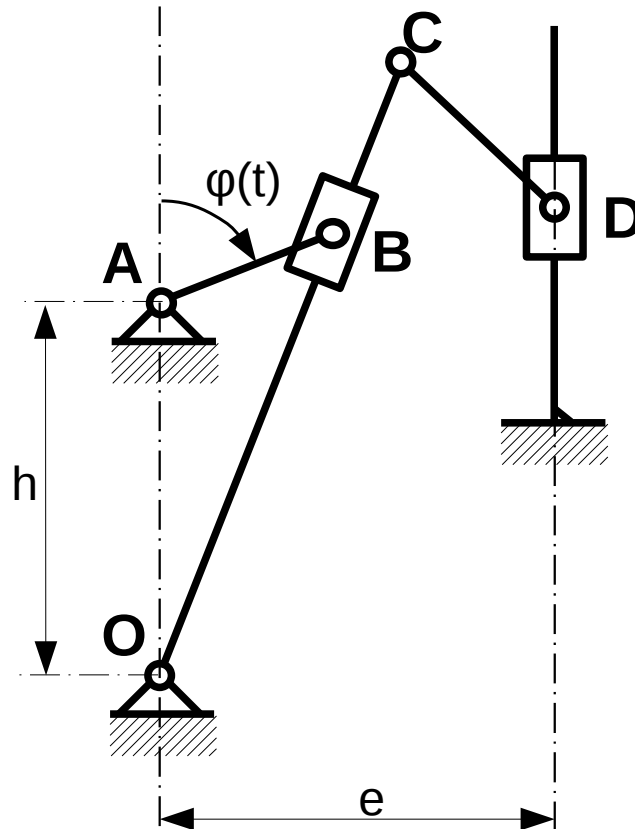


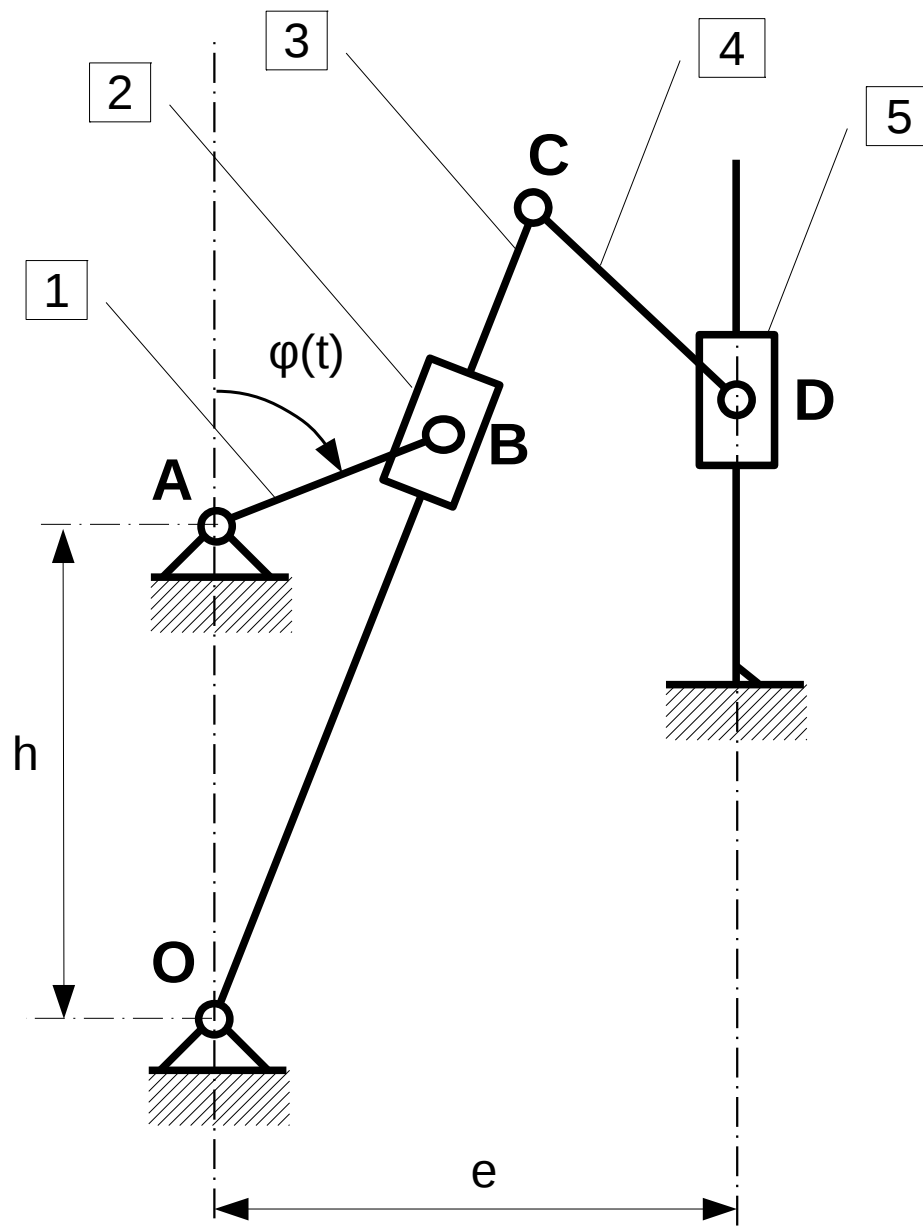
PAiT - zima 2017/2018

Prędkości i przyspieszenia w mechanizmach płaskich Przykład – metoda analityczna

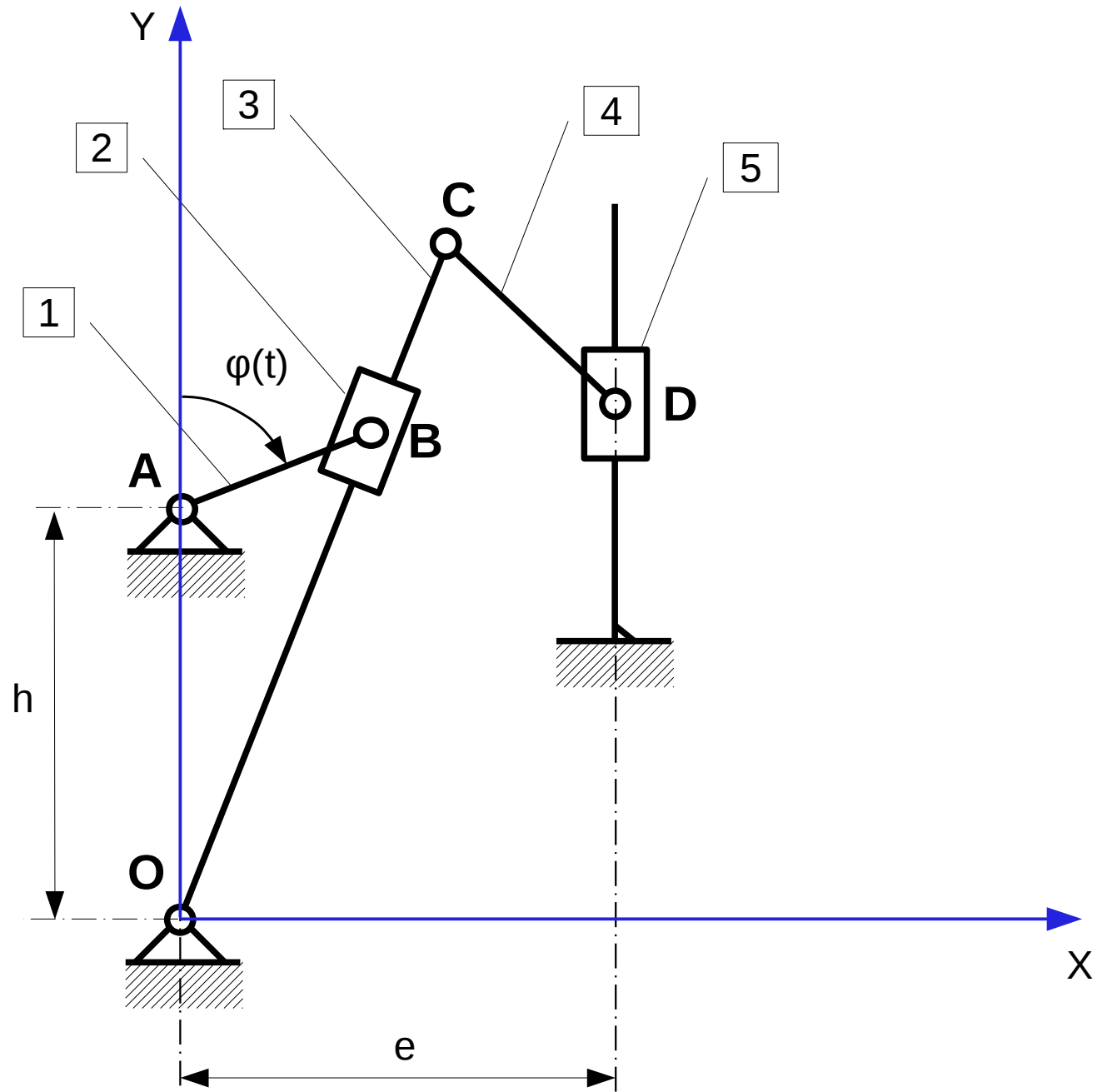
Dane: geometria mechanizmu (długości członów)
oraz kąt $\varphi(t)$ członu napędowego. $\varphi(t) = \omega t$, $\omega = \text{const}$.

Szukane: prędkość i przyspieszenie suwaka D jako funkcja kąta $\varphi(t)$.

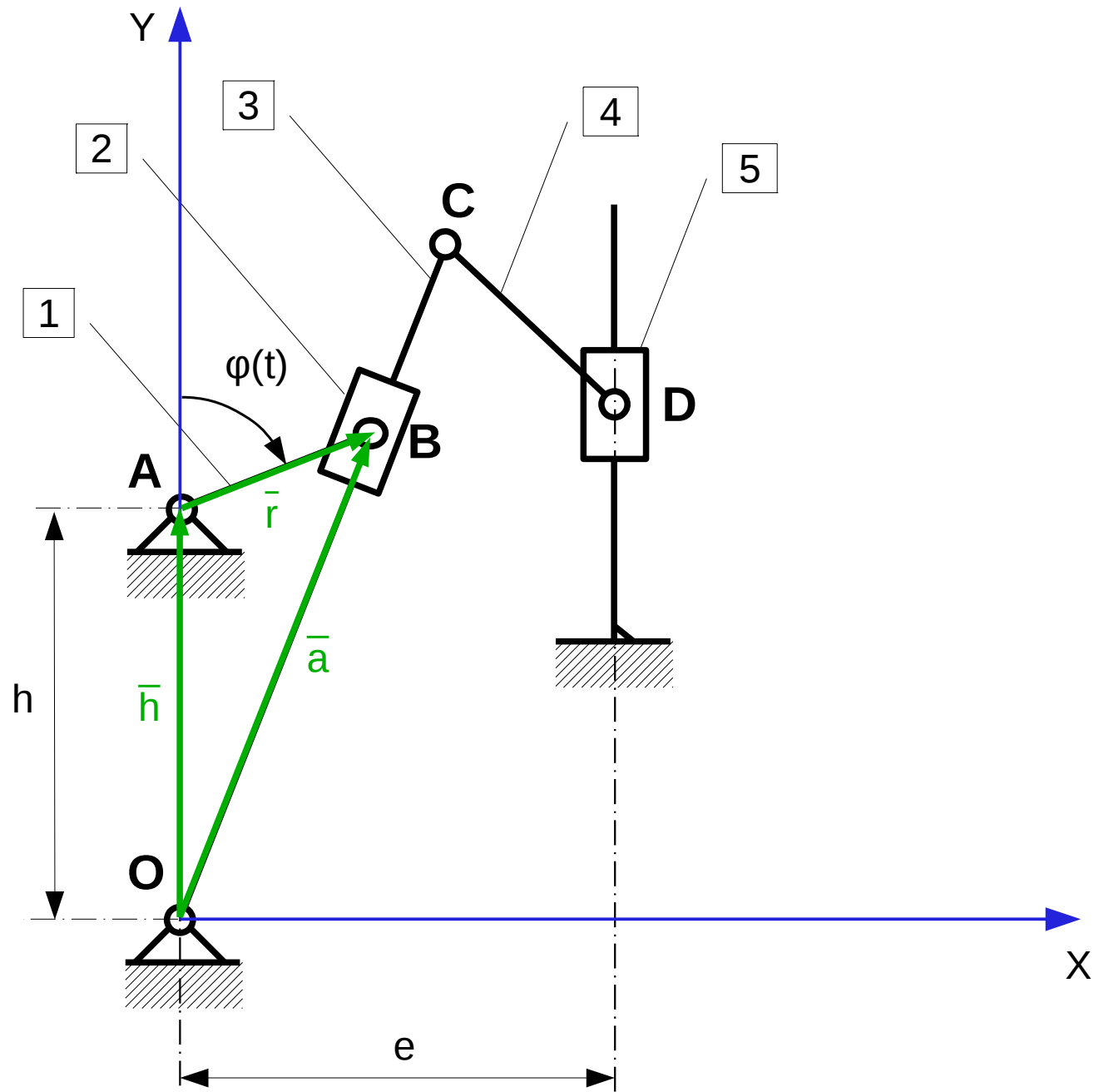




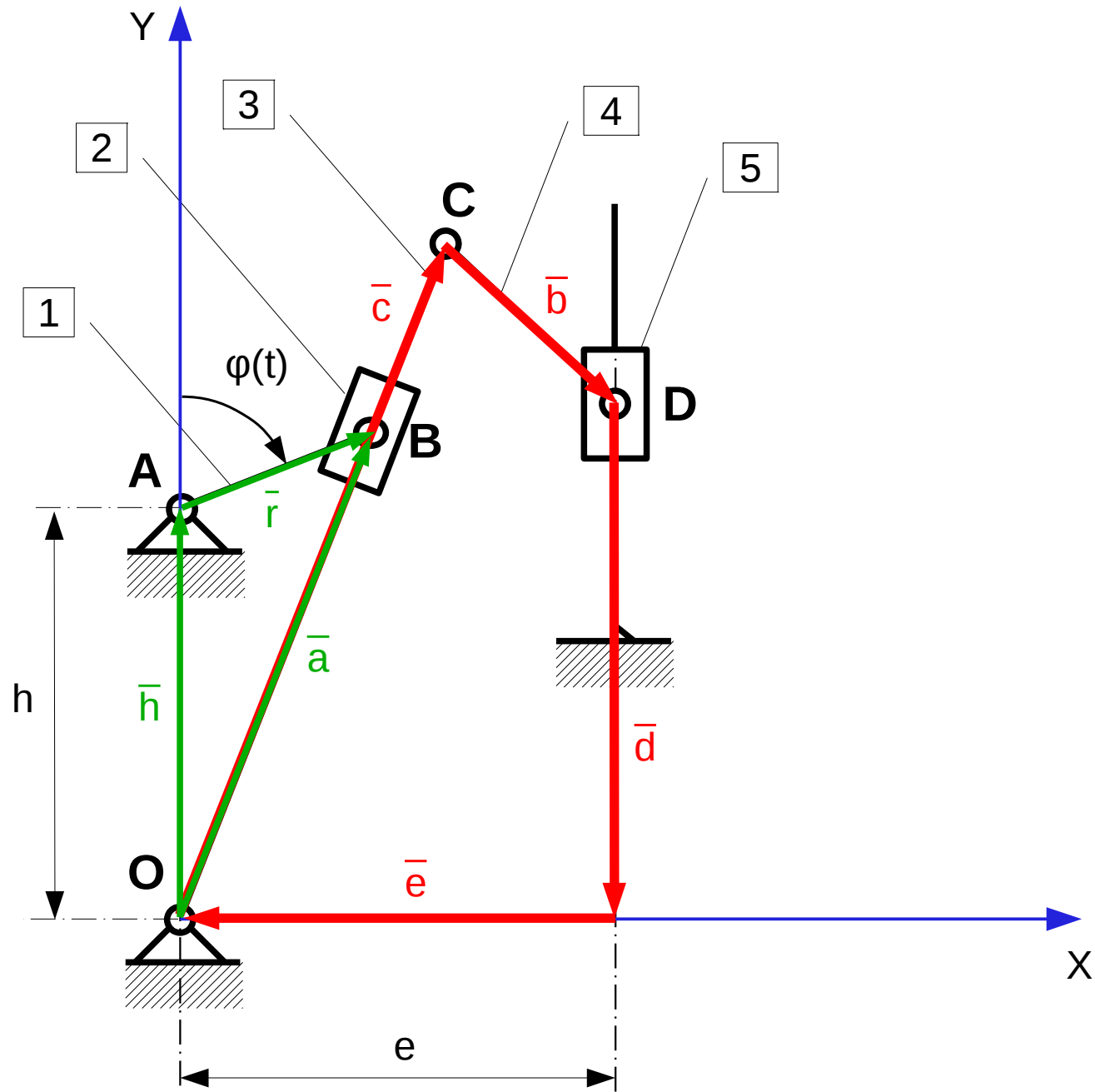
1. Wprowadzamy układ współrzędnych

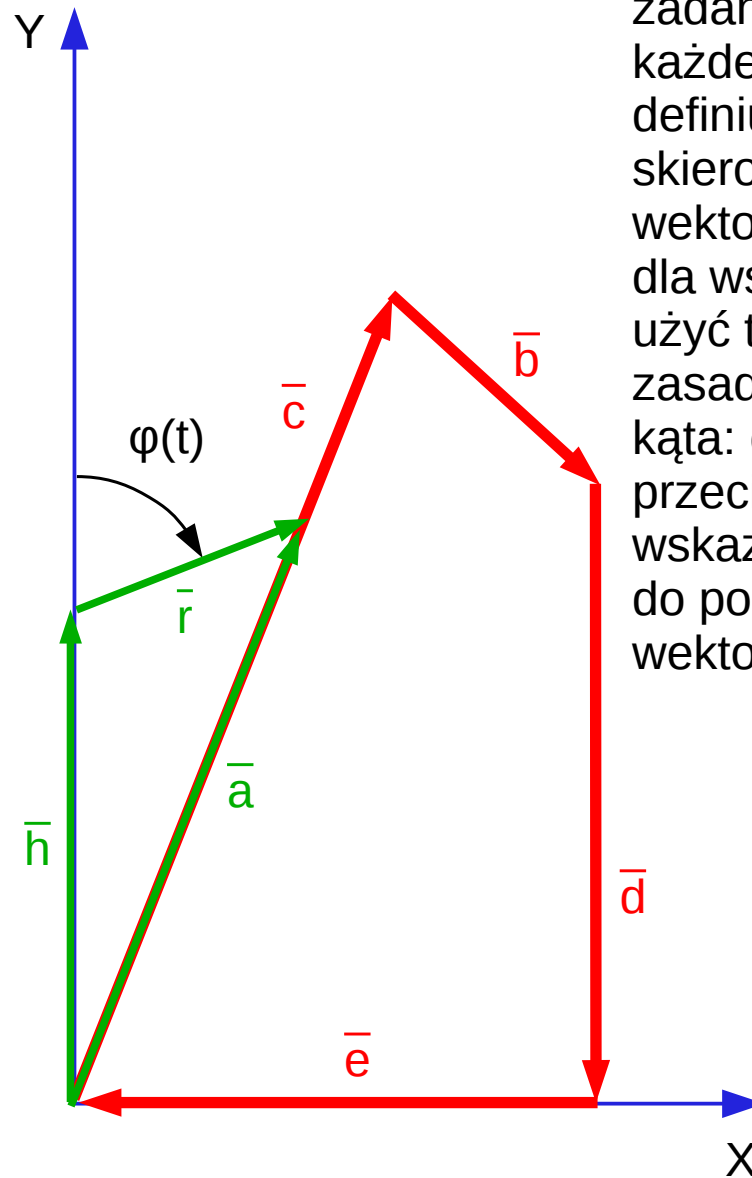
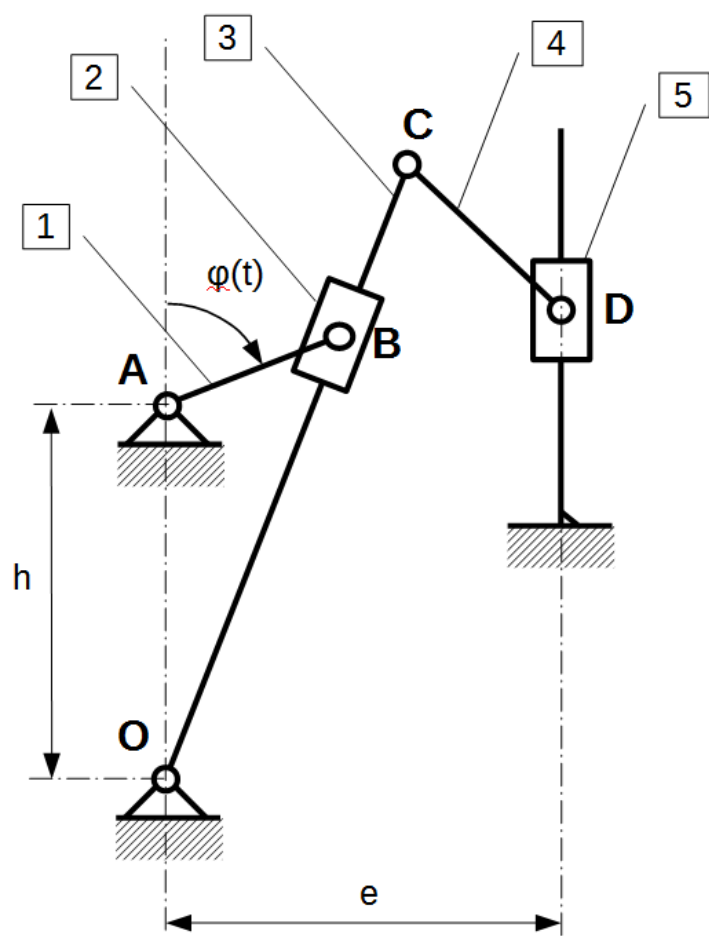


2. Wprowadzamy zamknięte wieloboki wektorów. Wektory te rozpinane są pomiędzy punktami charakterystycznymi mechanizmu. Kiedy mechanizm pracuje to wektory te mogą zmieniać swoje położenie, orientację i długość.

