Warsaw University of Technology

The Faculty of Automotive
and Construction Machinery Engineering
Institute of Machine Design Fundamentals
Department of Mechanics
http://www.ipbm.simr.pw.edu.pl/

Theory of Machines and Automatic Control

Winter 2017/2018

Lecturer: Sebastian Korczak, PhD Eng.

Lecture 9

Frequency response.
Classification of basic automatic systems.

Transfer function

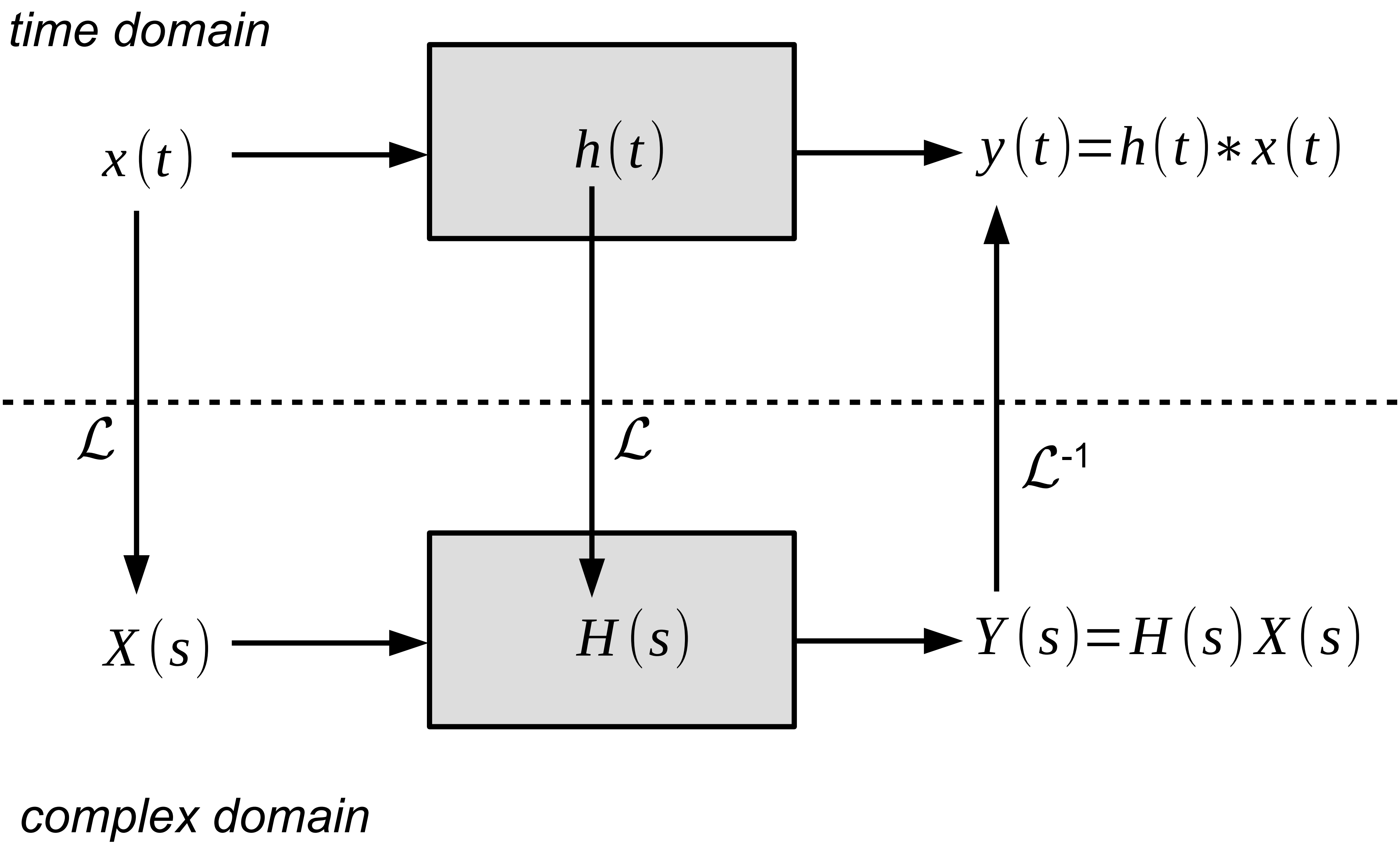
Linear time-invariant SISO system for continous-time input signal *x(t)* and output *y(t)* in a form

Transfer function	$H(s)=\frac{Y(s)}{X(s)}$

$Y(s)$ - Laplace'a transform of an output

$X(s)$ - Laplace'a transform of an input

Input and output



Transmitancja operatorowa i widmowa

Transfer function
(Laplace domain)

$$H(s)$$

Full system description
(for every possible input)

$s = j\omega$ ↓

Frequency response
(Fourier domain)

$$H(j\omega)$$

Description of a system in steady state with harmonic input

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A\sin(\omega t + \varphi)$

harmonic excitation

steady state!!!