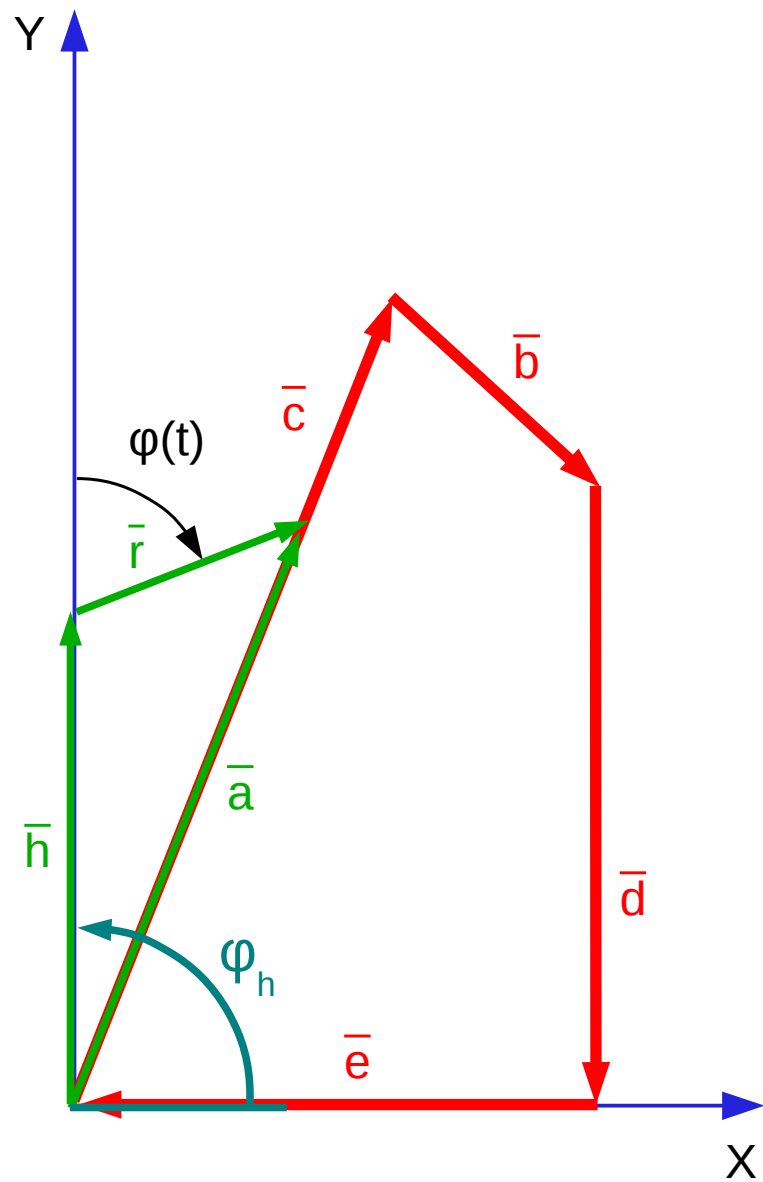
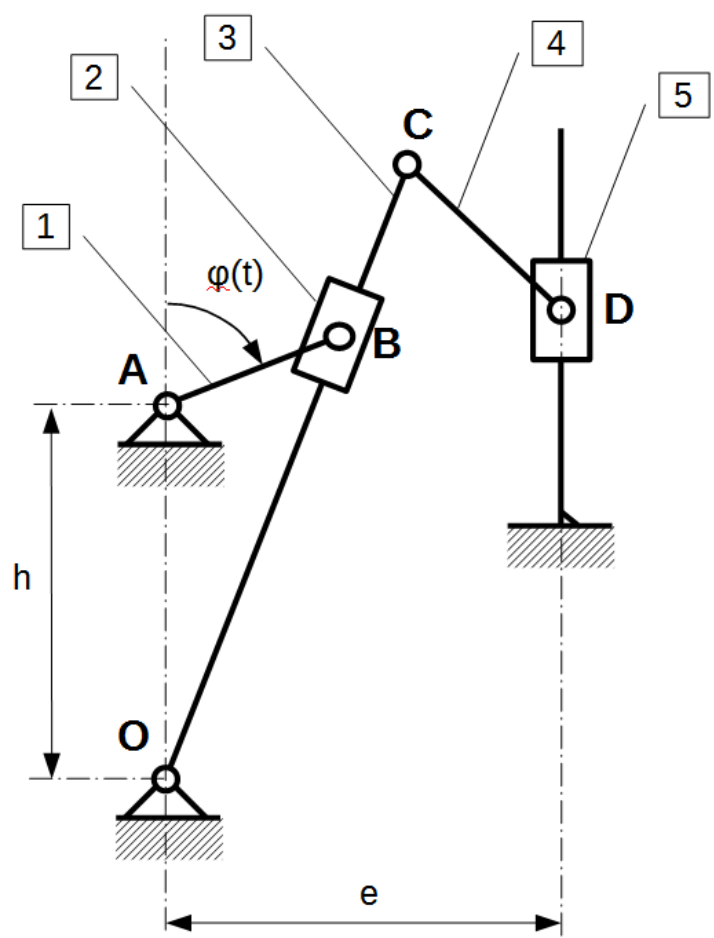


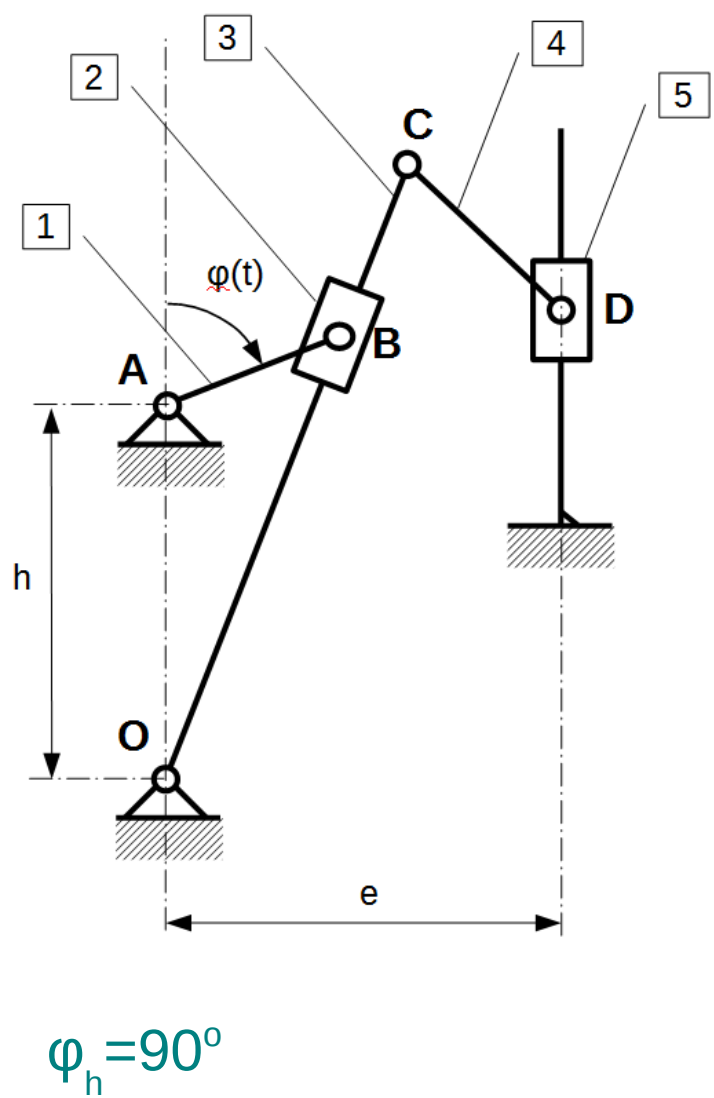
3. Tworzymy jak najłatwiejsze wieloboki wektorów, których będzie tyle, ile potrzeba, aby każdy punkt mechanizmu był uwzględniony.



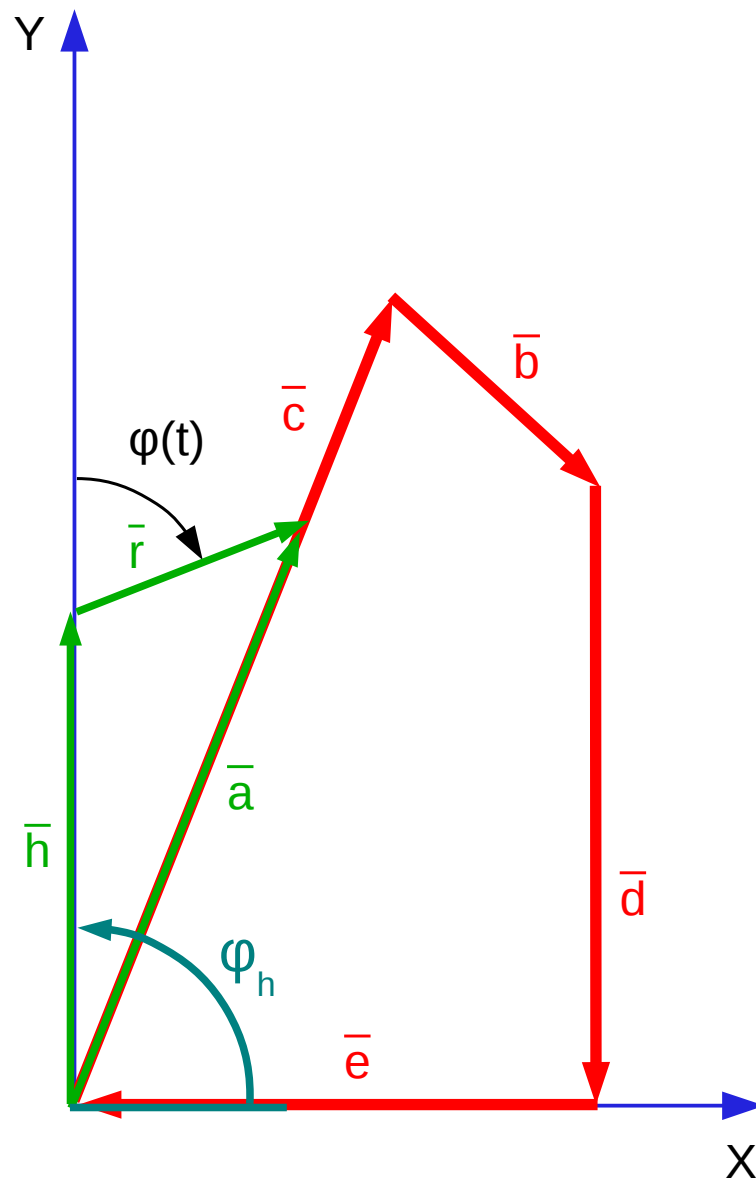


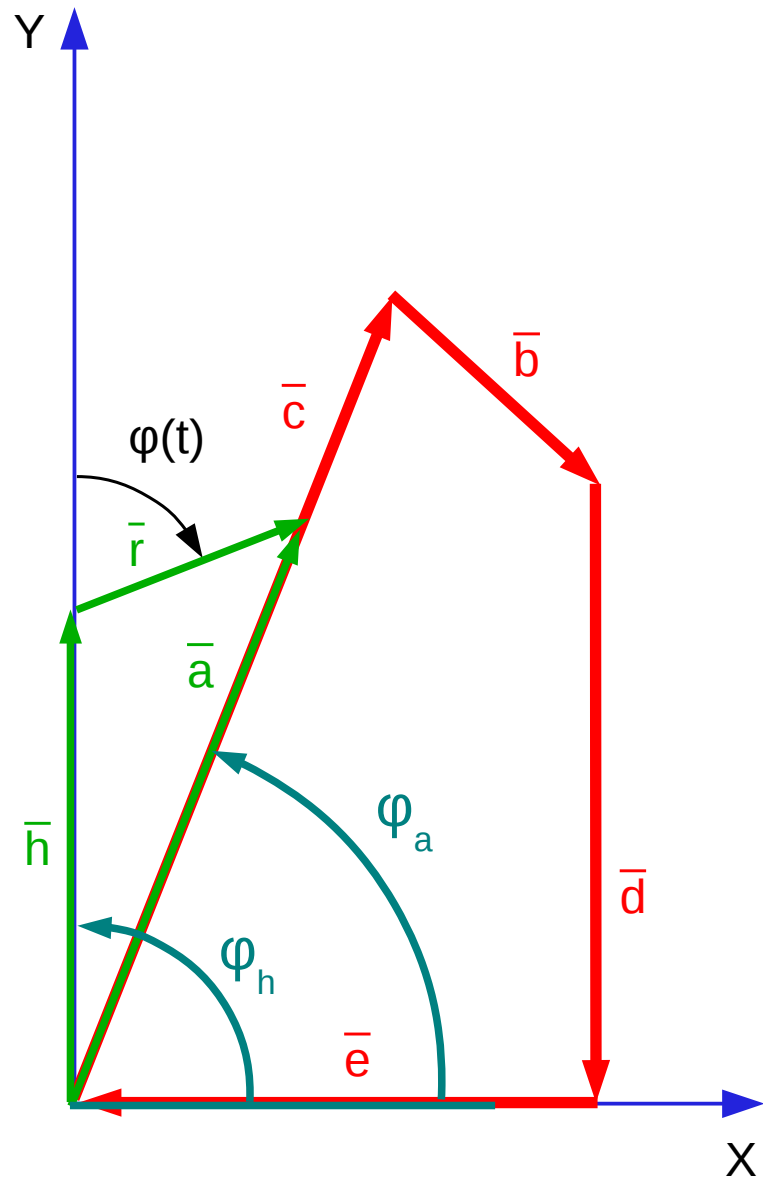
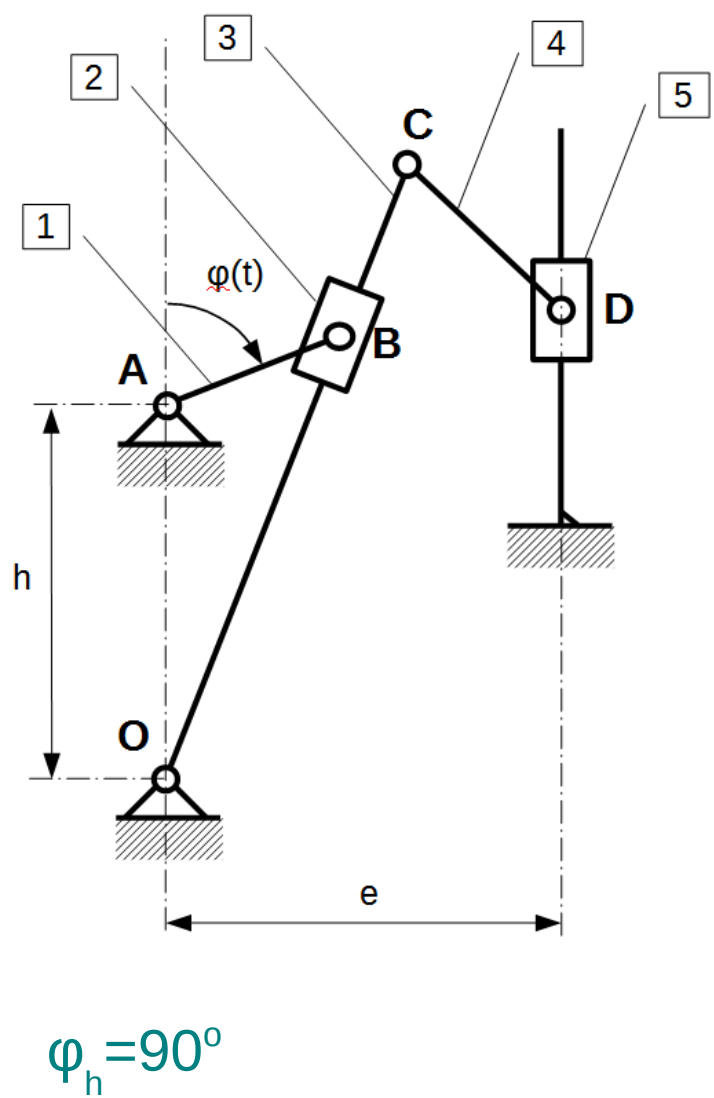
4. Oznaczamy zadany kąt $\varphi(t)$. Dla każdego wektora definiujemy kąt skierowania tego wektora. Najlepiej dla wszystkich kątów użyć tej samej zasady określania kąta: obrót osi **X** przeciwnie do ruchu wskazówek zegara do pokrycia z wektorem.

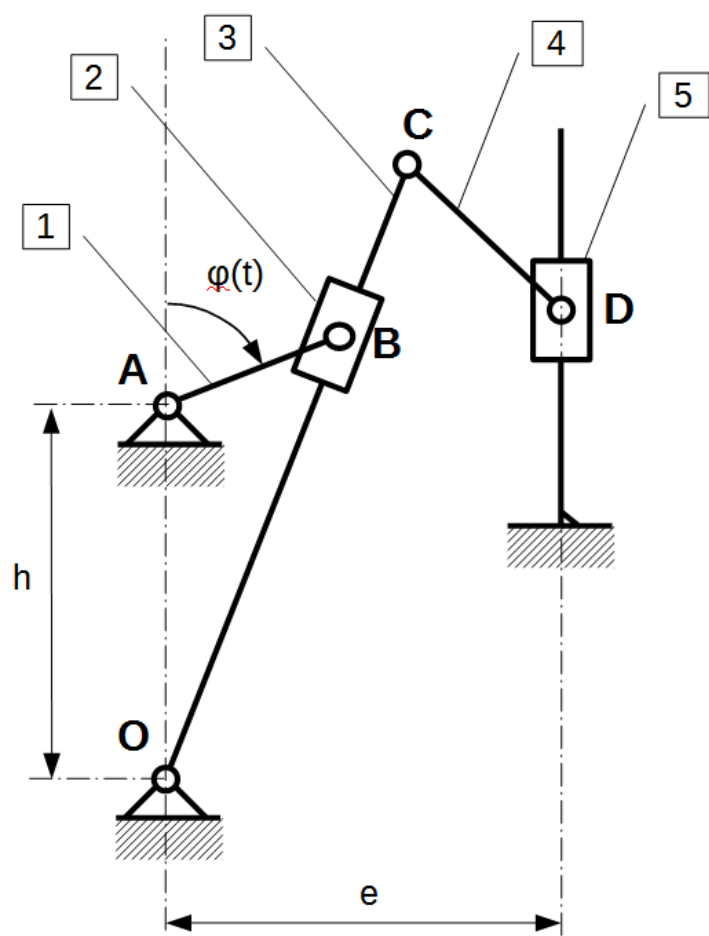




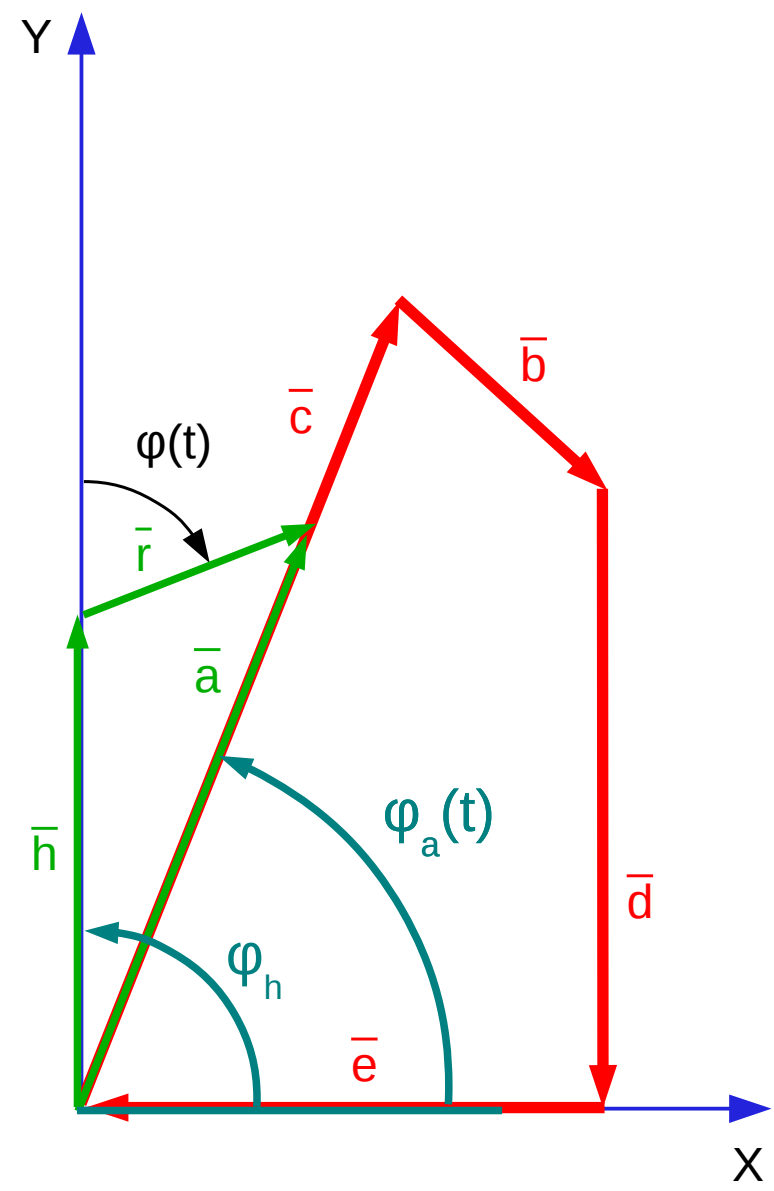
$$\varphi_h = 90^\circ$$

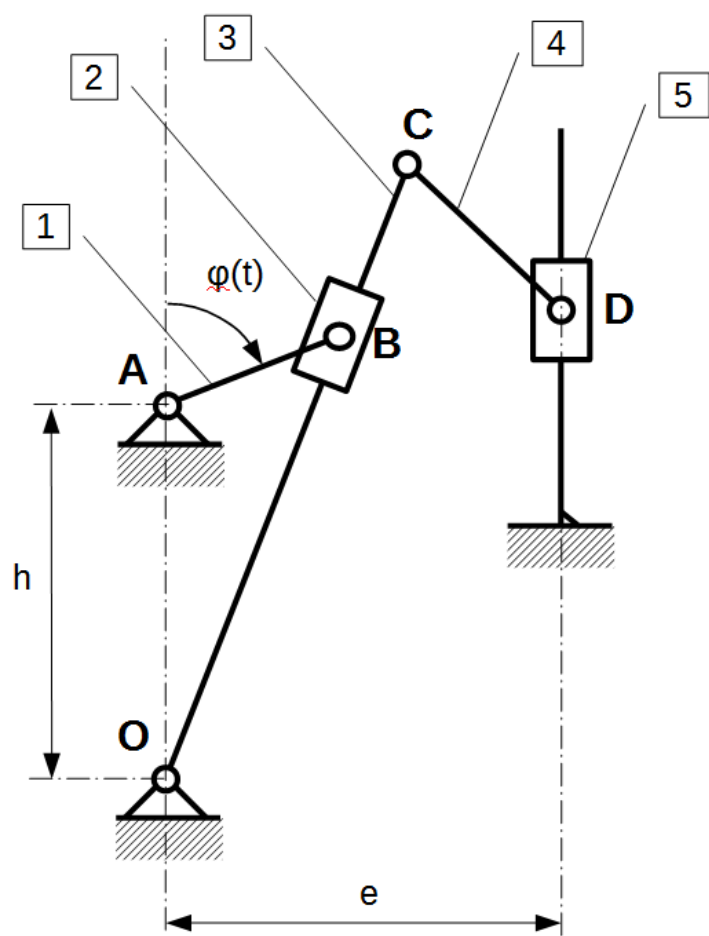




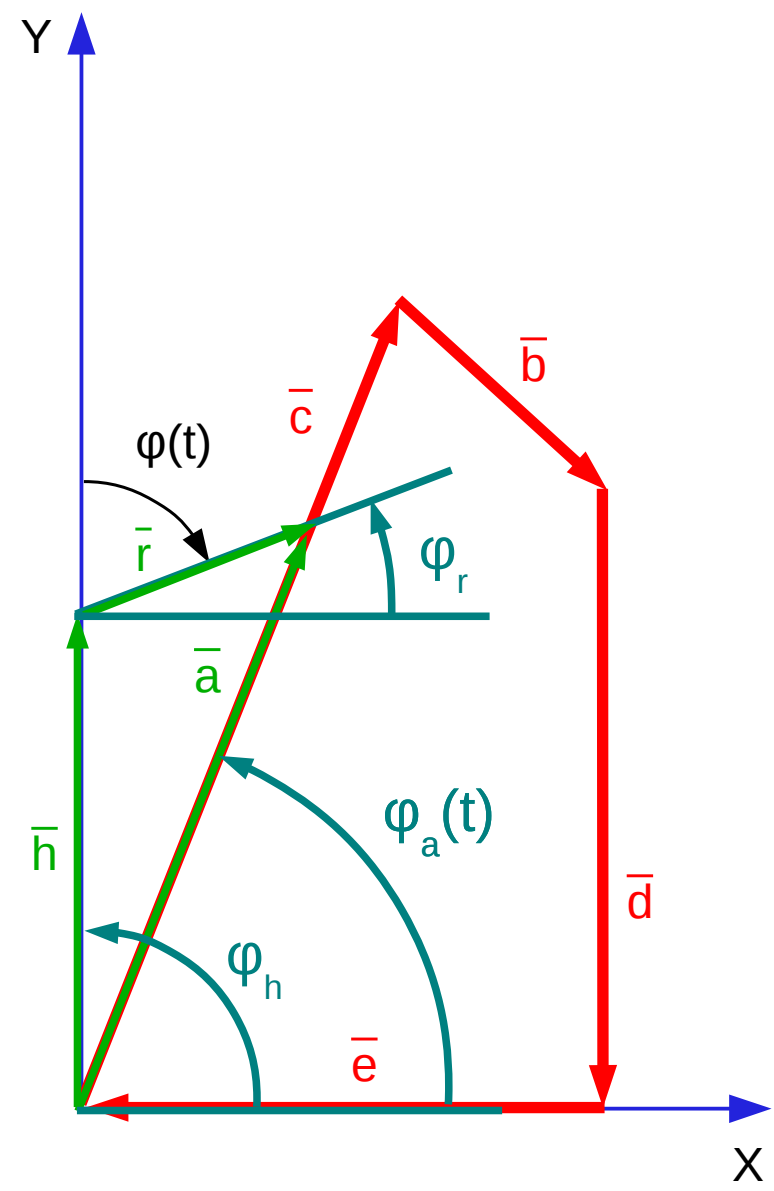


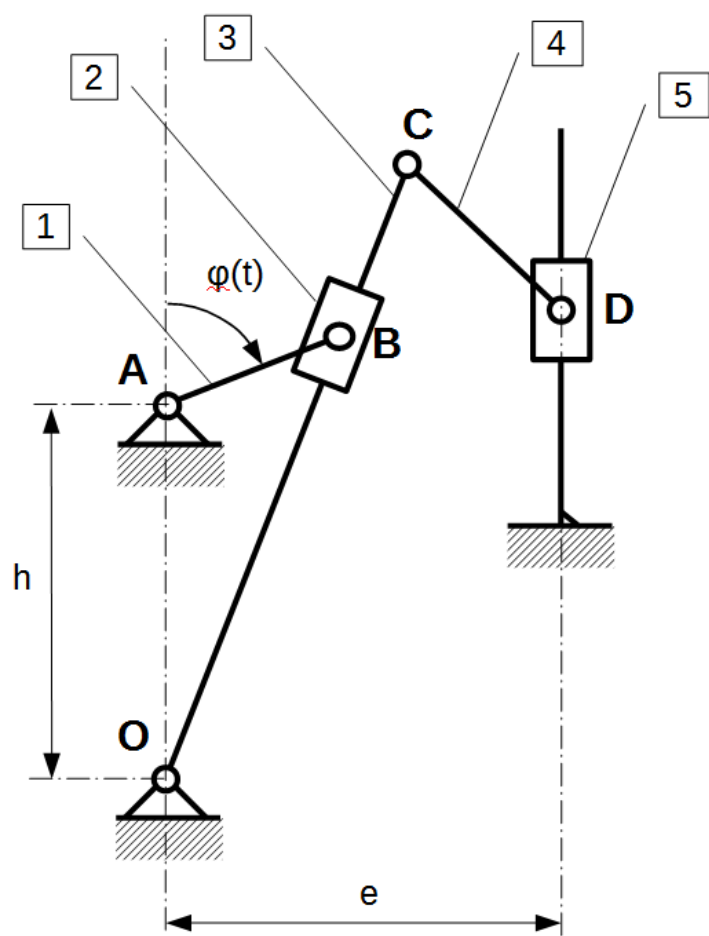
$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$



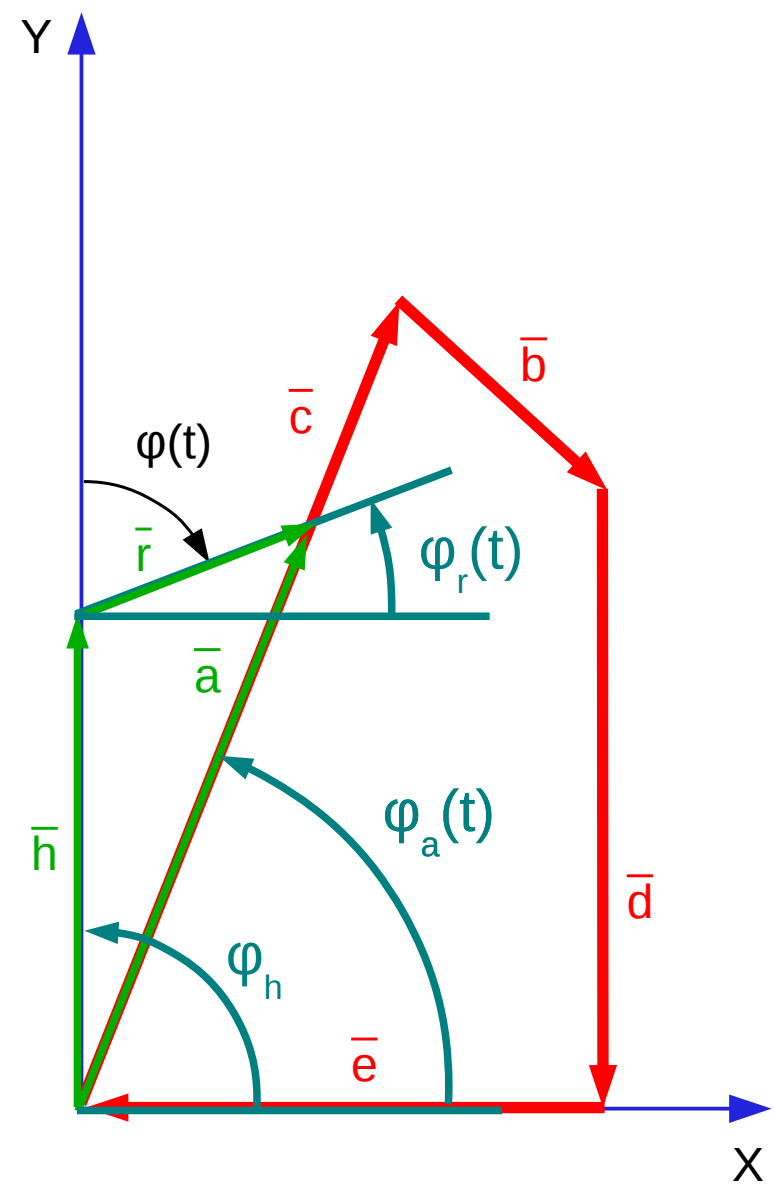


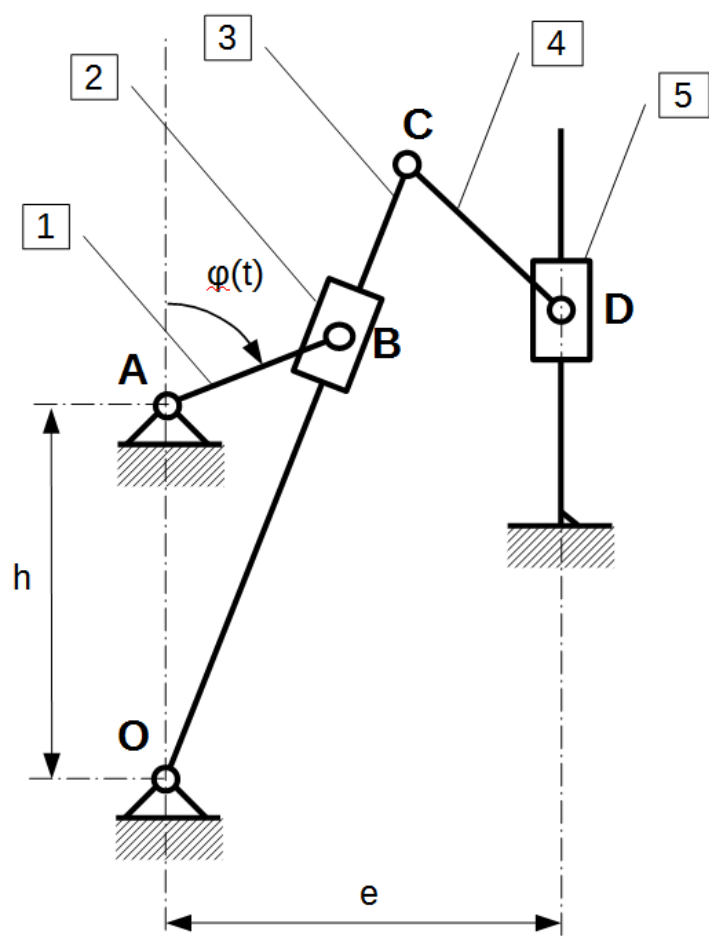
$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$



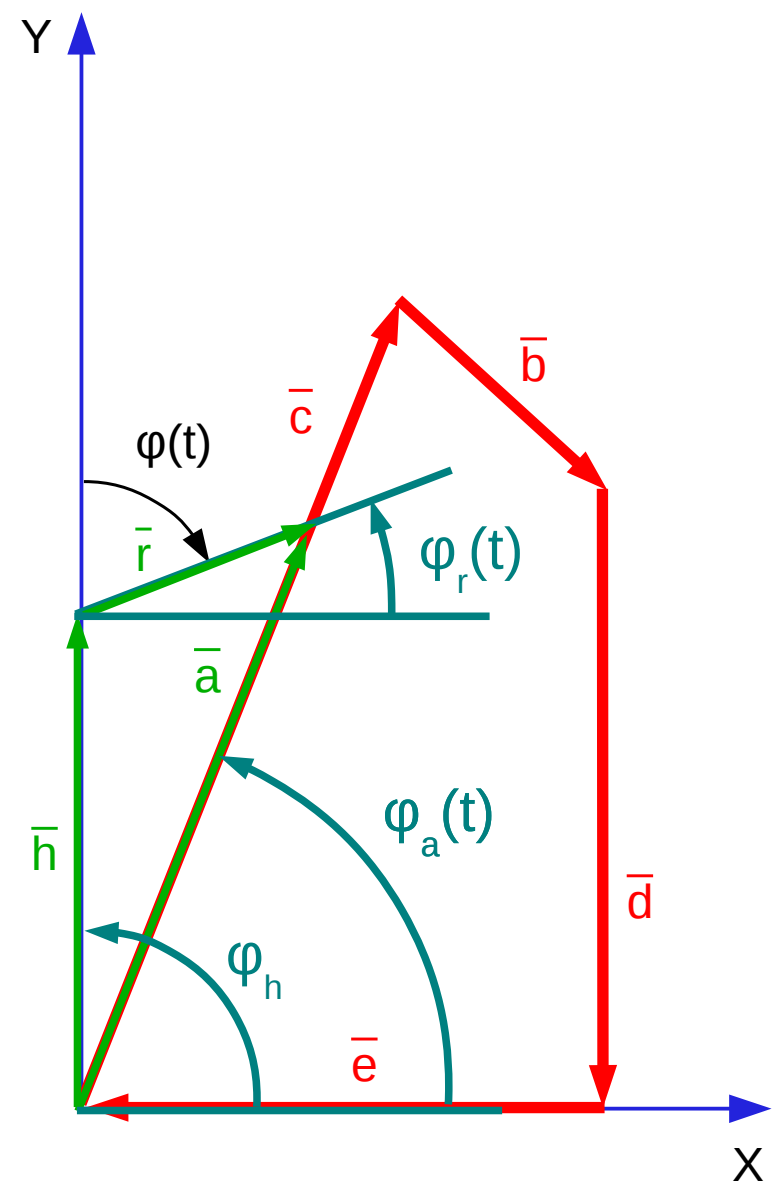


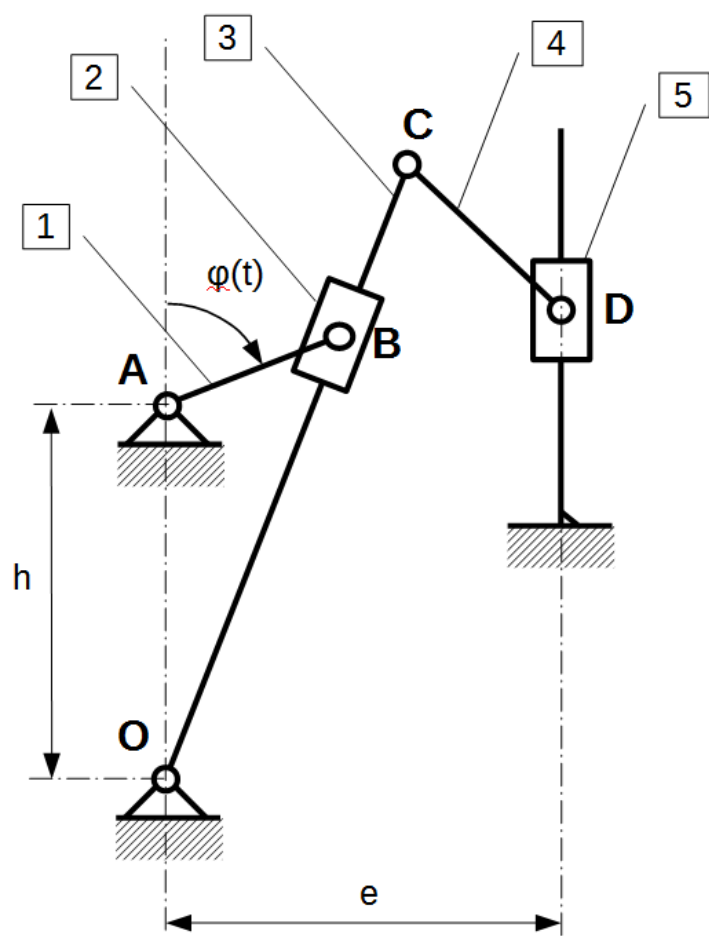
$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$



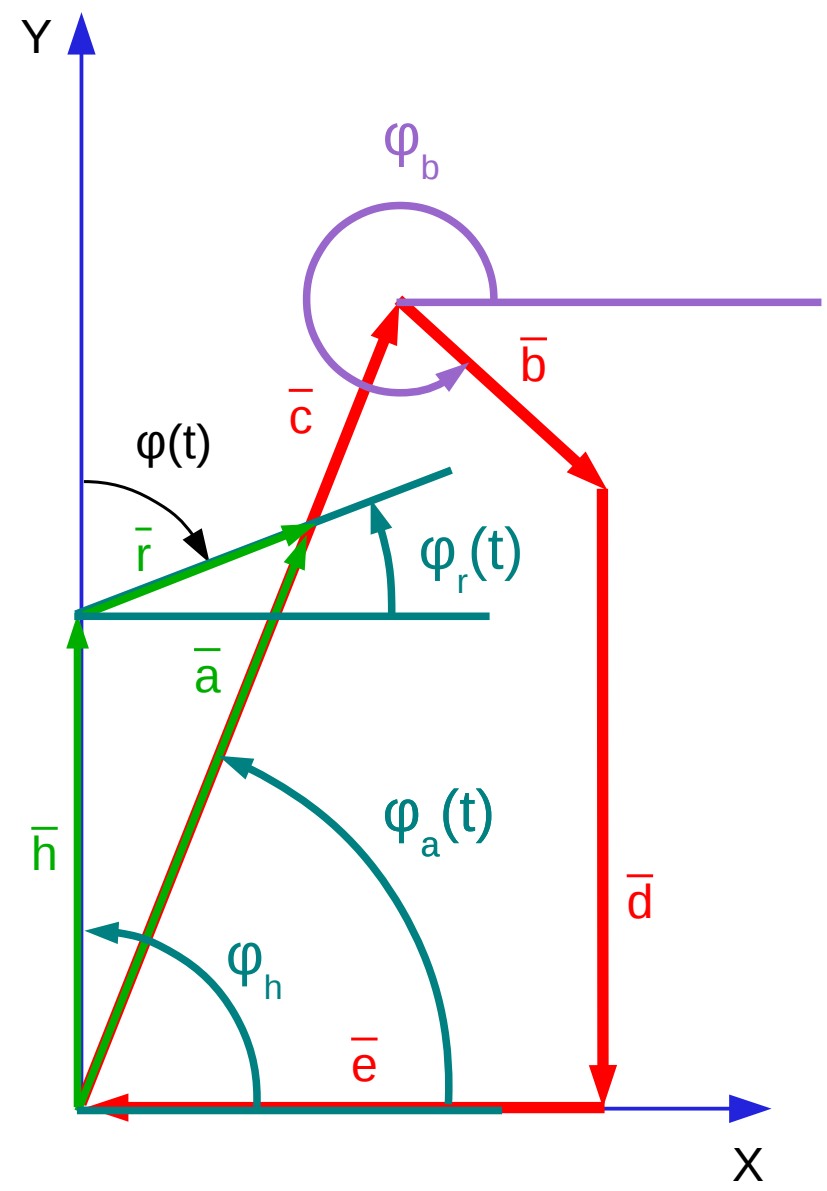


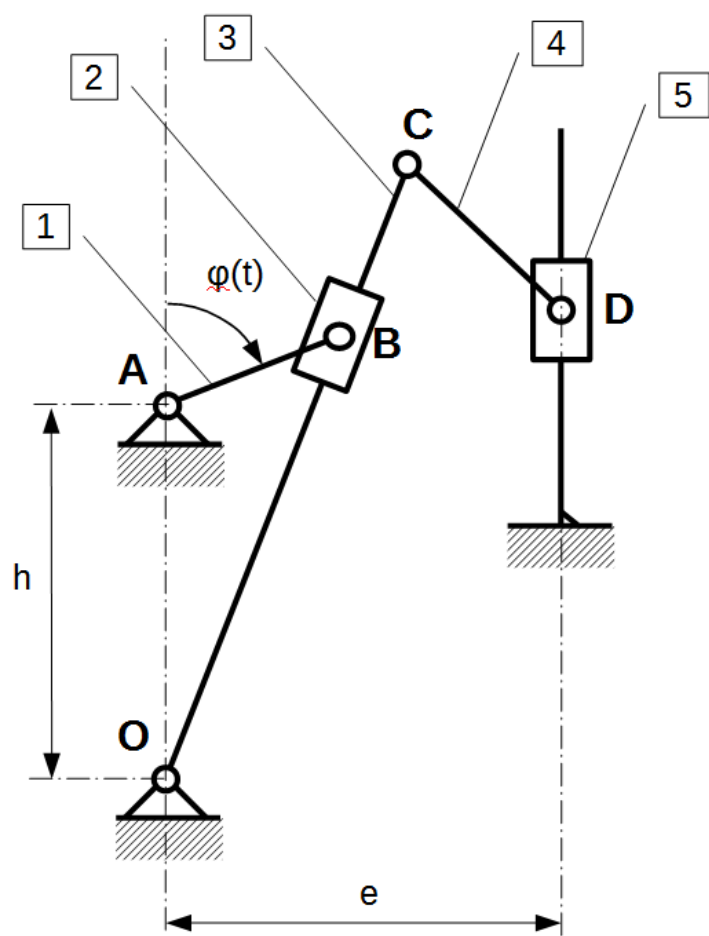
$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$
 $\varphi_r(t) = 90^\circ - \varphi(t)$