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

$$H(s)\xrightarrow{s=j\omega}$$

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

$$H(s)\xrightarrow{s=j\omega} H(j\omega)=P(\omega)+jQ(\omega)$$

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

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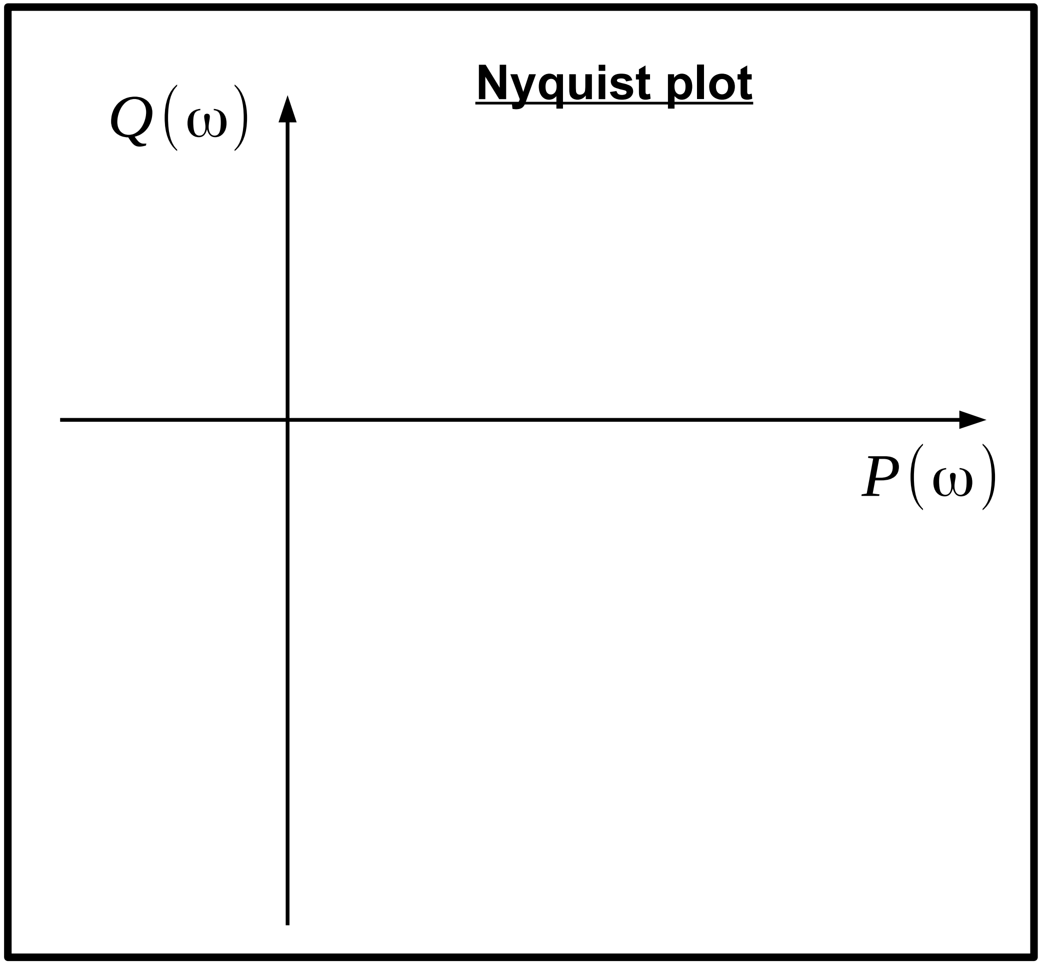

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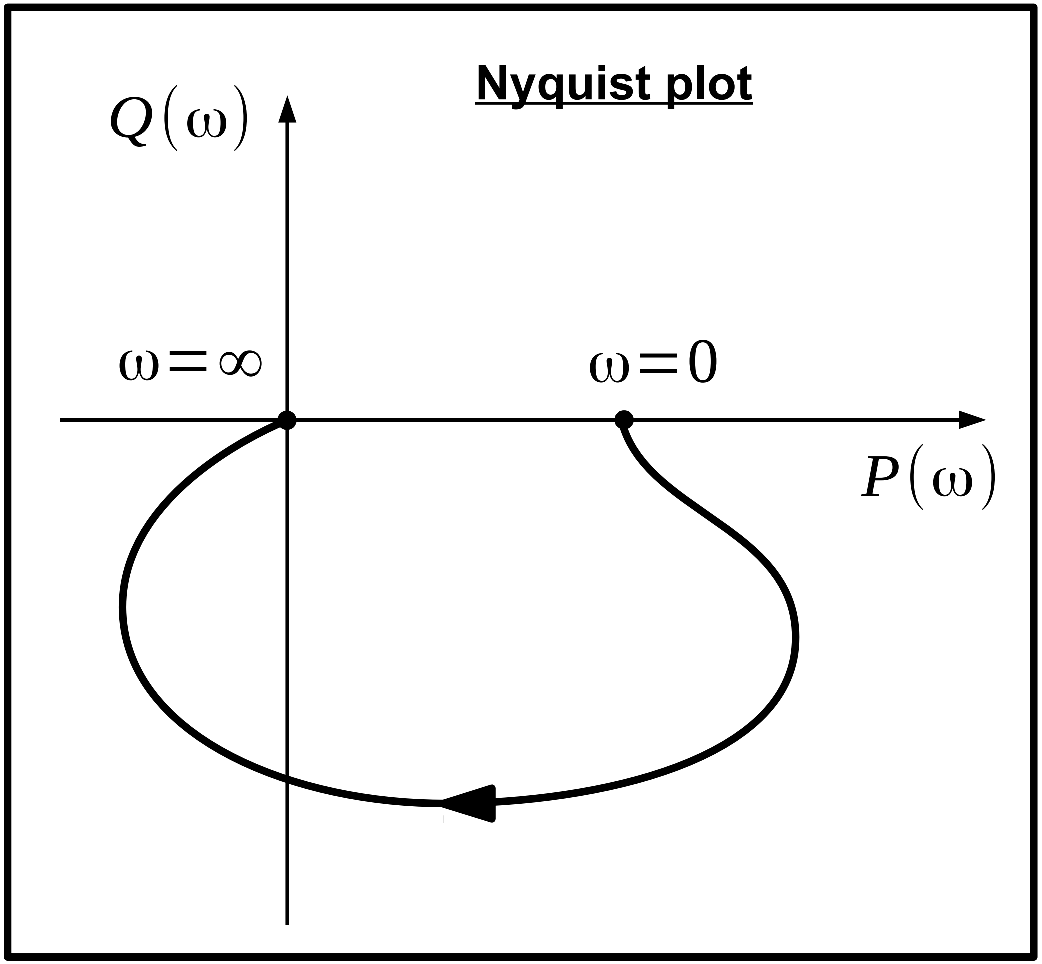