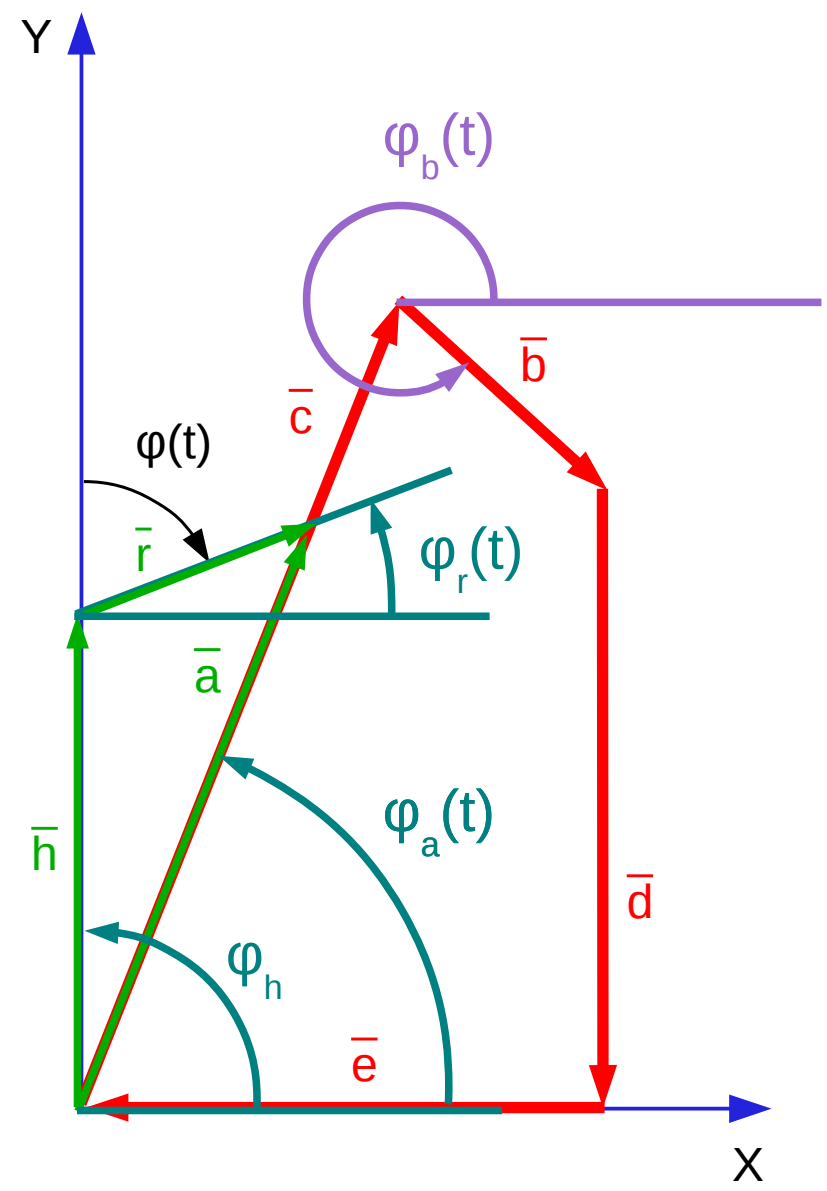


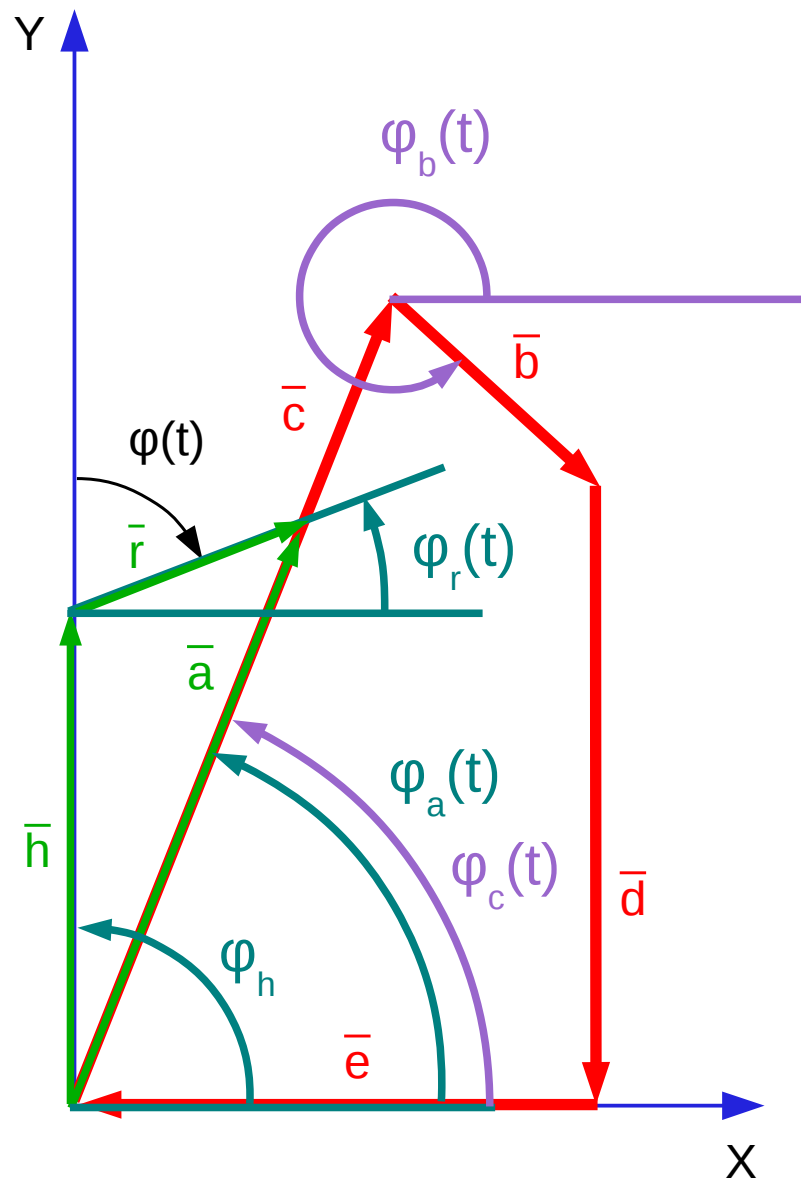
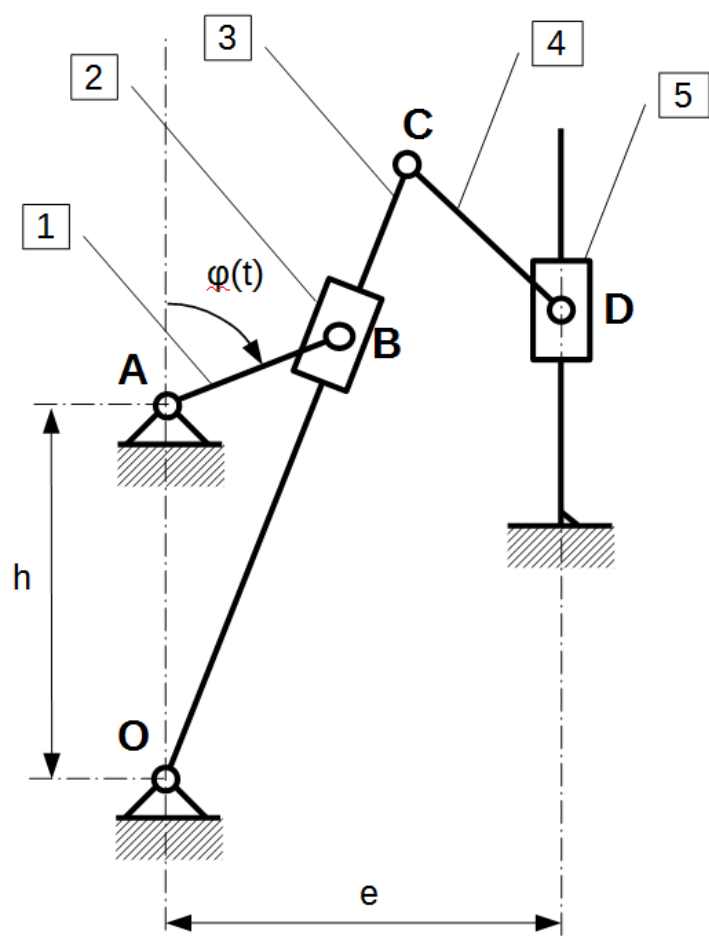
$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$
 $\varphi_r(t) = 90^\circ - \varphi(t)$





$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$
 $\varphi_r(t) = 90^\circ - \varphi(t)$
 $\varphi_b(t) \neq \text{const.}$





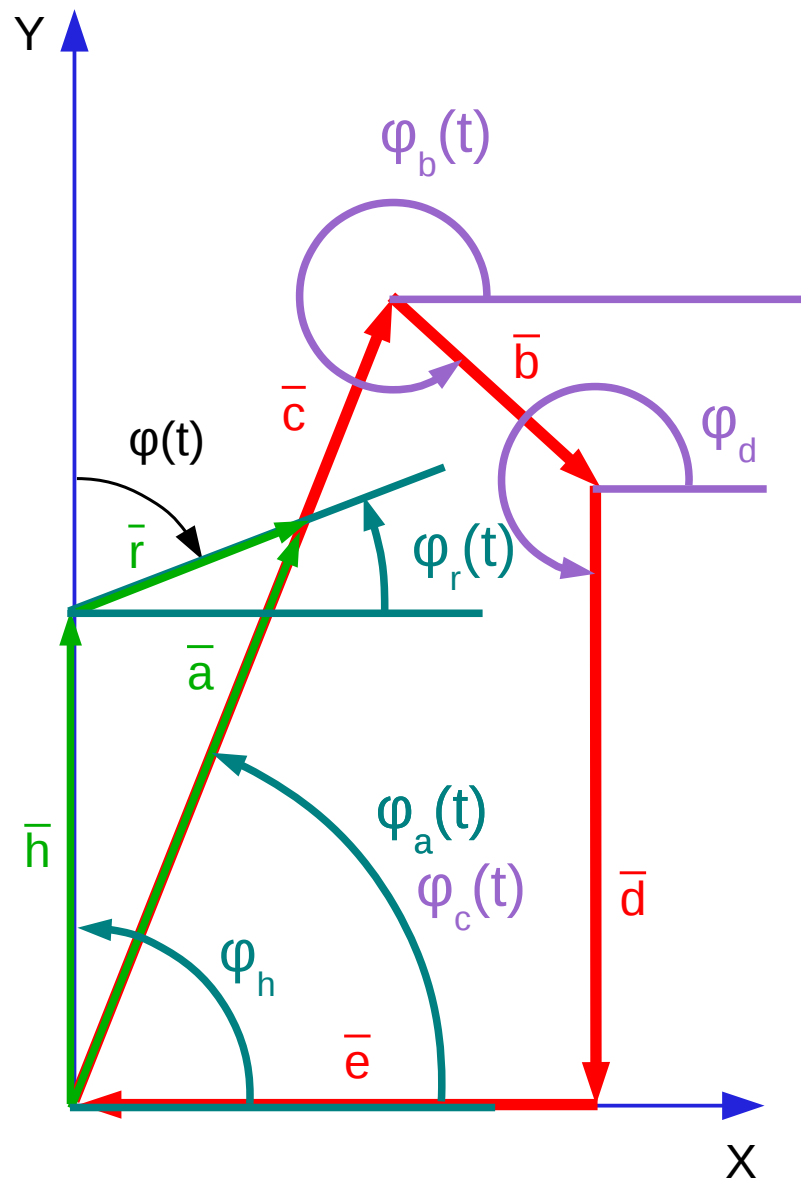
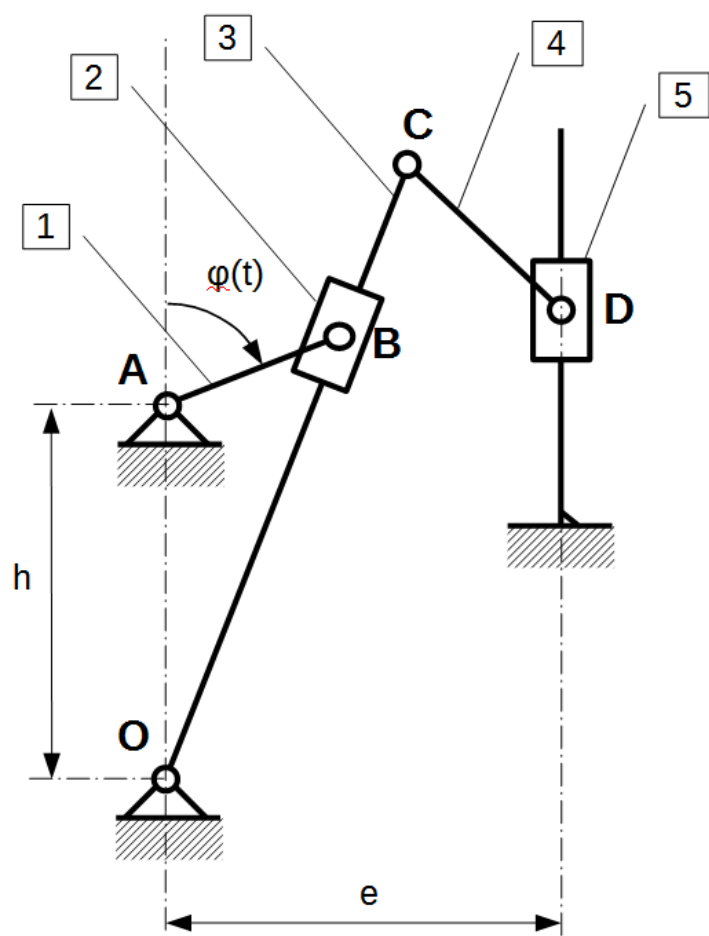
$\varphi_h = 90^\circ$

$\varphi_a(t) \neq \text{const.}$

$\varphi_r(t) = 90^\circ - \varphi(t)$

$\varphi_b(t) \neq \text{const.}$

$\varphi_c(t) = \varphi_a(t)$



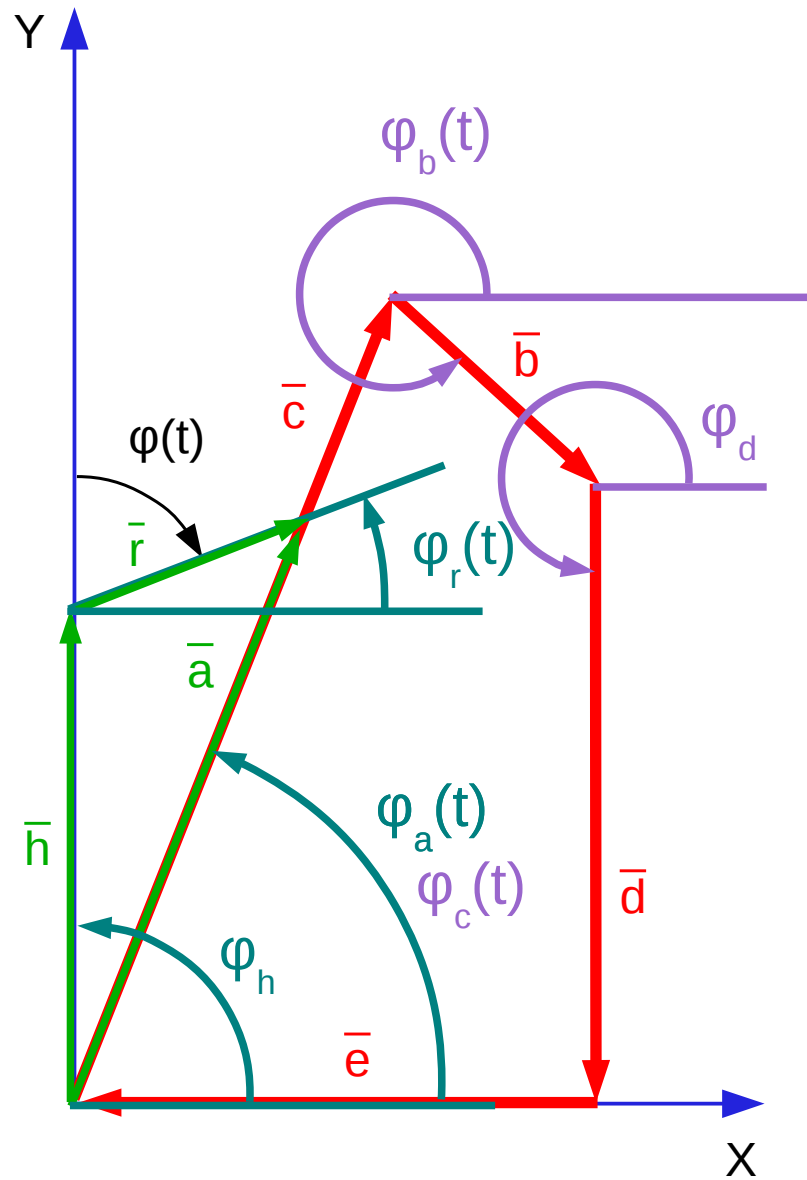
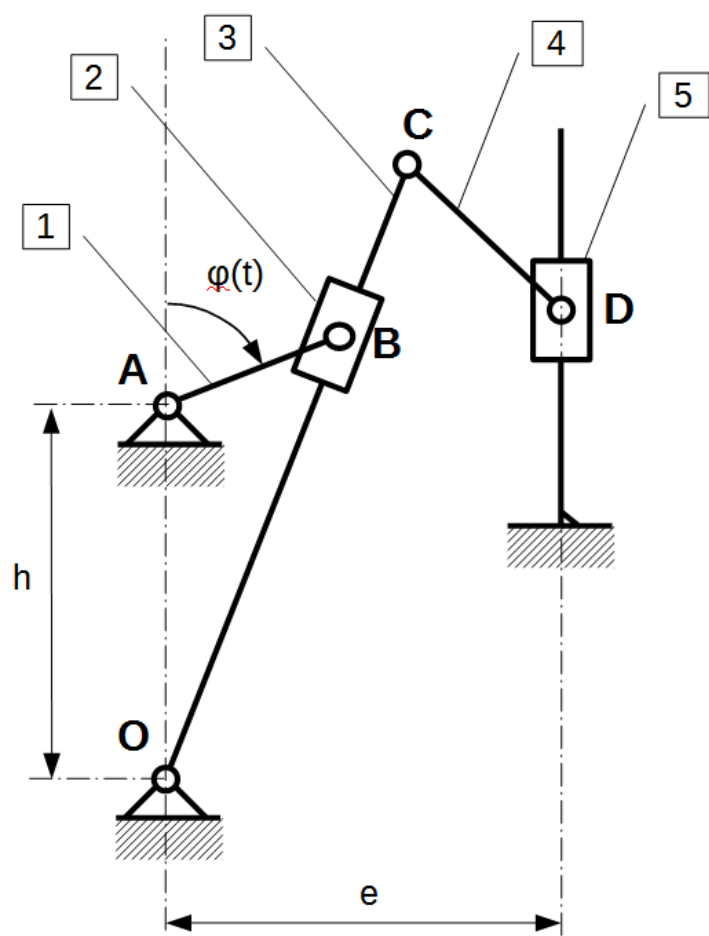
$$\varphi_h = 90^\circ$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$



$$\varphi_h = 90^\circ$$

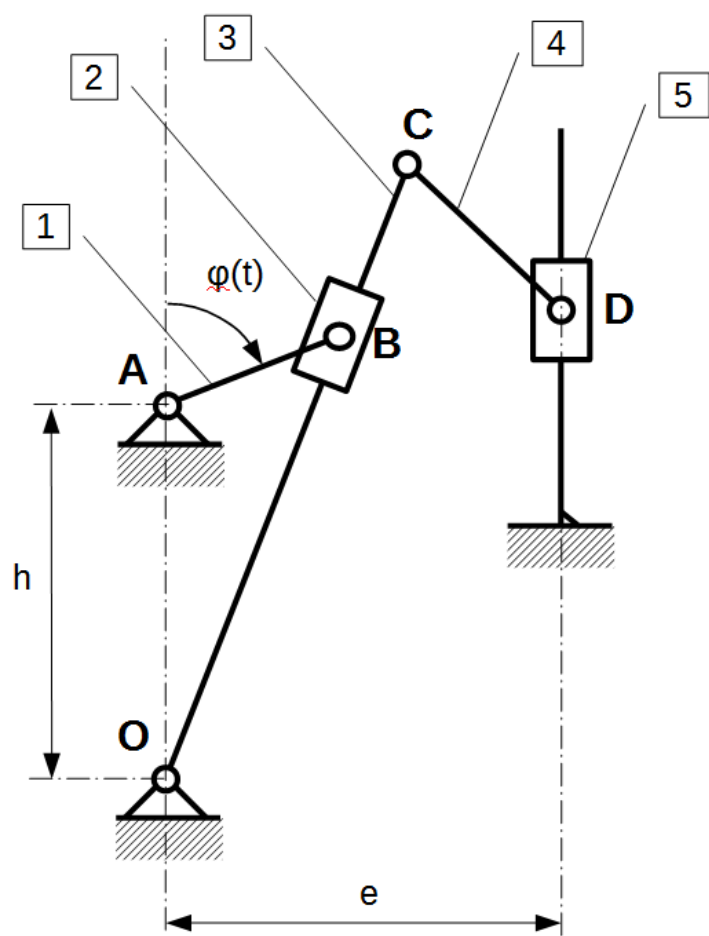
$$\varphi_d = 270^\circ$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$



$$\varphi_h = 90^\circ$$

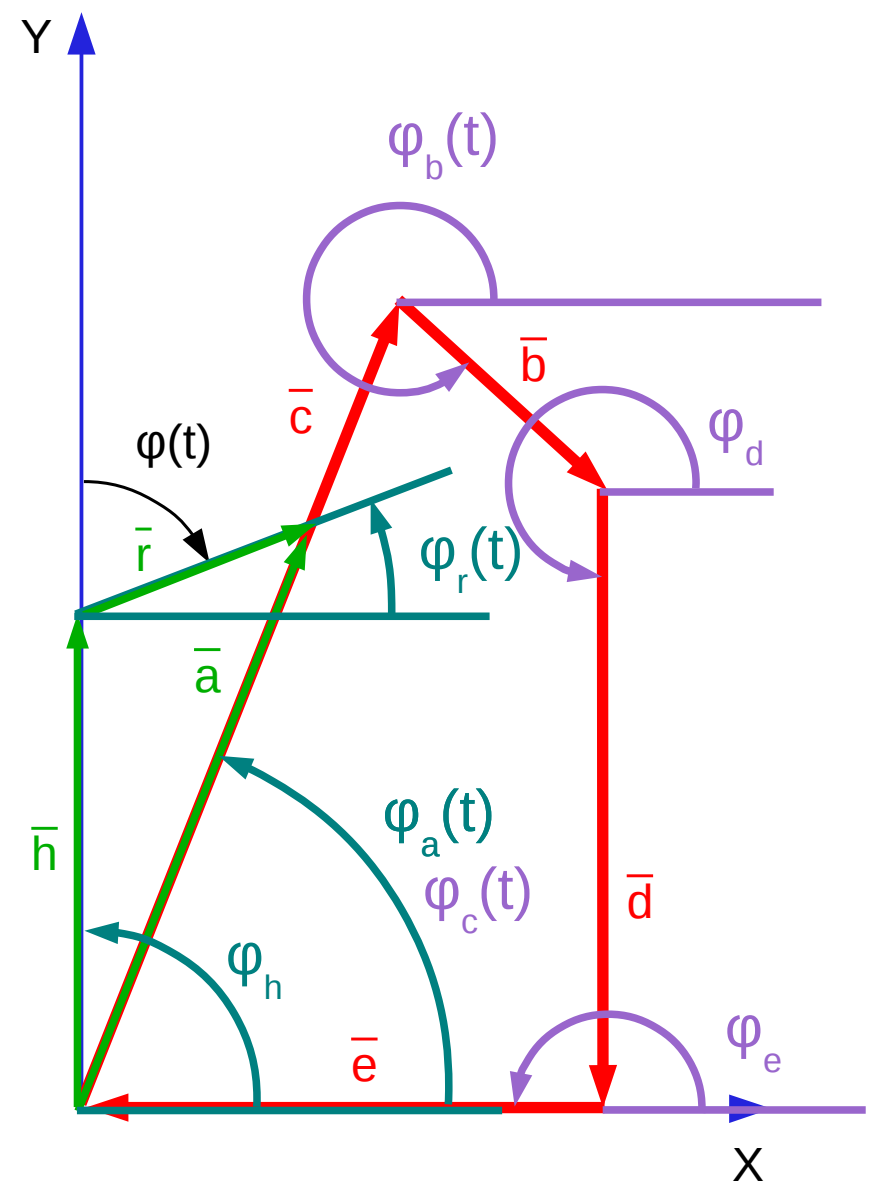
$$\varphi_a(t) \neq \text{const.}$$

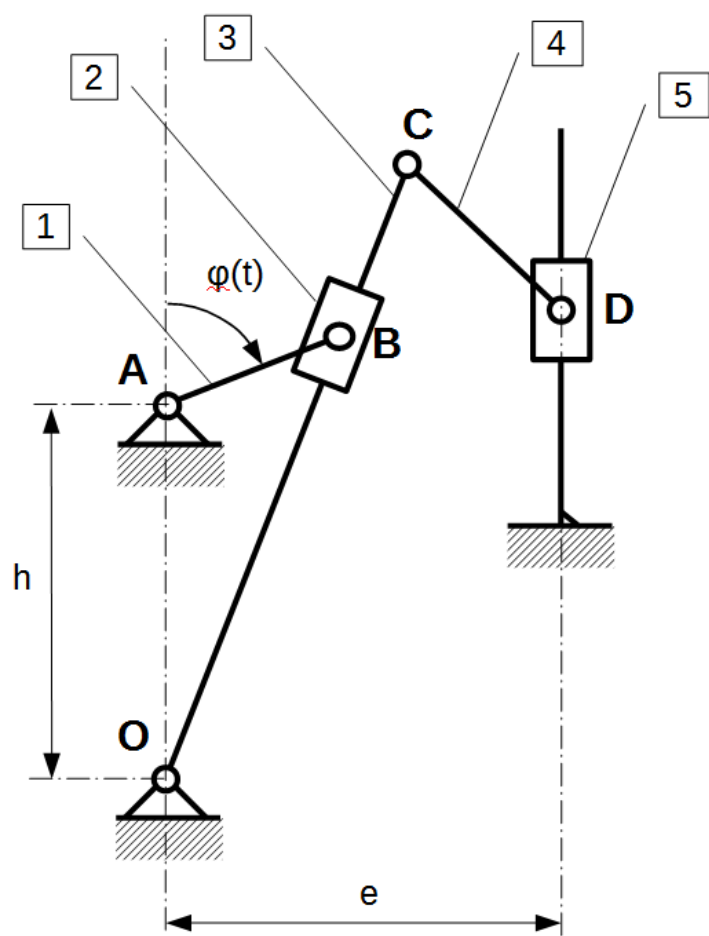
$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$

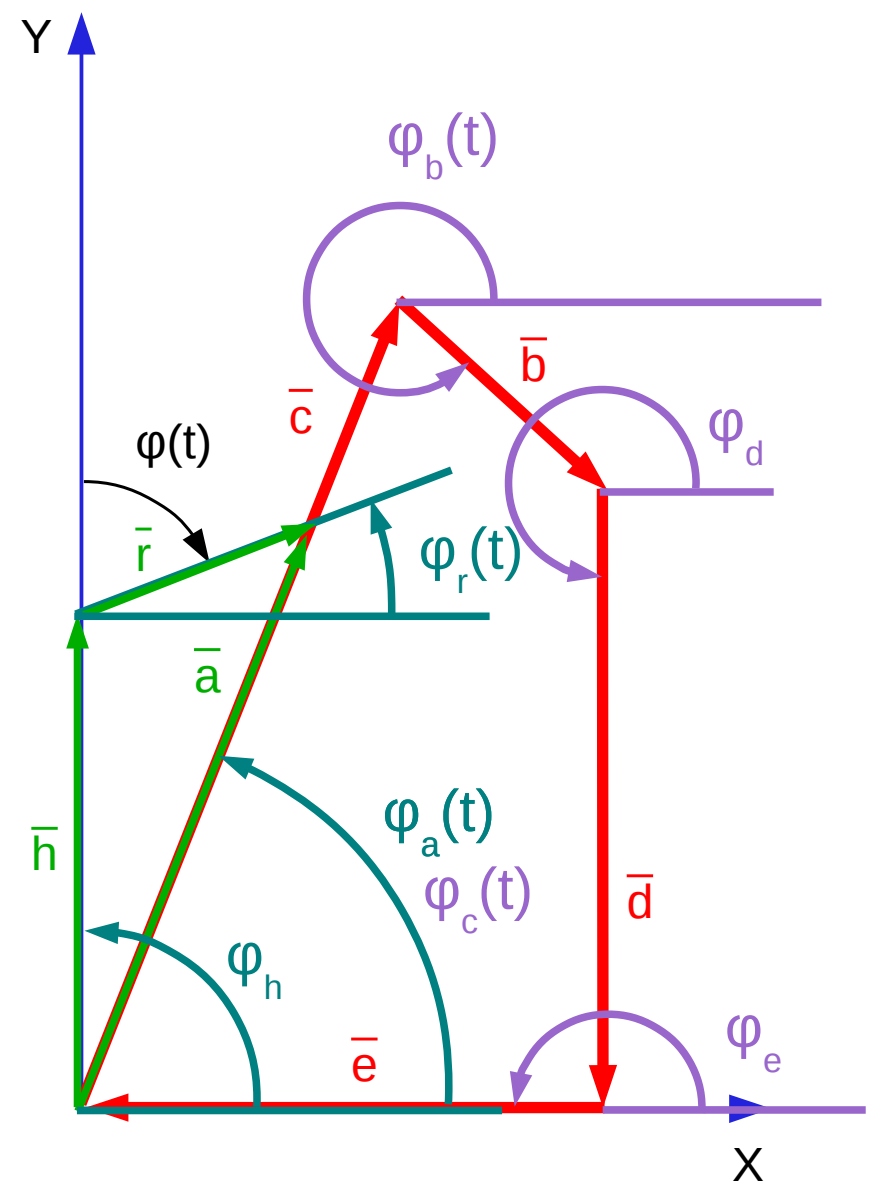
$$\varphi_d = 270^\circ$$

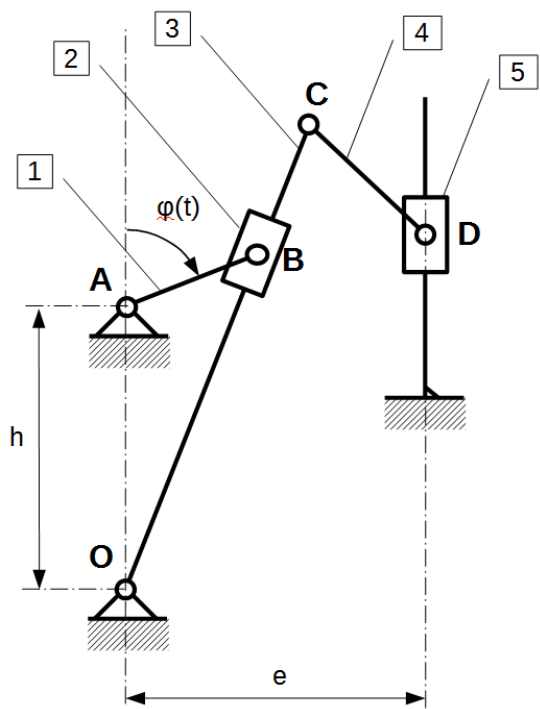




$\varphi_h = 90^\circ$
 $\varphi_a(t) \neq \text{const.}$
 $\varphi_r(t) = 90^\circ - \varphi(t)$
 $\varphi_b(t) \neq \text{const.}$
 $\varphi_c(t) = \varphi_a(t)$

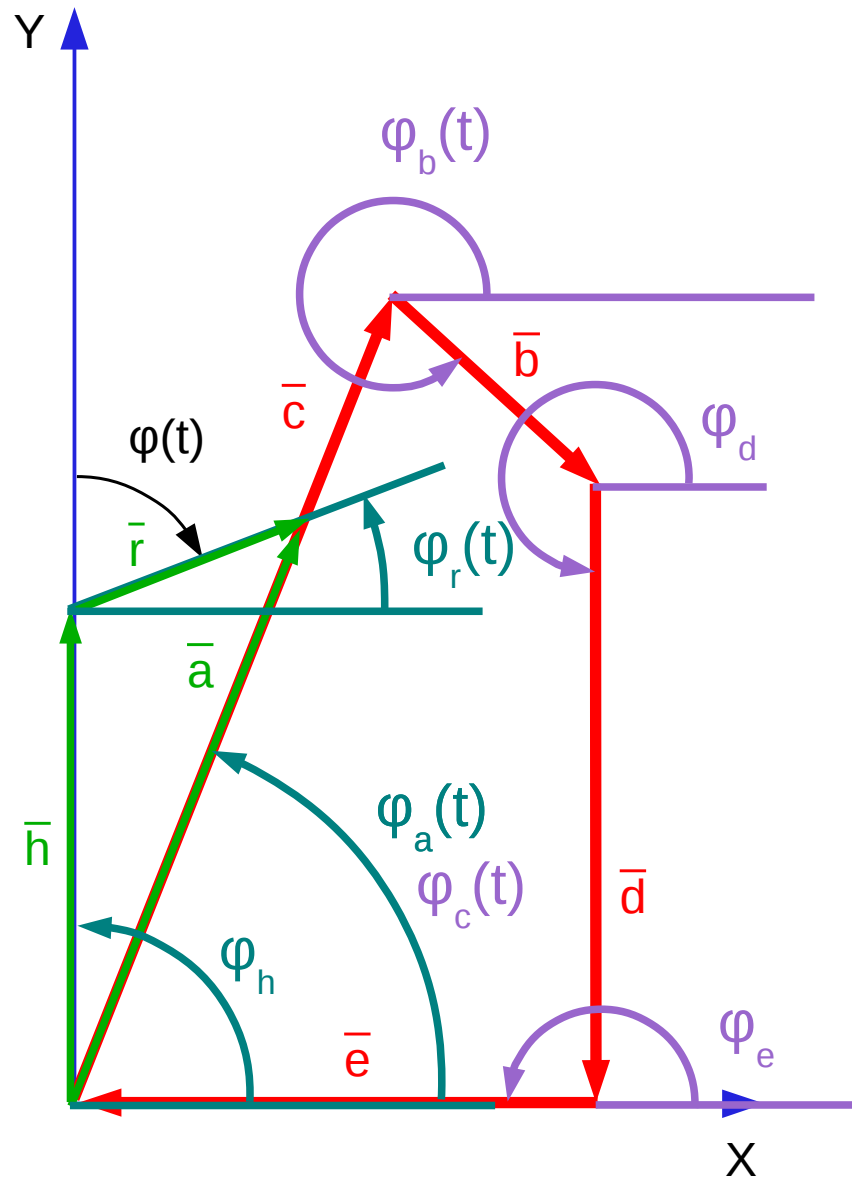
$\varphi_d = 270^\circ$
 $\varphi_e = 180^\circ$

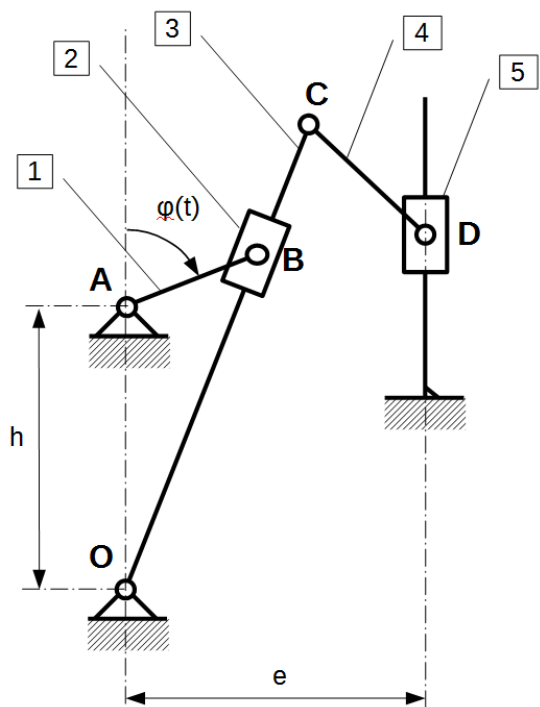




- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

5. Określiliśmy wszystkie kąty, podkreślając, które mają stałą wartość a które zmieniają się w czasie pracy mechanizmu.

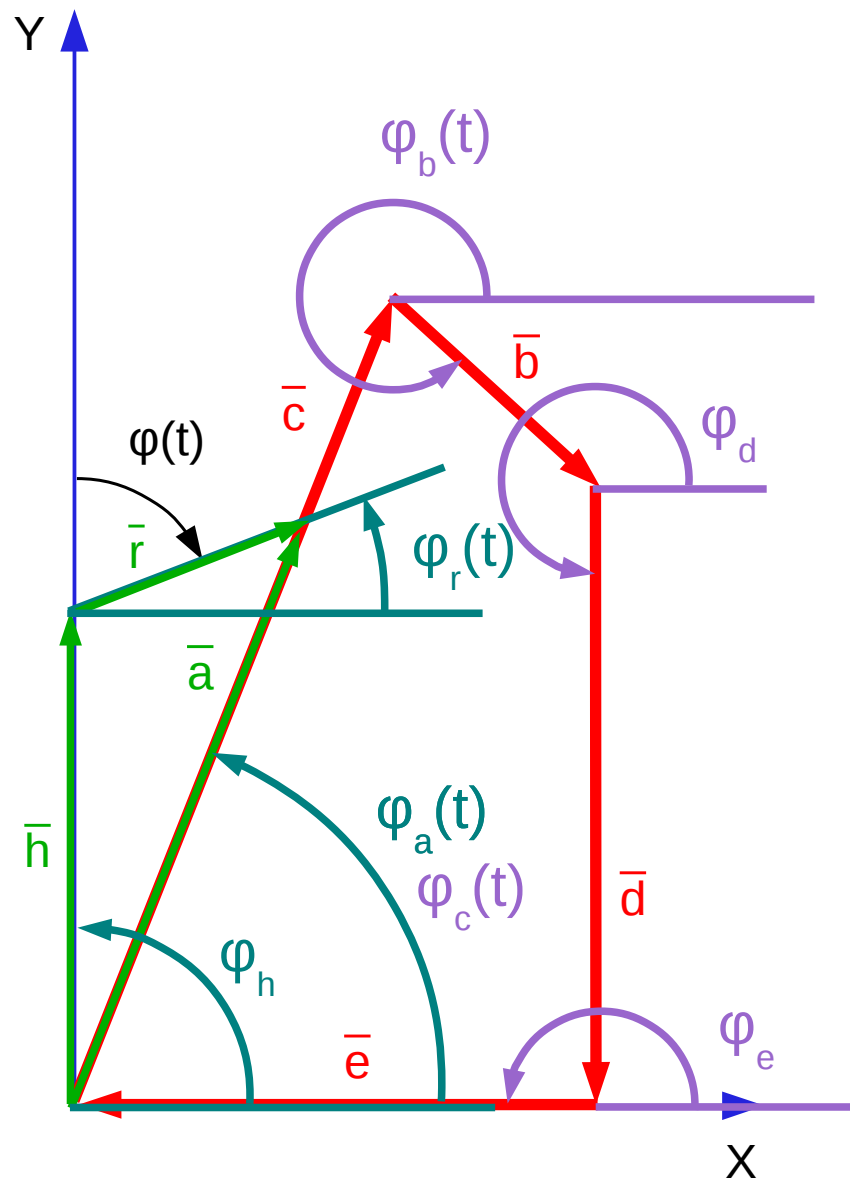


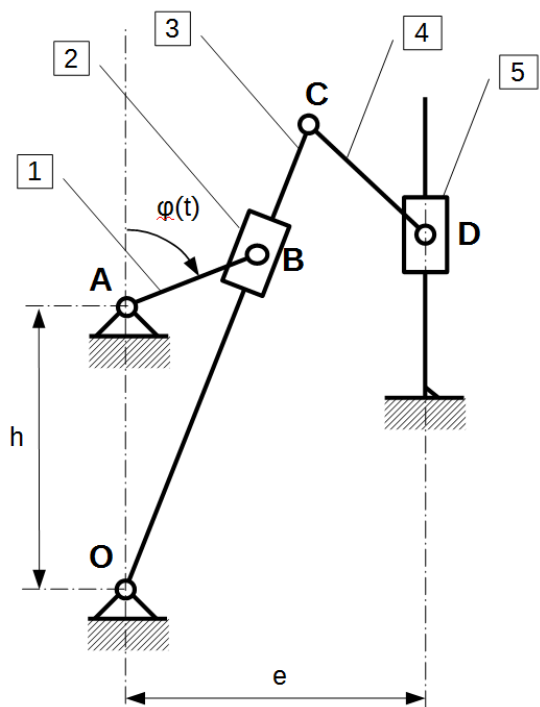


6. Teraz określamy długości wszystkich wektorów:

$$|\bar{r}| = |AB| = r = \text{const.}$$

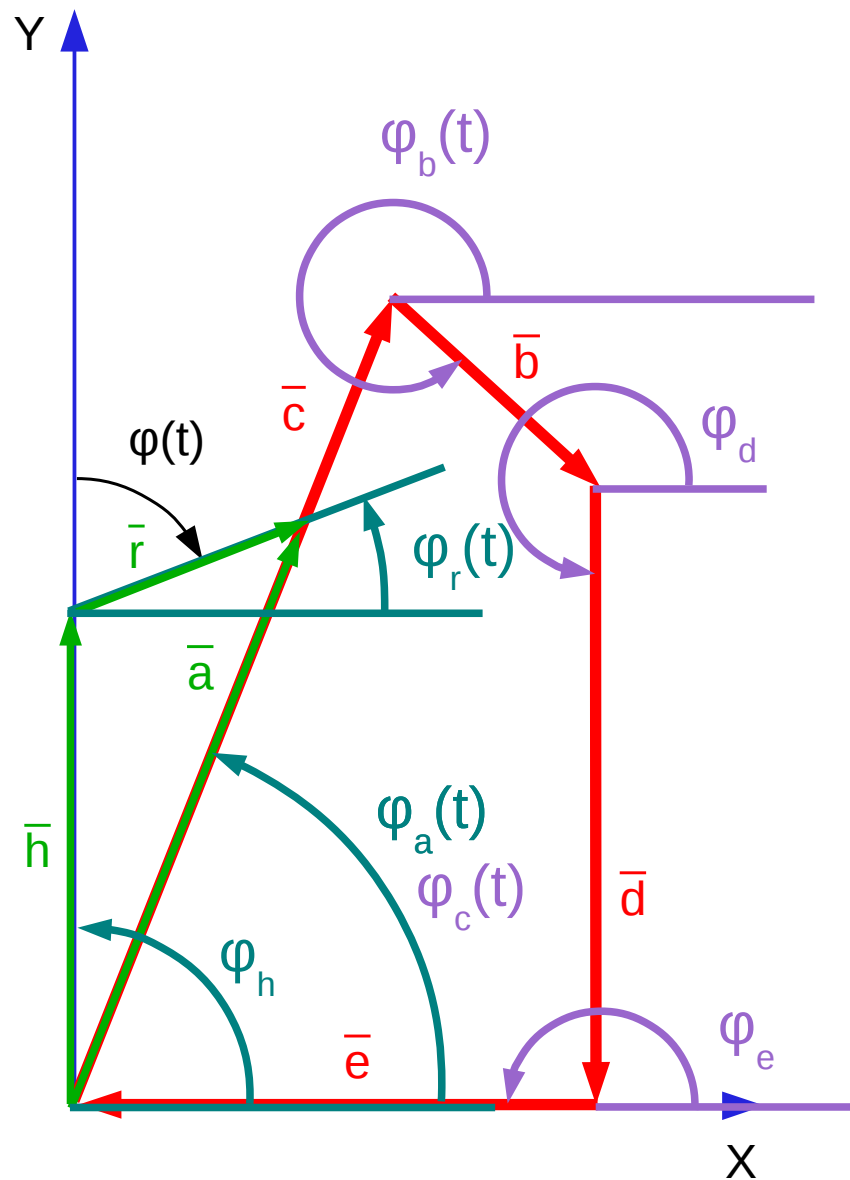
- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