Transfer function – frequency response

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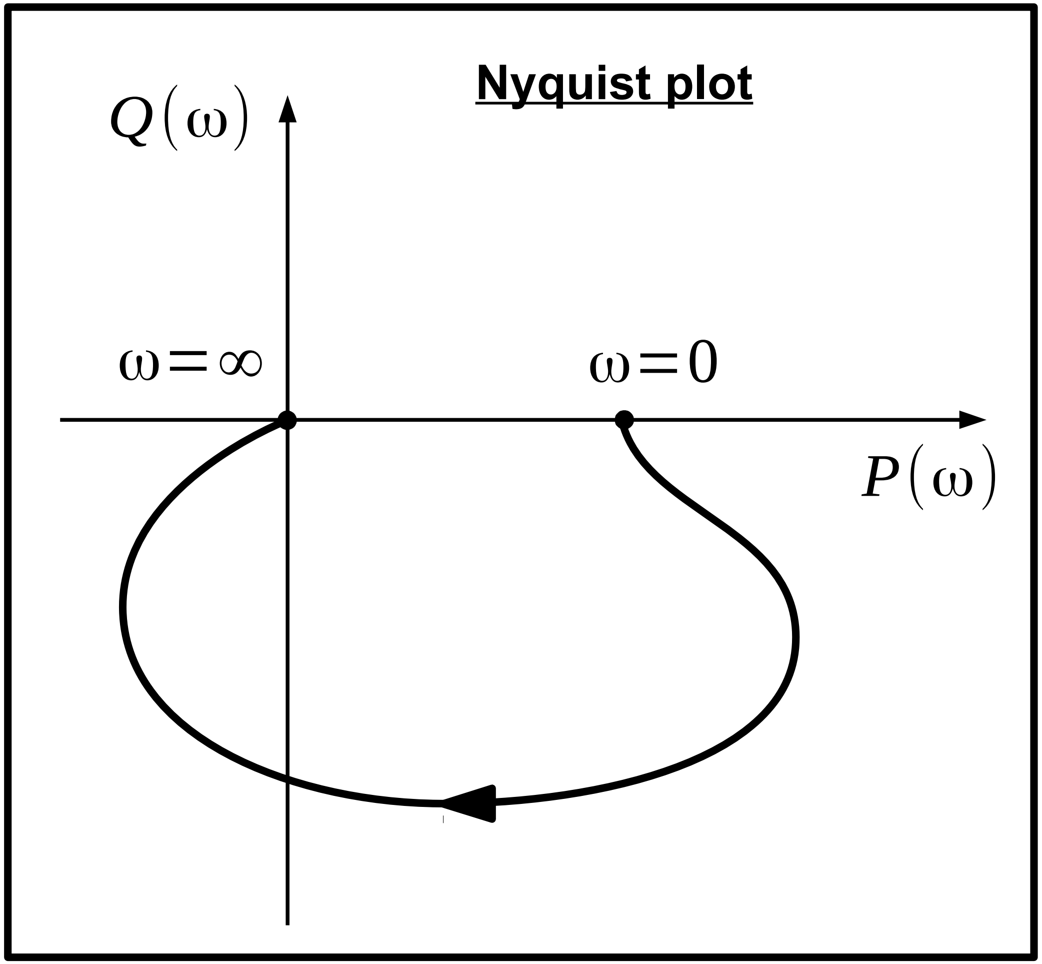
$$A(\omega)=|H(j\omega)|=\sqrt{P^2(\omega)+Q^2(\omega)}$$



Transfer function – frequency response

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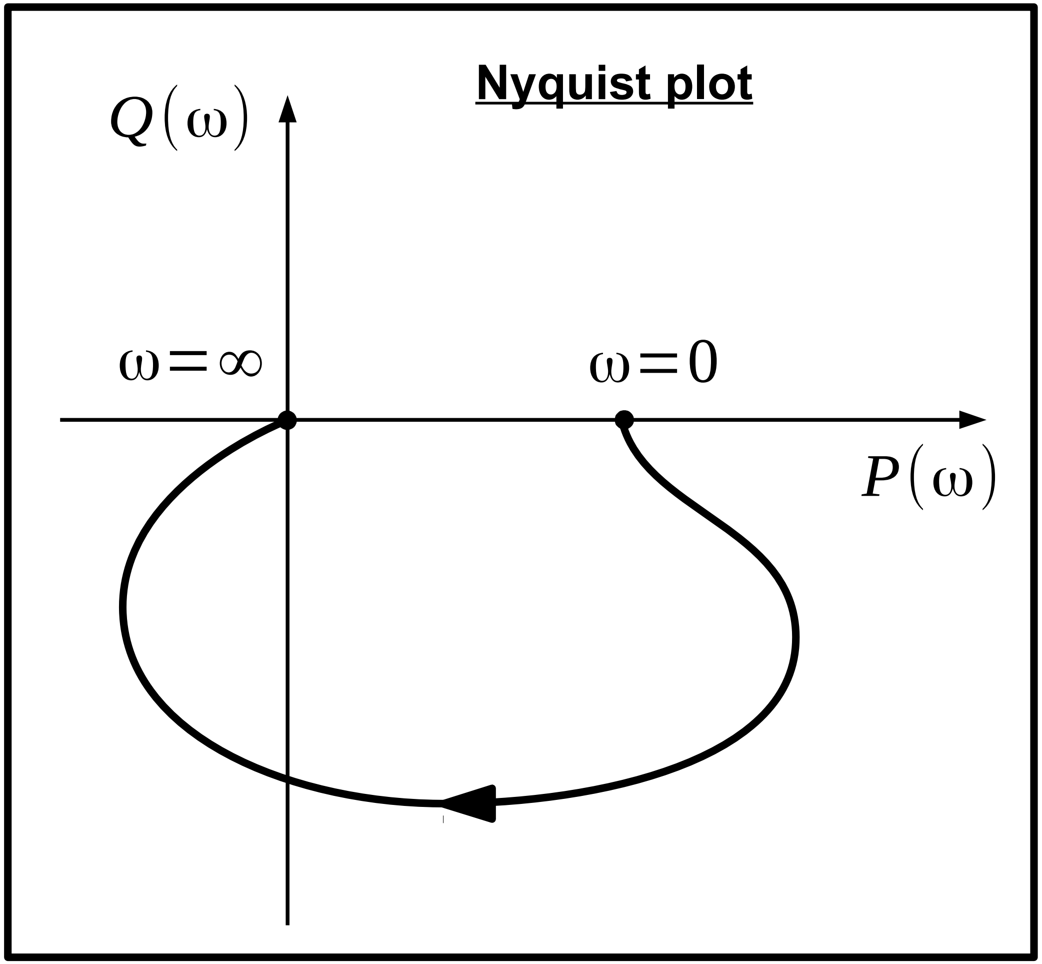
$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