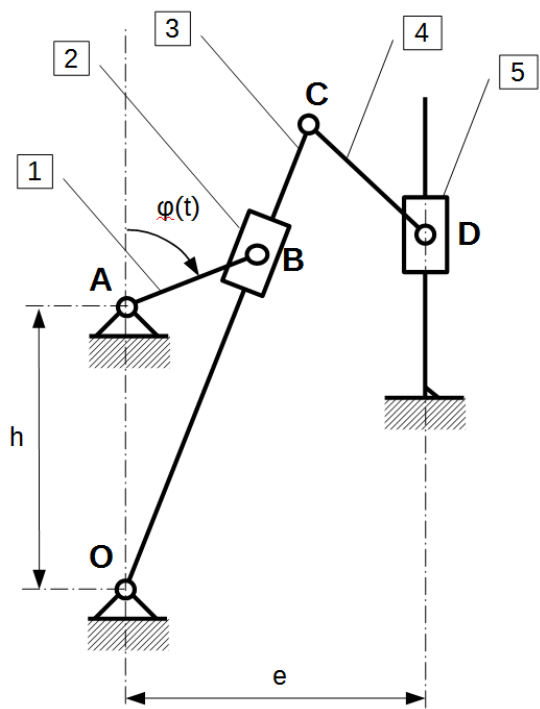




- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

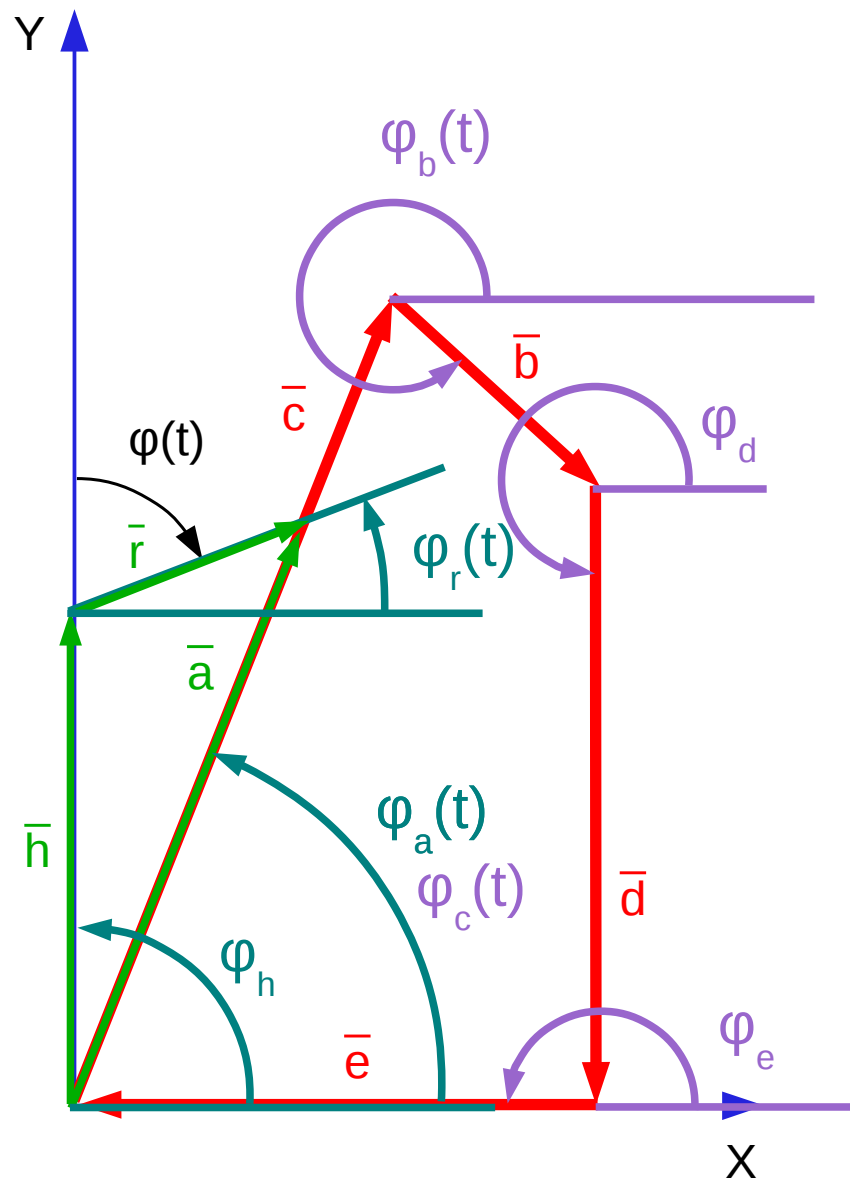
$|\bar{r}| = |AB| = r = \text{const.}$
 $|\bar{h}| = |OA| = h = \text{const.}$

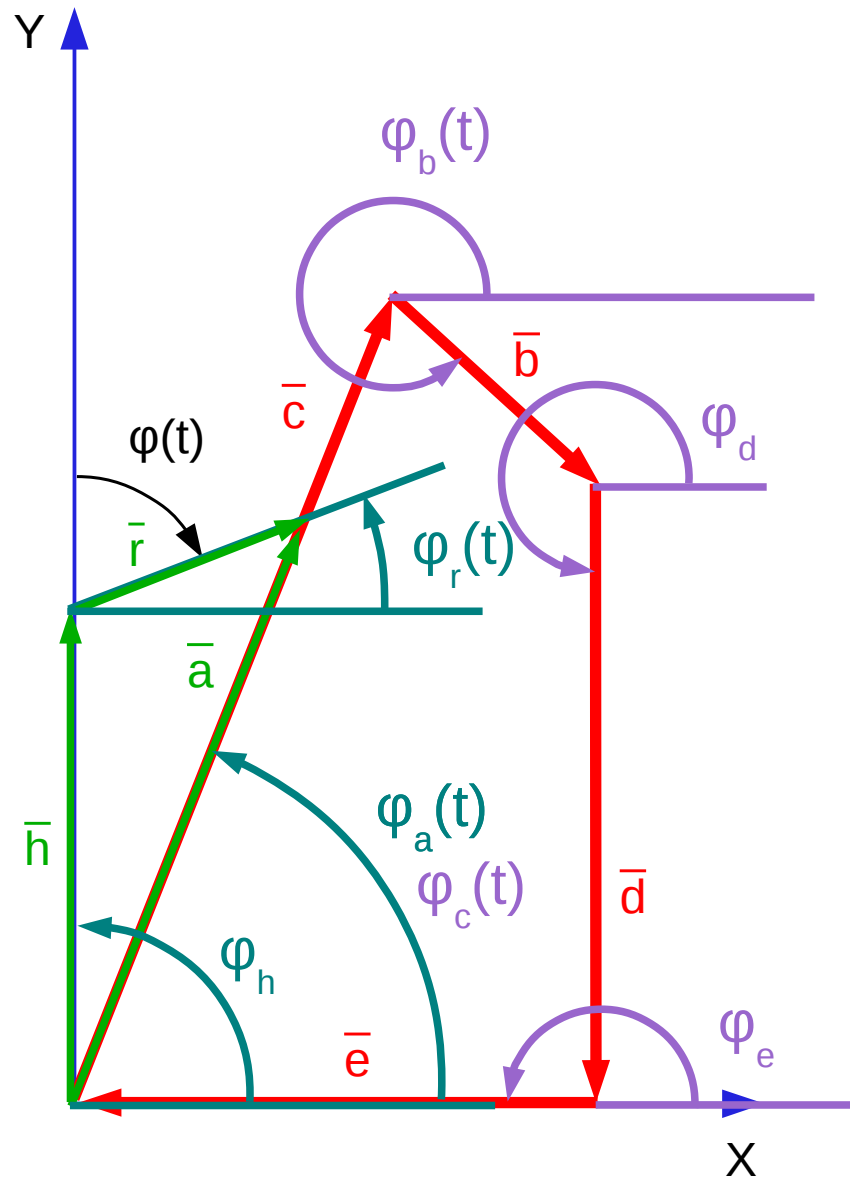
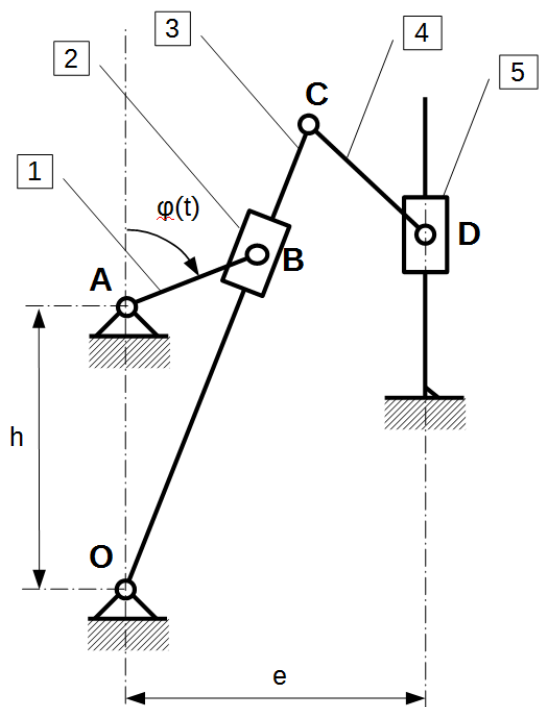




- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

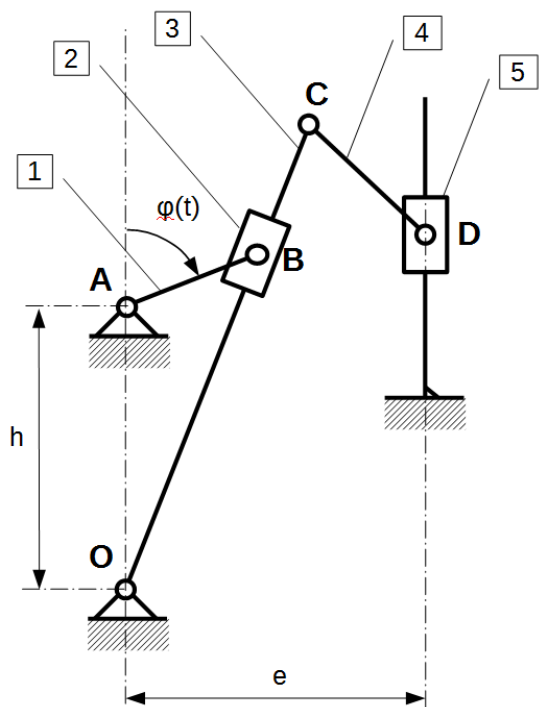
- $|\bar{r}| = |AB| = r = \text{const.}$
- $|\bar{h}| = |OA| = h = \text{const.}$
- $|\bar{a}| = |OB| = a(t)$





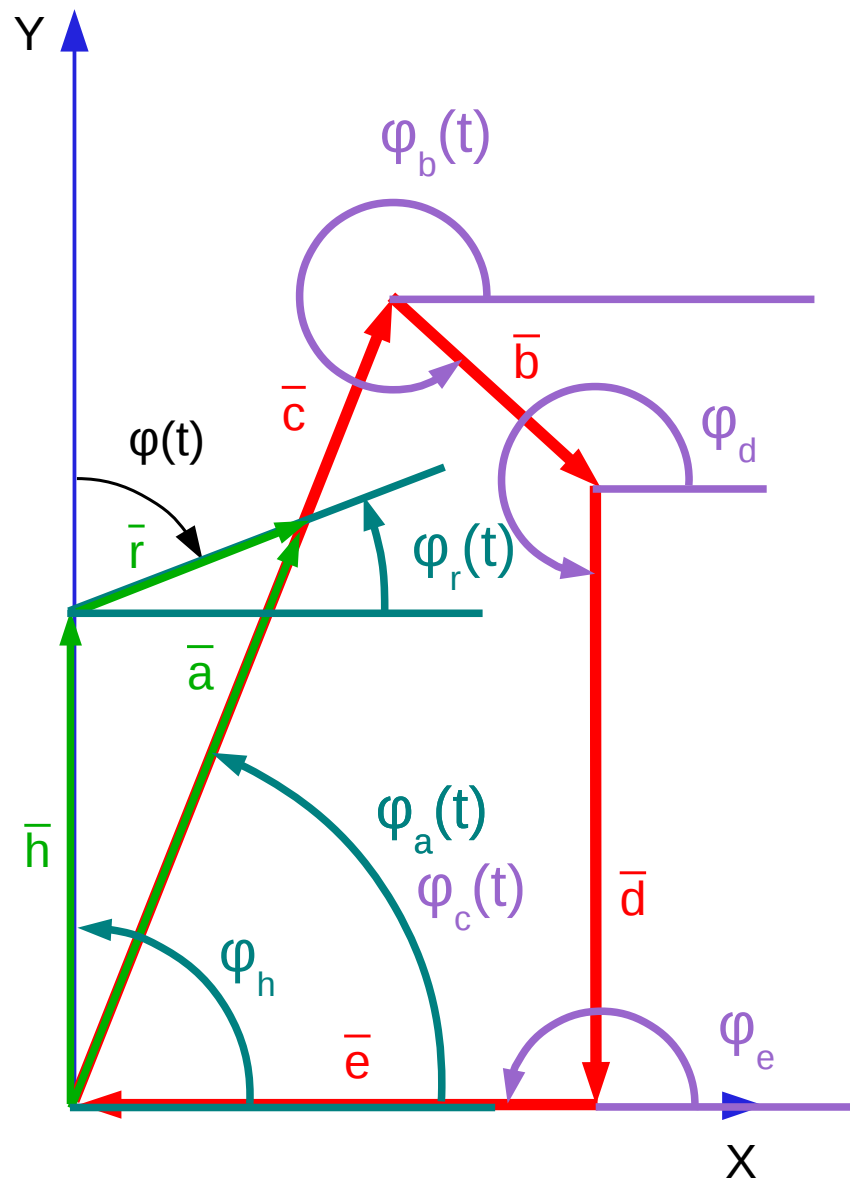
- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

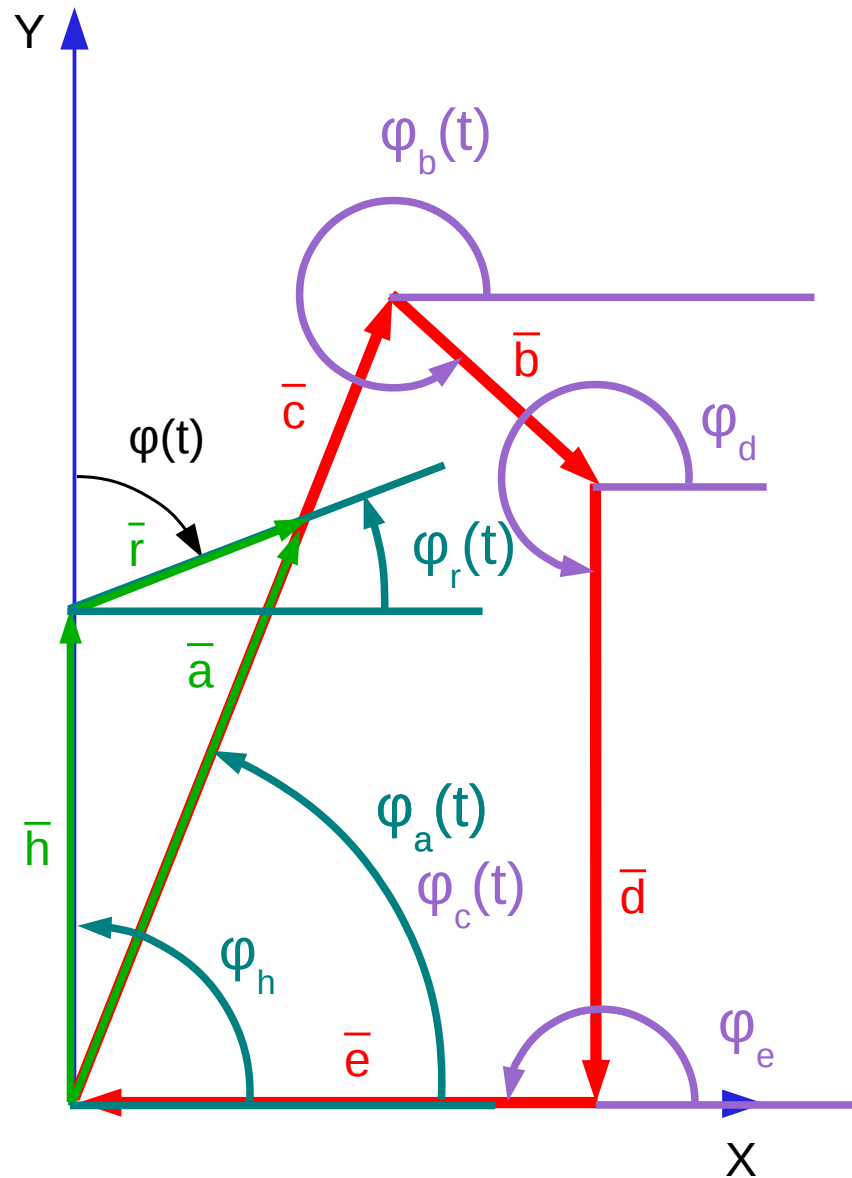
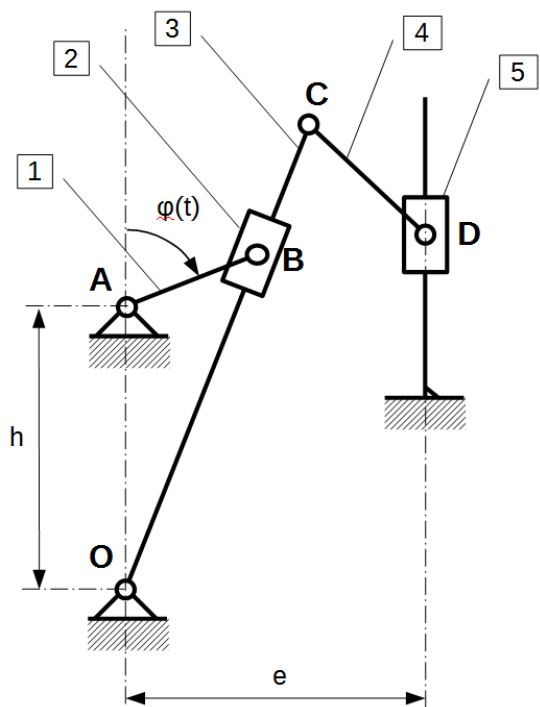
- $|\vec{r}| = |AB| = r = \text{const.}$
- $|\vec{h}| = |OA| = h = \text{const.}$
- $|\vec{a}| = |OB| = a(t)$
- $|\vec{c}| = |OC| = c = \text{const.}$



- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

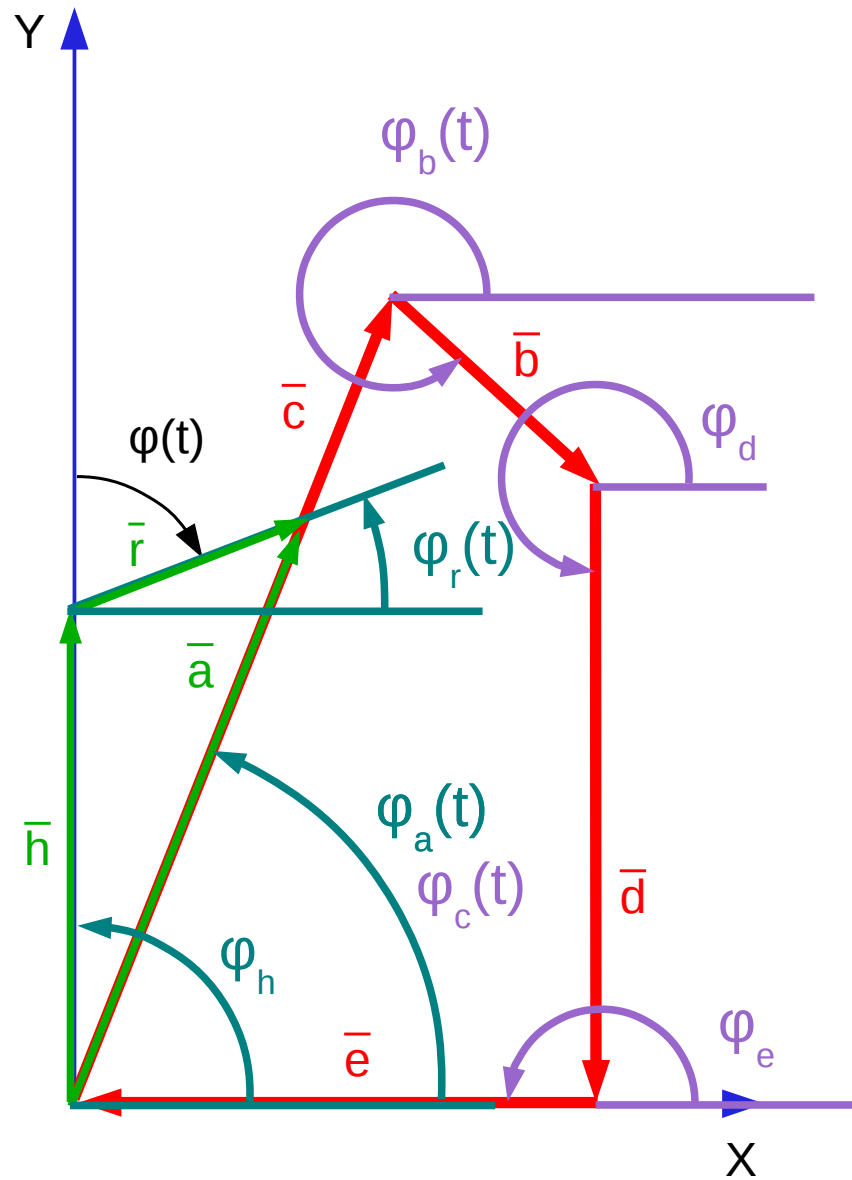
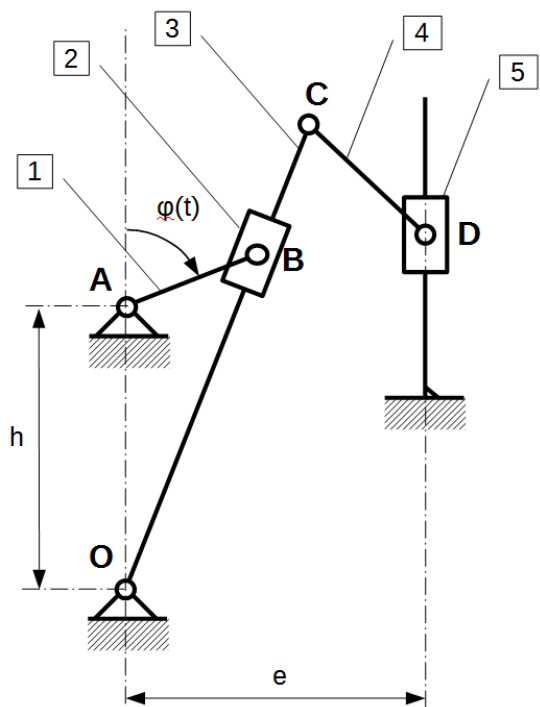
- $|\bar{r}| = |AB| = r = \text{const.}$
- $|\bar{h}| = |OA| = h = \text{const.}$
- $|\bar{a}| = |OB| = a(t)$
- $|\bar{c}| = |OC| = c = \text{const.}$
- $|\bar{b}| = |CD| = b = \text{const.}$





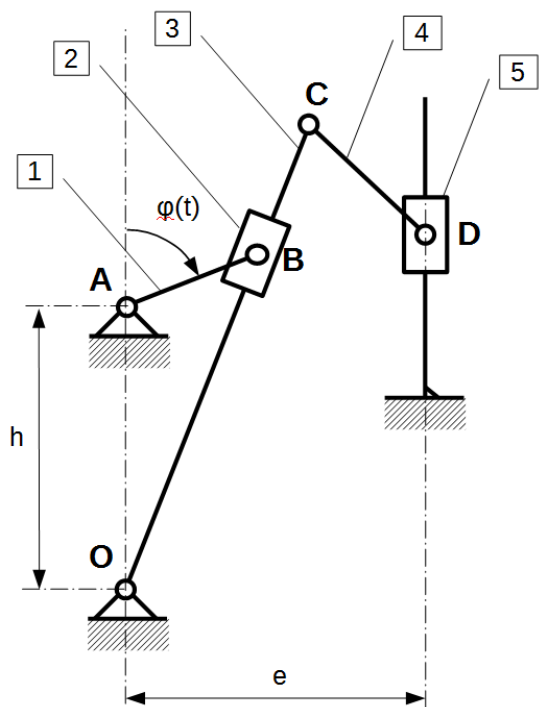
- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

- $|\bar{r}| = |AB| = r = \text{const.}$
- $|\bar{h}| = |OA| = h = \text{const.}$
- $|\bar{a}| = |OB| = a(t)$
- $|\bar{c}| = |OC| = c = \text{const.}$
- $|b| = |CD| = b = \text{const.}$
- $|d| = d(t)$



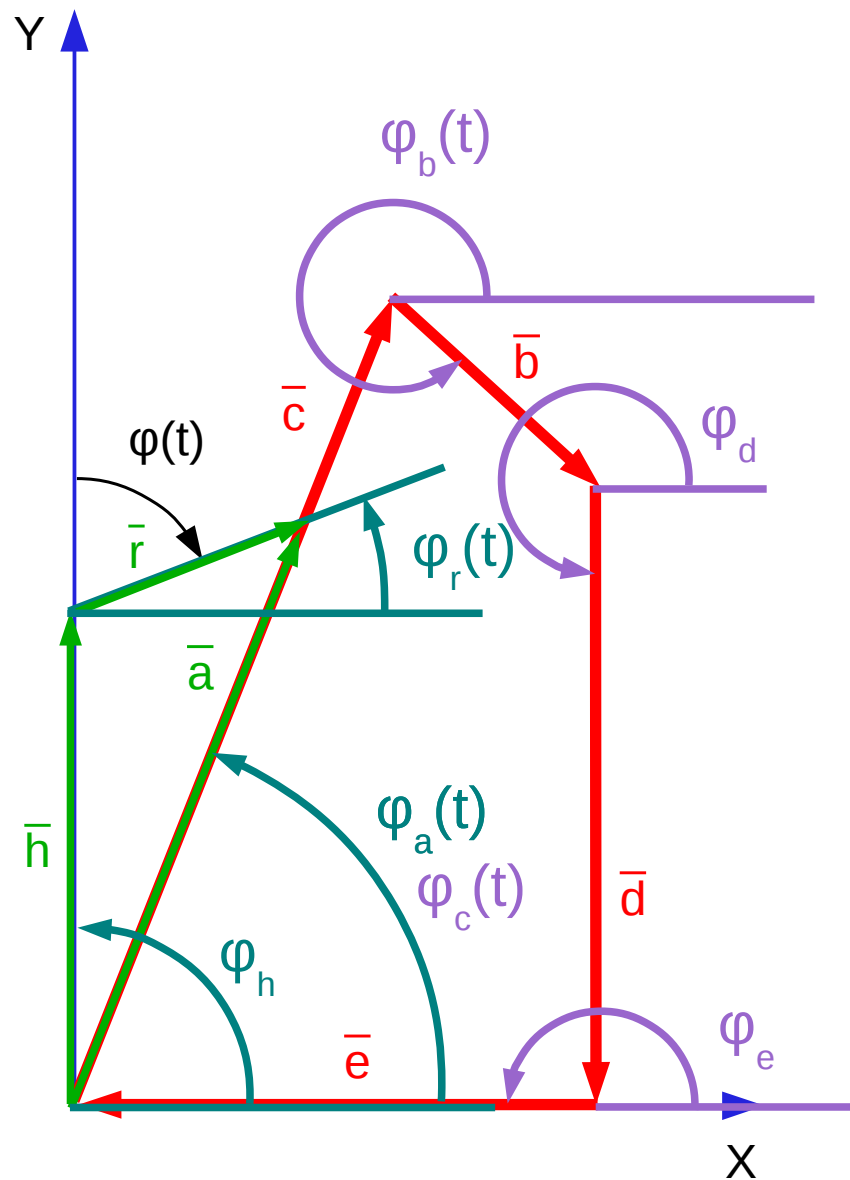
- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

- $|\vec{r}| = |AB| = r = \text{const.}$
- $|\vec{h}| = |OA| = h = \text{const.}$
- $|\vec{a}| = |OB| = a(t)$
- $|\vec{c}| = |OC| = c = \text{const.}$
- $|b| = |CD| = b = \text{const.}$
- $|d| = d(t)$
- $|\vec{e}| = e = \text{const.}$



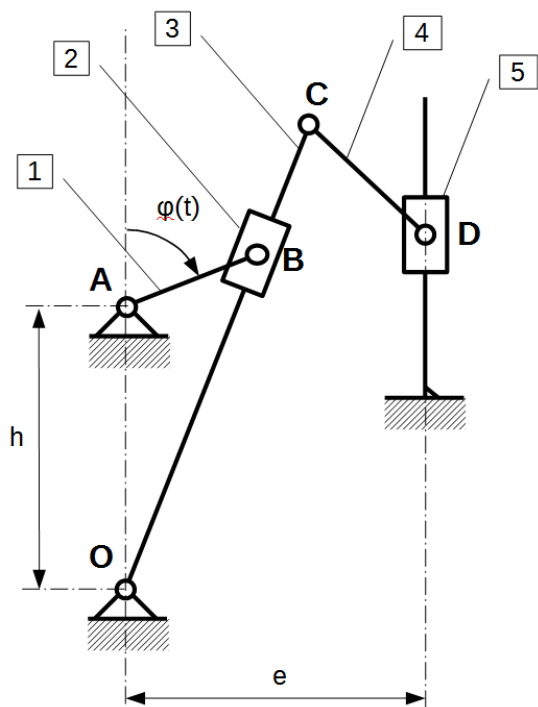
- $\varphi_h = 90^\circ$
- $\varphi_a(t) \neq \text{const.}$
- $\varphi_b(t) \neq \text{const.}$
- $\varphi_c(t) = \varphi_a(t)$
- $\varphi_r(t) = 90^\circ - \varphi(t)$
- $\varphi_d = 270^\circ$
- $\varphi_e = 180^\circ$

- $|\bar{r}| = |AB| = r = \text{const.}$
- $|\bar{h}| = |OA| = h = \text{const.}$
- $|\bar{a}| = |OB| = a(t)$
- $|\bar{c}| = |OC| = c = \text{const.}$
- $|b| = |CD| = b = \text{const.}$
- $|d| = d(t)$
- $|\bar{e}| = e = \text{const.}$



$$\begin{matrix} \bar{c} + \bar{b} + \bar{h} + \bar{r} = \bar{a} \\ \bar{c} + \bar{b} + \bar{d} + \bar{e} = \bar{0} \end{matrix}$$

7. Piszemy równania wektorowe dla wieloboków



$$\varphi_h = 90^\circ$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_d = 270^\circ$$

$$\varphi_e = 180^\circ$$

$$|\bar{h}| = |OA| = h = \text{const.}$$

$$|\bar{r}| = |AB| = r = \text{const.}$$

$$|\bar{a}| = |OB| = a(t)$$

$$|\bar{c}| = |OC| = c = \text{const.}$$

$$|\bar{b}| = |CD| = b = \text{const.}$$

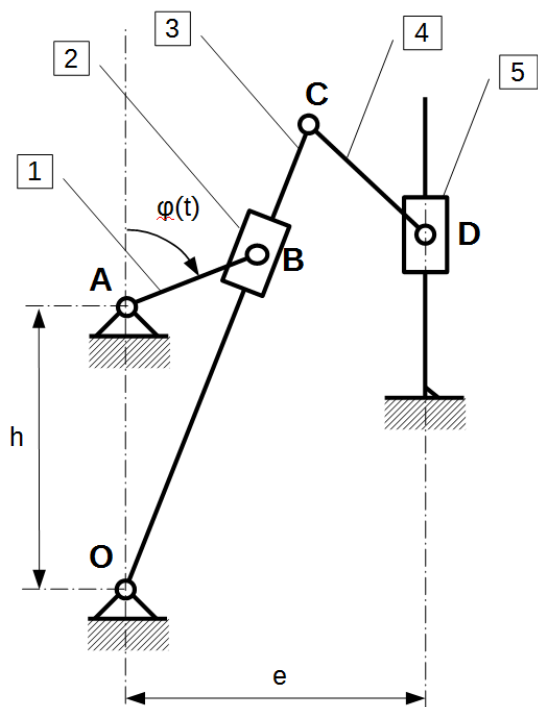
$$|\bar{d}| = d(t)$$

$$|\bar{e}| = e = \text{const.}$$

$$\bar{c} + \bar{b} + \bar{d} + \bar{e} = \bar{0}$$

$$\bar{h} + \bar{r} = \bar{a}$$

8. Rozwiązujemy to równanie wektorowe rzutując wektory na osie układu współrzędnych.



$$\varphi_h = 90^\circ$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_d = 270^\circ$$

$$\varphi_e = 180^\circ$$

$$|\bar{h}| = |OA| = h = \text{const.}$$

$$|\bar{r}| = |AB| = r = \text{const.}$$

$$|\bar{a}| = |OB| = a(t)$$

$$|\bar{c}| = |OC| = c = \text{const.}$$

$$|b| = |CD| = b = \text{const.}$$

$$|d| = d(t)$$

$$|\bar{e}| = e = \text{const.}$$

$$\bar{c} + \bar{b} + \bar{d} + \bar{e} = \bar{0}$$

$$\bar{h} + \bar{r} = \bar{a}$$

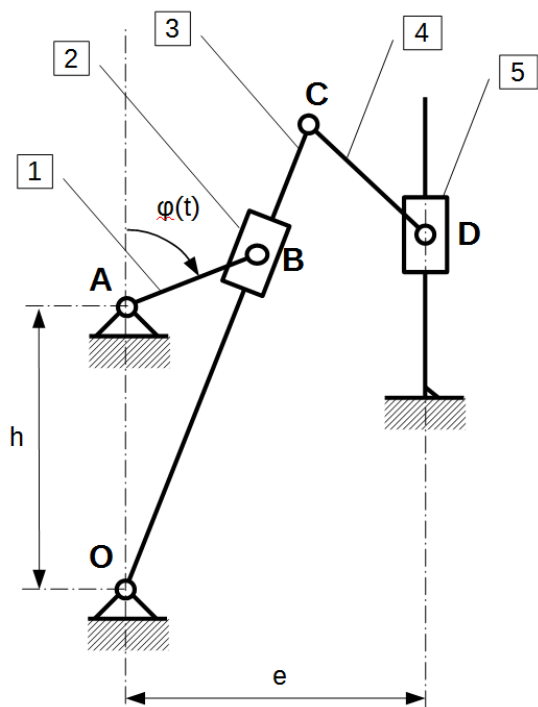
$$x: h \cos \phi_h + r \cos \phi_r(t) = a(t) \cos \phi_a(t)$$

$$y: h \sin \phi_h + r \sin \phi_r(t) = a(t) \sin \phi_a(t)$$

$$x: c \cos \phi_c(t) + b \cos \phi_b(t) + d(t) \cos \phi_d + e \cos \phi_e = 0$$

$$y: c \sin \phi_c(t) + b \sin \phi_b(t) + d(t) \sin \phi_d + e \sin \phi_e = 0$$

9. Dzięki temu, że wszystkie kąty oznaczyliśmy tą samą metodą, rzuty są odpowiednio z kosinusami i sinusami.



$$\varphi_h = 90^\circ$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_d = 270^\circ$$

$$\varphi_e = 180^\circ$$

$$|\bar{h}| = |OA| = h = \text{const.}$$

$$|\bar{r}| = |AB| = r = \text{const.}$$

$$|\bar{a}| = |OB| = a(t)$$

$$|\bar{c}| = |OC| = c = \text{const.}$$

$$|b| = |CD| = b = \text{const.}$$

$$|d| = d(t)$$

$$|\bar{e}| = e = \text{const.}$$

$$\bar{c} + \bar{b} + \bar{d} + \bar{e} = \bar{0}$$

10. Upraszczamy równania rzutów z użyciem informacji o kątach.

$$x: h \cos \varphi_h + r \cos \varphi_r(t) = a(t) \cos \varphi_a(t)$$

$$y: h \sin \varphi_h + r \sin \varphi_r(t) = a(t) \sin \varphi_a(t)$$

$$x: c \cos \varphi_c(t) + b \cos \varphi_b(t) + d(t) \cos \varphi_d + e \cos \varphi_e = 0$$

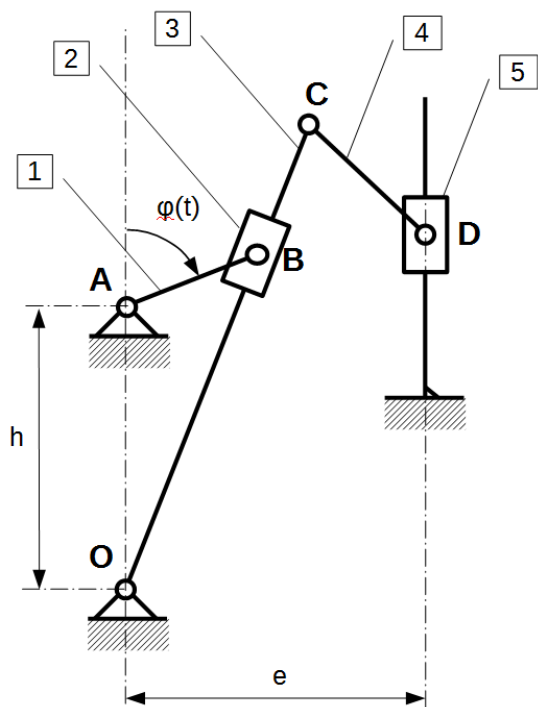
$$y: c \sin \varphi_c(t) + b \sin \varphi_b(t) + d(t) \sin \varphi_d + e \sin \varphi_e = 0$$

$$x: h \cos 90^\circ + r \cos (90^\circ - \varphi(t)) = a(t) \cos \varphi_a(t)$$

$$y: h \sin 90^\circ + r \sin (90^\circ - \varphi(t)) = a(t) \sin \varphi_a(t)$$

$$x: c \cos \varphi_a(t) + b \cos \varphi_b(t) + d(t) \cos 270^\circ + e \cos 180^\circ = 0$$

$$y: c \sin \varphi_a(t) + b \sin \varphi_b(t) + d(t) \sin 270^\circ + e \sin 180^\circ = 0$$



$$\varphi_h = 90^\circ$$

$$\varphi_r(t) = 90^\circ - \varphi(t)$$

$$\varphi_a(t) \neq \text{const.}$$

$$\varphi_c(t) = \varphi_a(t)$$

$$\varphi_b(t) \neq \text{const.}$$

$$\varphi_d = 270^\circ$$

$$\varphi_e = 180^\circ$$

$$|\bar{h}| = |OA| = h = \text{const.}$$

$$|\bar{r}| = |AB| = r = \text{const.}$$

$$|\bar{a}| = |OB| = a(t)$$

$$|\bar{c}| = |OC| = c = \text{const.}$$

$$|b| = |CD| = b = \text{const.}$$

$$|d| = d(t)$$

$$|\bar{e}| = e = \text{const.}$$

$$\bar{c} + \bar{b} + \bar{d} + \bar{e} = \bar{0}$$

$$\bar{h} + \bar{r} = \bar{a}$$

$$x: h \cos \phi_h + r \cos \phi_r(t) = a(t) \cos \phi_a(t)$$

$$y: h \sin \phi_h + r \sin \phi_r(t) = a(t) \sin \phi_a(t)$$

$$x: c \cos \phi_c(t) + b \cos \phi_b(t) + d(t) \cos \phi_d + e \cos \phi_e = 0$$

$$y: c \sin \phi_c(t) + b \sin \phi_b(t) + d(t) \sin \phi_d + e \sin \phi_e = 0$$

$$x: h \cos 90^\circ + r \cos(90^\circ - \phi(t)) = a(t) \cos \phi_a(t)$$

$$y: h \sin 90^\circ + r \sin(90^\circ - \phi(t)) = a(t) \sin \phi_a(t)$$

$$x: c \cos \phi_a(t) + b \cos \phi_b(t) + d(t) \cos 270^\circ + e \cos 180^\circ = 0$$

$$y: c \sin \phi_a(t) + b \sin \phi_b(t) + d(t) \sin 270^\circ + e \sin 180^\circ = 0$$

$$x: r \sin \phi(t) = a(t) \cos \phi_a(t)$$

$$y: h + r \cos \phi(t) = a(t) \sin \phi_a(t)$$

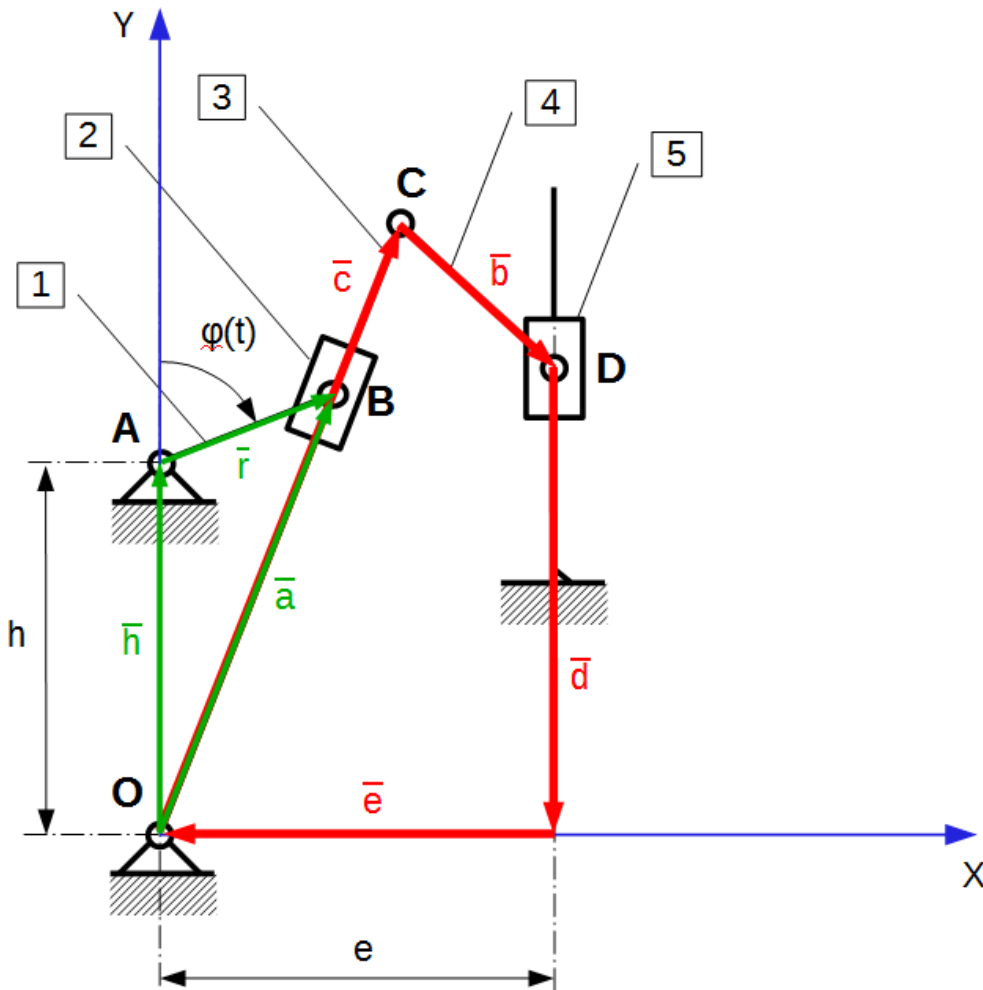
$$x: c \cos \phi_a(t) + b \cos \phi_b(t) - e = 0$$

$$y: c \sin \phi_a(t) + b \sin \phi_b(t) - d(t) = 0$$

11. Po wszystkich uproszczeniach mamy układ 4 równań z 4 niewiadomymi. Rozwiązujemy go, uważając na funkcje typu *arcus*, które nie zawsze zwracają wartości w interesującym nas zakresie kątów.