$$A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$

$$\varphi(\omega) = \mathrm{Arg}\, H(j\omega) = \arctan\frac{Q}{P}$$



Transfer function – frequency response

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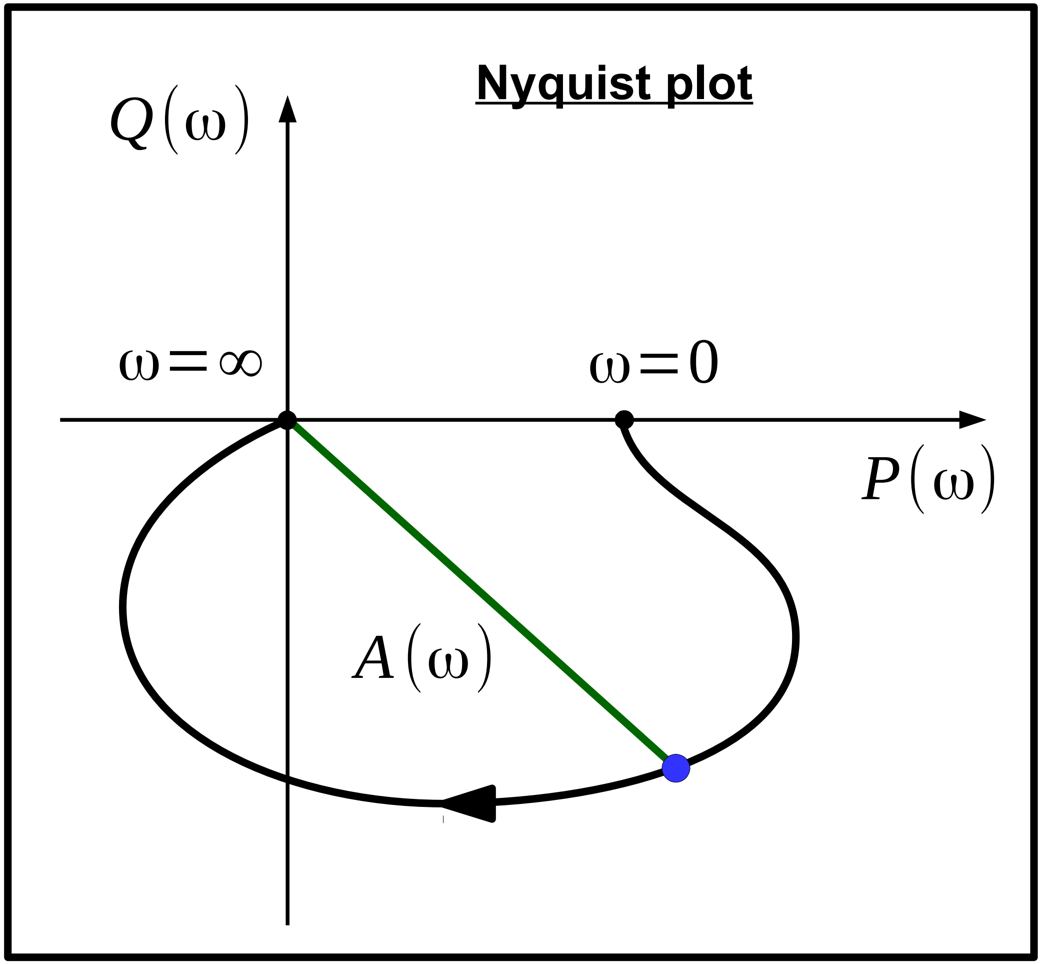
$$A(\omega)=|H(j\omega)|=\sqrt{P^2(\omega)+Q^2(\omega)}$$