$$\begin{aligned} \textcircled{1} \quad x: & \quad r \sin \phi(t) = a(t) \cos \phi_a(t) \\ \textcircled{2} \quad y: & \quad h + r \cos \phi(t) = a(t) \sin \phi_a(t) \\ \textcircled{3} \quad x: & \quad c \cos \phi_a(t) + b \cos \phi_b(t) - e = 0 \\ \textcircled{4} \quad y: & \quad c \sin \phi_a(t) + b \sin \phi_b(t) - d(t) = 0 \end{aligned}$$

12. Po rozwiązaniu tego układu skupiamy się na funkcji $d(t)$ która opisuje odległość suwaka od podpory.



- ① $x: r \sin \phi(t) = a(t) \cos \phi_a(t)$
- ② $y: h + r \cos \phi(t) = a(t) \sin \phi_a(t)$
- ③ $x: c \cos \phi_a(t) + b \cos \phi_b(t) - e = 0$
- ④ $y: c \sin \phi_a(t) + b \sin \phi_b(t) - d(t) = 0$

$$a(t) = \sqrt{r^2 \sin^2(t) + (h + r \cos \phi(t))^2}$$

$$\phi_a(t) = \operatorname{atan} \left(\frac{h + r \cos \phi(t)}{r \sin \phi(t)} \right)$$

$$\phi_b(t) = \arccos \frac{e - c \cos \phi_a(t)}{b}$$

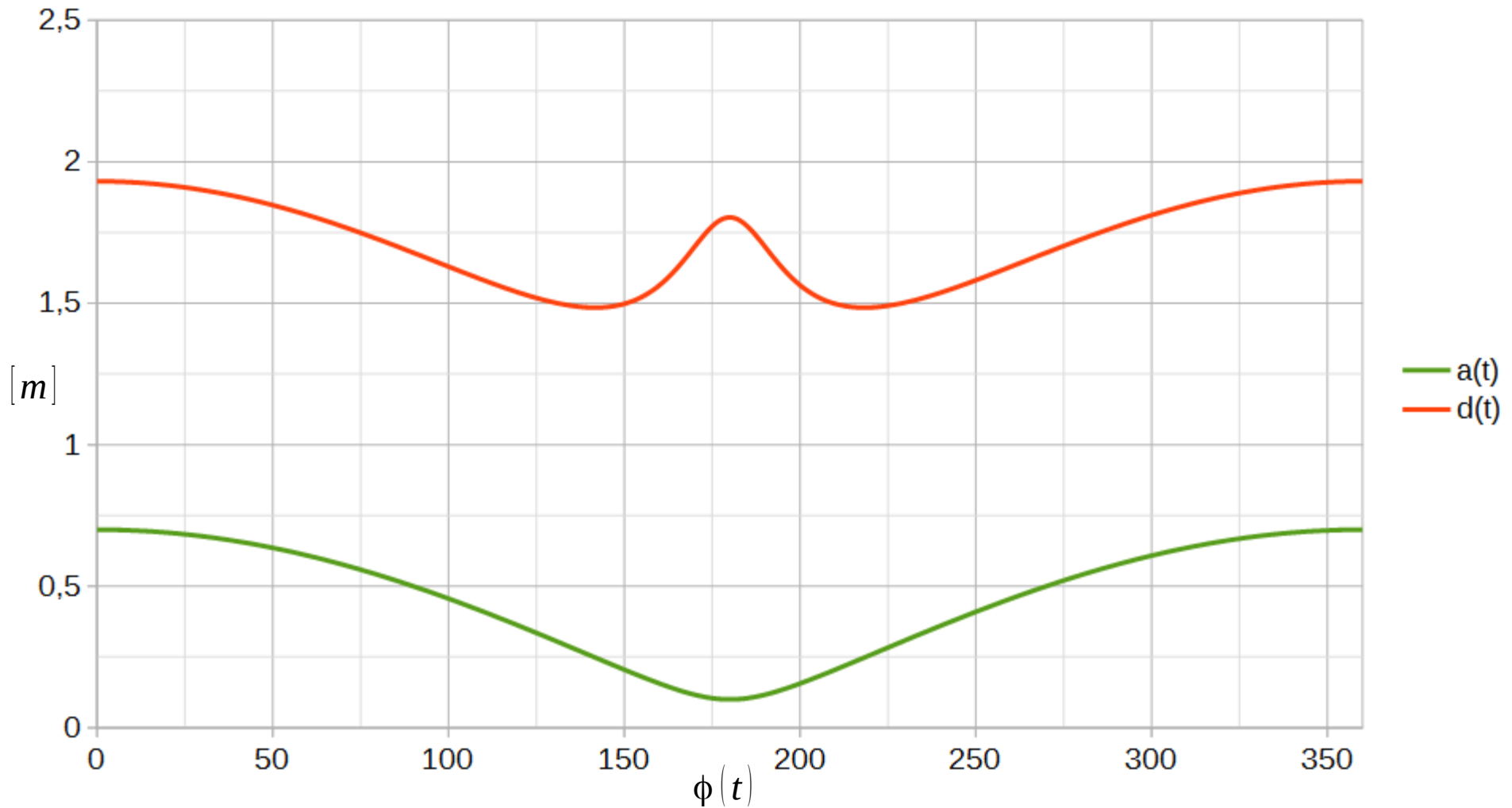
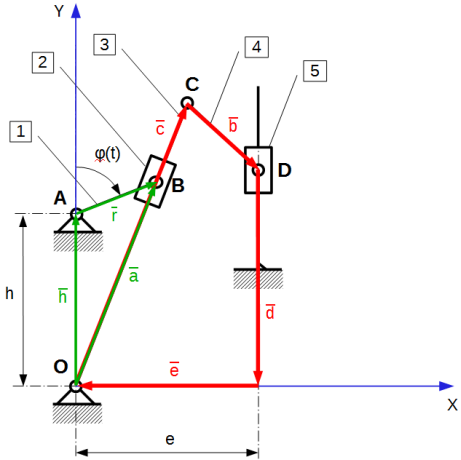
$$d(t) = c \sin \phi_a(t) + b \sqrt{1 - \left(\frac{e - c \cos \phi_a(t)}{b} \right)^2}$$

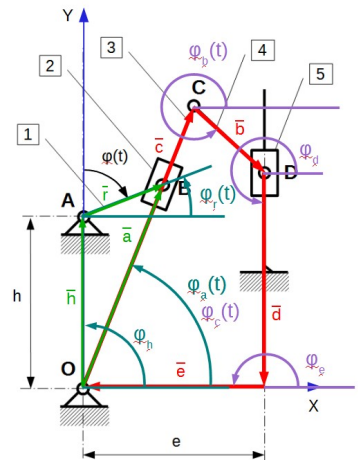
13. Dla przykładowych danych (tabela obok) przedstawiono wykresy dla funkcji:

r=	0,3	[m]
h=	0,4	[m]
c=	1	[m]
e=	0,4	[m]
b=	1	[m]
ω=	1	[rad/s]

$$a(t) = \sqrt{r^2 \sin^2(t) + (h + r \cos \phi(t))^2}$$

$$d(t) = c \sin \phi_a(t) + b \sqrt{1 - \left(\frac{e - c \cos \phi_a(t)}{b} \right)^2}$$

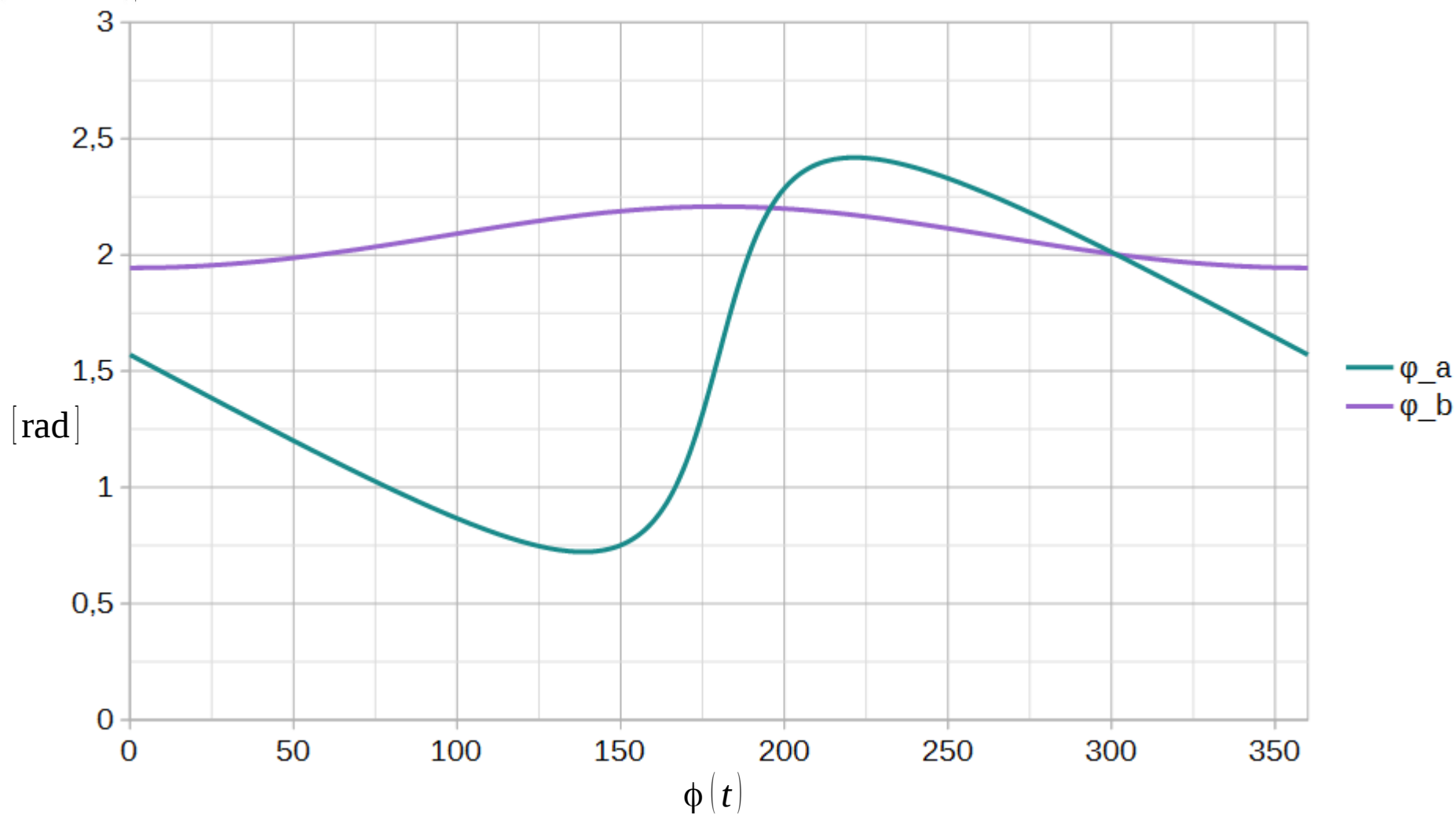




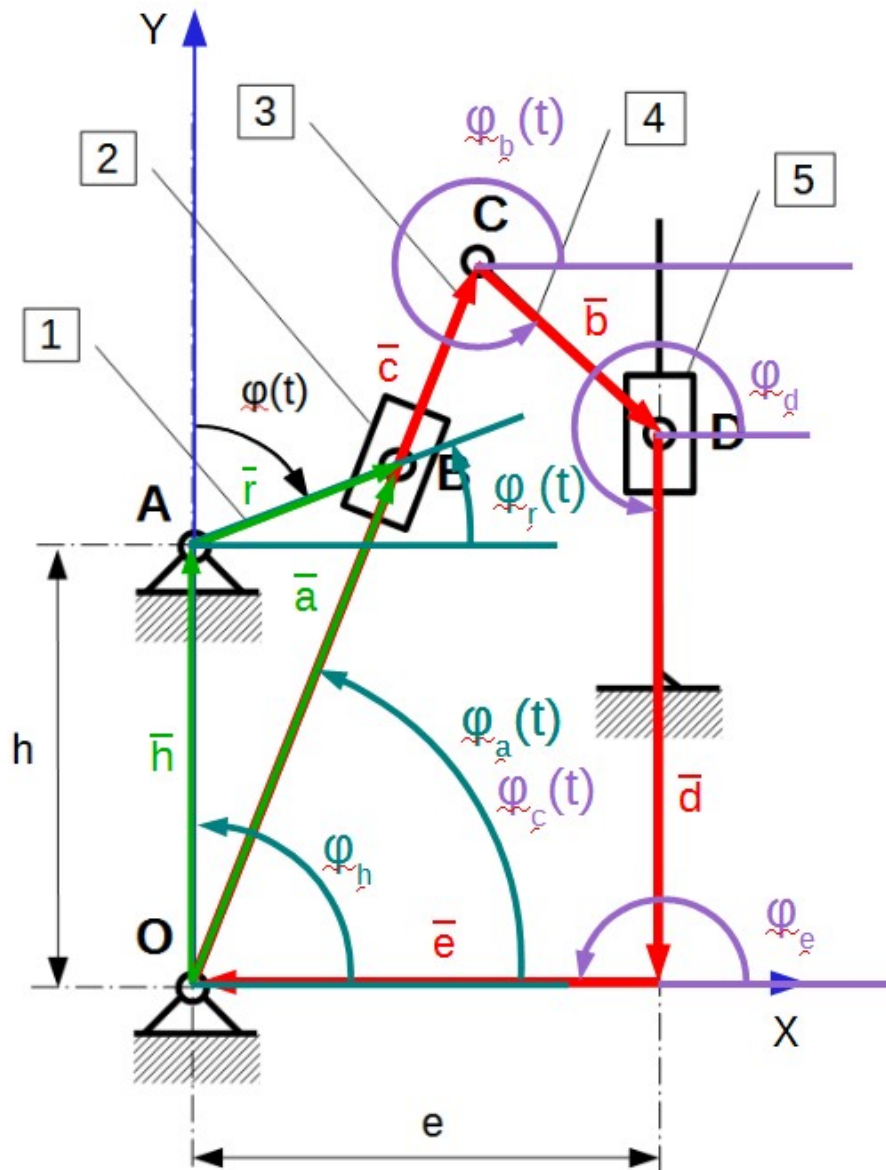
$$\phi_a(t) = \text{atan} \left(\frac{h + r \cos \phi(t)}{r \sin \phi(t)} \right)$$

$$\phi_b(t) = \arccos \frac{e - c \cos \phi_a(t)}{b}$$

r=	0,3	[m]
h=	0,4	[m]
c=	1	[m]
e=	0,4	[m]
b=	1	[m]
ω=	1	[rad/s]



14. Poszukiwane: prędkość i przyspieszenie suwaka D otrzymamy z różniczkowań po czasie funkcji $d(t)$. Różniczkowanie to, można wykonać "na papierze", ale ze względu na skomplikowaną formę $d(t)$ lepiej użyć oprogramowania do obliczeń symbolicznych (Mathematica, wxMaxima, Mathcad). Można też wykonać przybliżone różniczkowanie numeryczne metodą ilorazu różnicowego.



- ① $x: r \sin \phi(t) = a(t) \cos \phi_a(t)$
- ② $y: h + r \cos \phi(t) = a(t) \sin \phi_a(t)$
- ③ $x: c \cos \phi_a(t) + b \cos \phi_b(t) - e = 0$
- ④ $y: c \sin \phi_a(t) + b \sin \phi_b(t) - d(t) = 0$

$$a(t) = \sqrt{r^2 \sin^2(t) + (h + r \cos \phi(t))^2}$$

$$\phi_a(t) = \text{atan} \left(\frac{h + r \cos \phi(t)}{r \sin \phi(t)} \right)$$

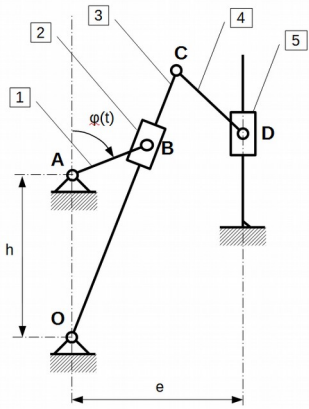
$$\phi_b(t) = \arccos \frac{e - c \cos \phi_a(t)}{b}$$

$$d(t) = c \sin \phi_a(t) + b \sqrt{1 - \left(\frac{e - c \cos \phi_a(t)}{b} \right)^2}$$

$$v_D(t) = \dot{d}(t)$$

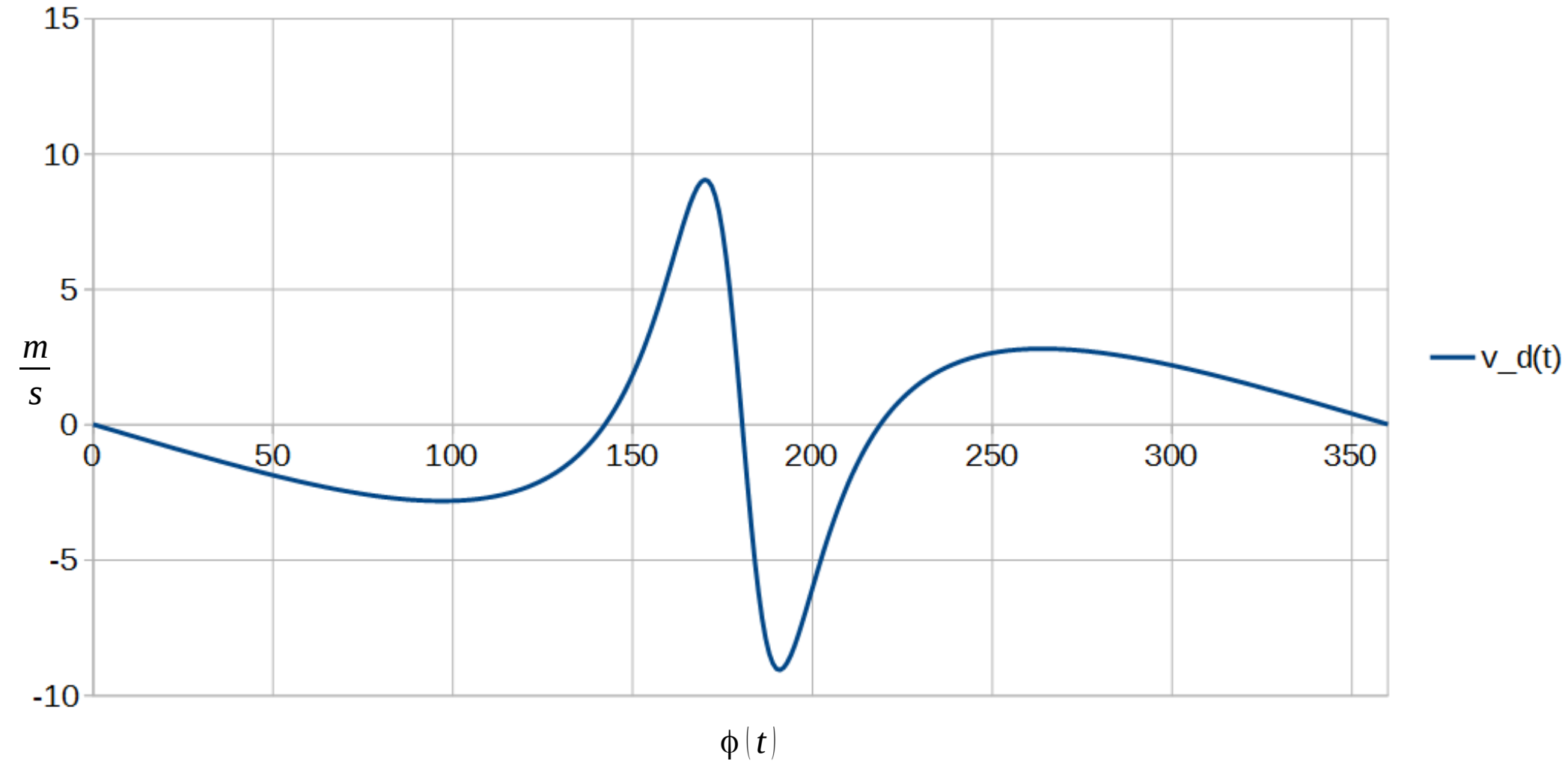
$$a_D(t) = \ddot{d}(t)$$

15. Wykres prędkości

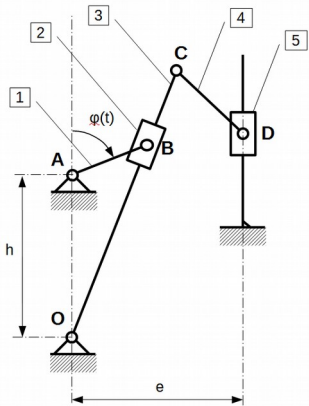


r=	0,3	[m]
h=	0,4	[m]
c=	1	[m]
e=	0,4	[m]
b=	1	[m]
ω=	1	[rad/s]

$$v_D(t) = \dot{d}(t)$$



16. Wykres przyspieszenia



r=	0,3	[m]
h=	0,4	[m]
c=	1	[m]
e=	0,4	[m]
b=	1	[m]
ω=	1	[rad/s]

$$a_D(t) = \ddot{d}(t)$$