GAIN

$$\varphi(\omega)=\mathrm{Arg}\,H(j\omega)=\arctan\frac{Q}{P}$$



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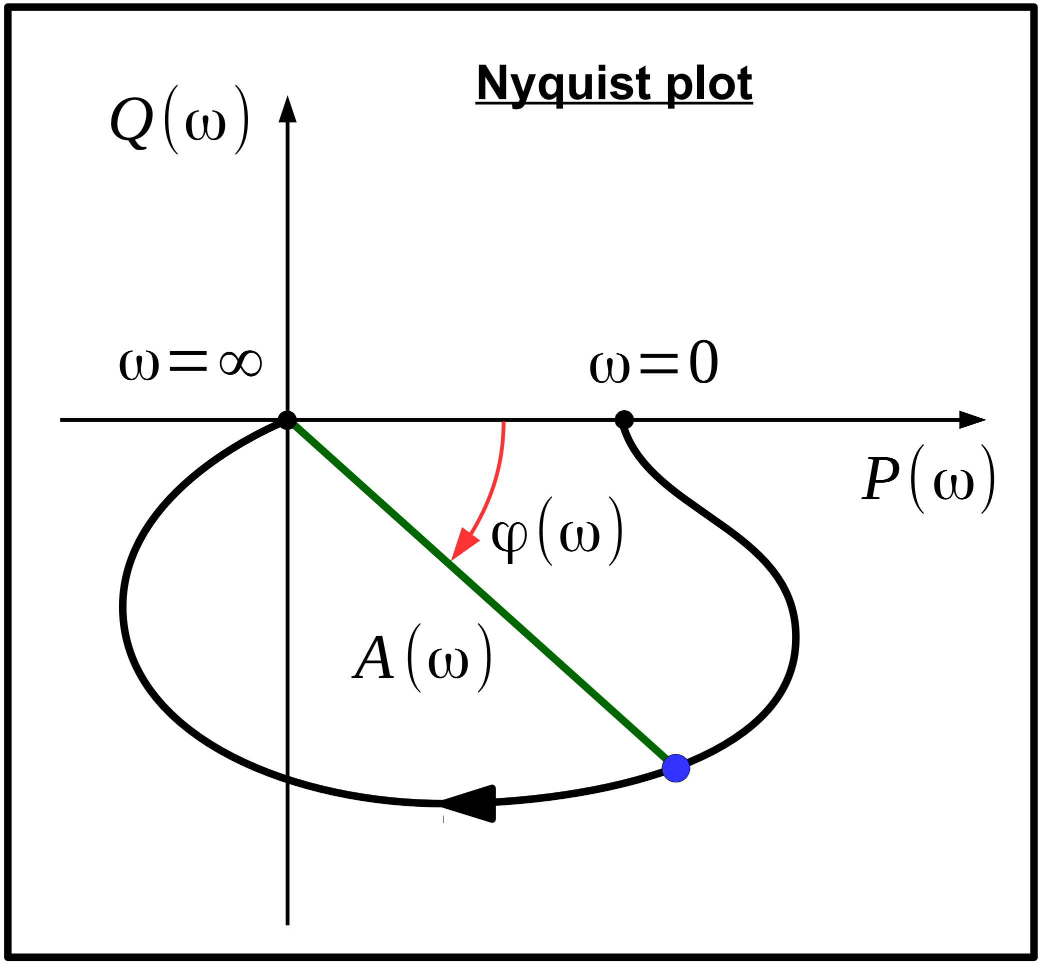
DELAY



Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

Bode Plot

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

Bode Plot

gain (magnitude) plot

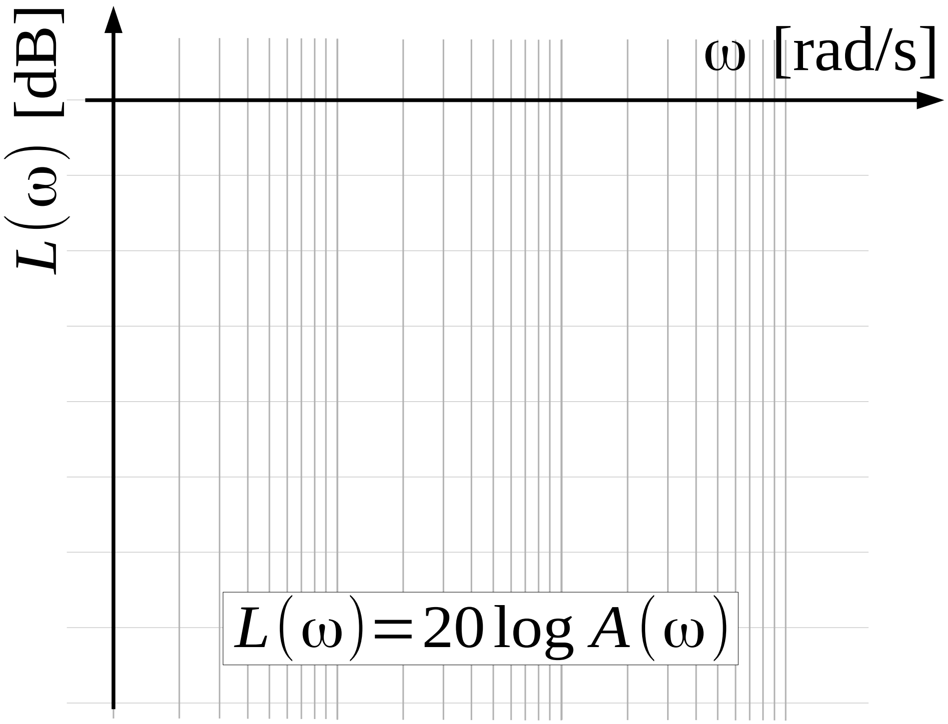
phase (phase shift) plot

Transfer function – frequency response

input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

Bode Plot

gain (magnitude) plot



$$L(\omega)=20\log A(\omega)$$

phase (phase shift) plot

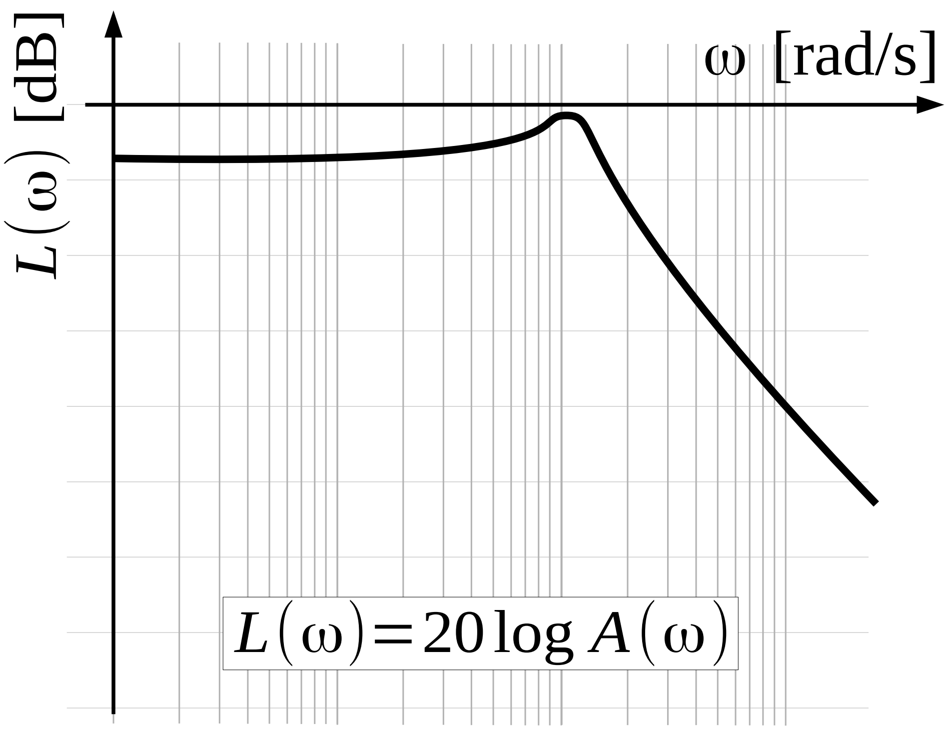
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A\sin(\omega t + \varphi)$

Bode Plot

gain (magnitude) plot

phase (phase shift) plot



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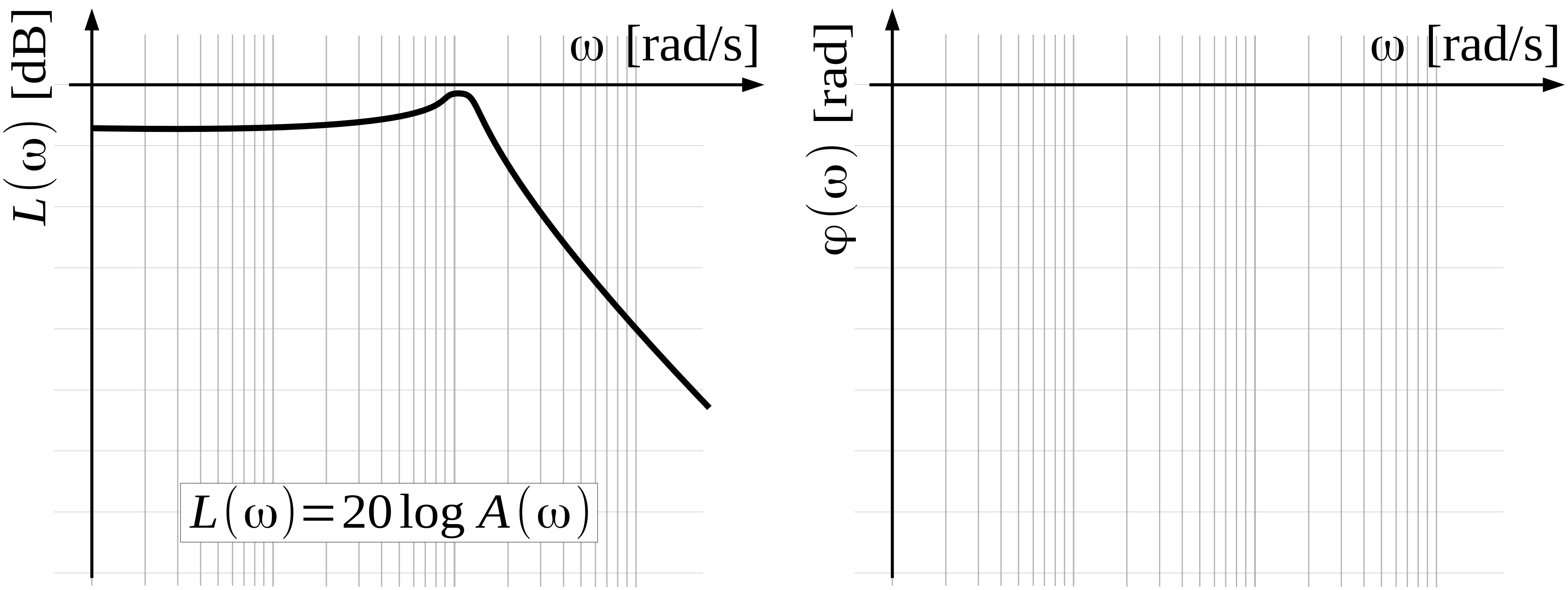
input: $x(t)=\sin(\omega t)$ transfer function: $H(s)$ output: $y(t)=A\sin(\omega t+\varphi)$

Bode Plot

gain (magnitude) plot

phase (phase shift) plot



Transfer function – frequency response

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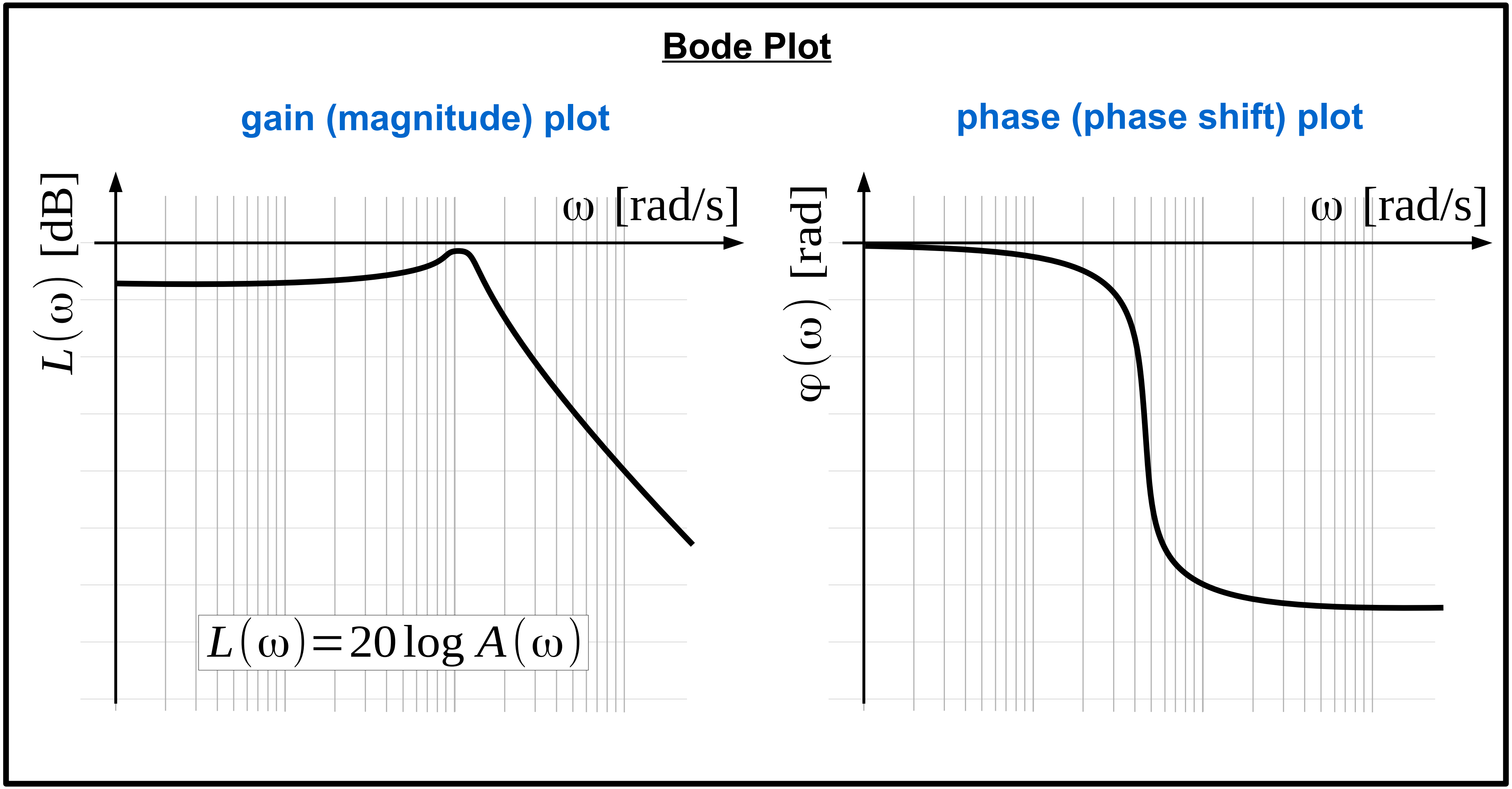

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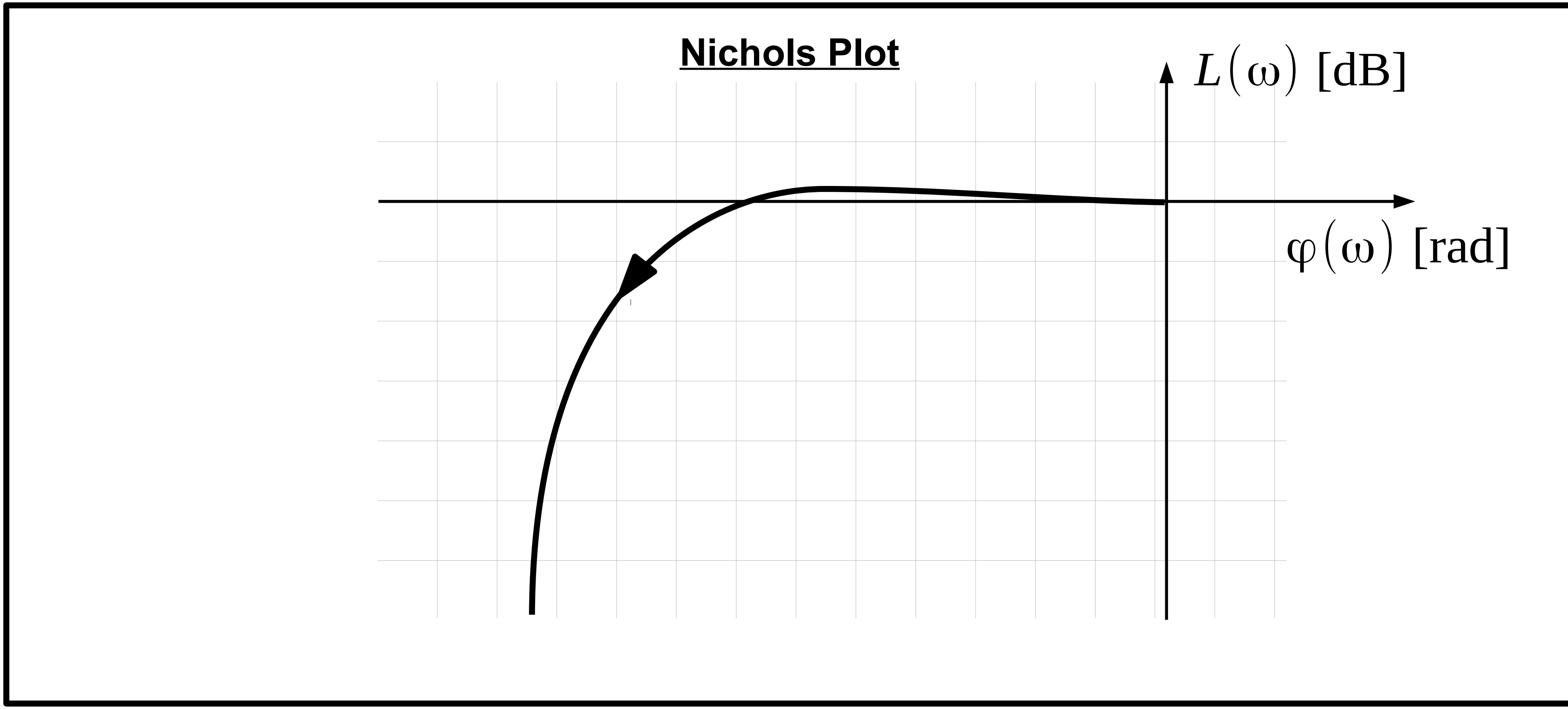


Transfer function – frequency response

A (gain)	20logA [dB]
1000	60
100	40
10	20
1	0
0.1	-20
0.01	-40
0.001	-60

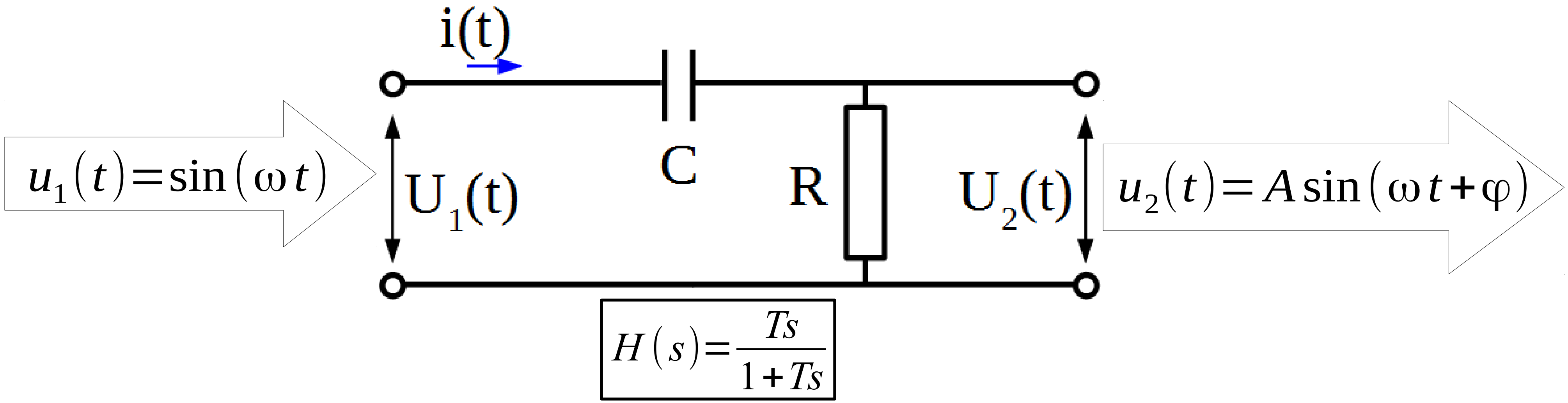
Transfer function – frequency response

Useful for:

- audio components (amplifiers, microphones, loudspeakers, cables)
- wireless components (antennas, amplifiers)
- vibrating systems (suspensions, drivetrains)
- control systems (regulators, objects)

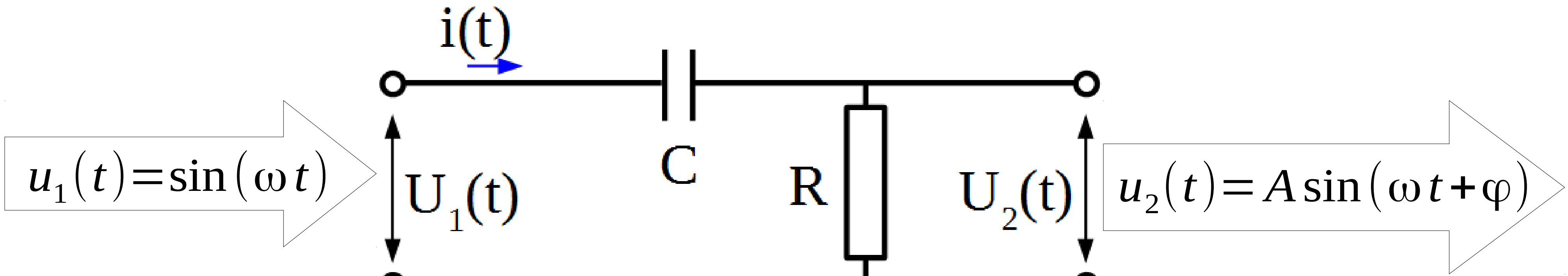
Transfer function – frequency response

RC circuit example



Transfer function – frequency response

RC circuit example



$$H(s)=\frac{Ts}{1+Ts}$$

$s=j\omega$

$$H(j\omega)=\frac{Tj\omega}{1+Tj\omega}=\frac{Tj\omega}{1+Tj\omega}\cdot\frac{1-Tj\omega}{1-Tj\omega}=\frac{Tj\omega-T^2j^2\omega^2}{1^2-T^2j^2\omega^2}=\frac{Tj\omega+T^2\omega^2}{1^2+T^2\omega^2}=\frac{T^2\omega^2}{1^2+T^2\omega^2}+j\frac{T\omega}{1^2+T^2\omega^2}$$

$$P(\omega)=\frac{T^2\omega^2}{1+T^2\omega^2}\qquad Q(\omega)=\frac{T\omega}{1+T^2\omega^2}\qquad A(\omega)=|H(j\omega)|=\sqrt{P^2(\omega)+Q^2(\omega)}=\frac{T\omega}{\sqrt{T^2\omega^2+1}}$$

$$L(\omega)=20\log A(\omega)=20\log\frac{T\omega}{\sqrt{T^2\omega^2+1}}=20\log T\omega-20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega)=\arg H(j\omega)=\arctan\frac{Q}{P}=\arctan\left(\frac{1}{T\omega}\right)$$

Transfer function – frequency response

RC circuit example

Nyquist plot

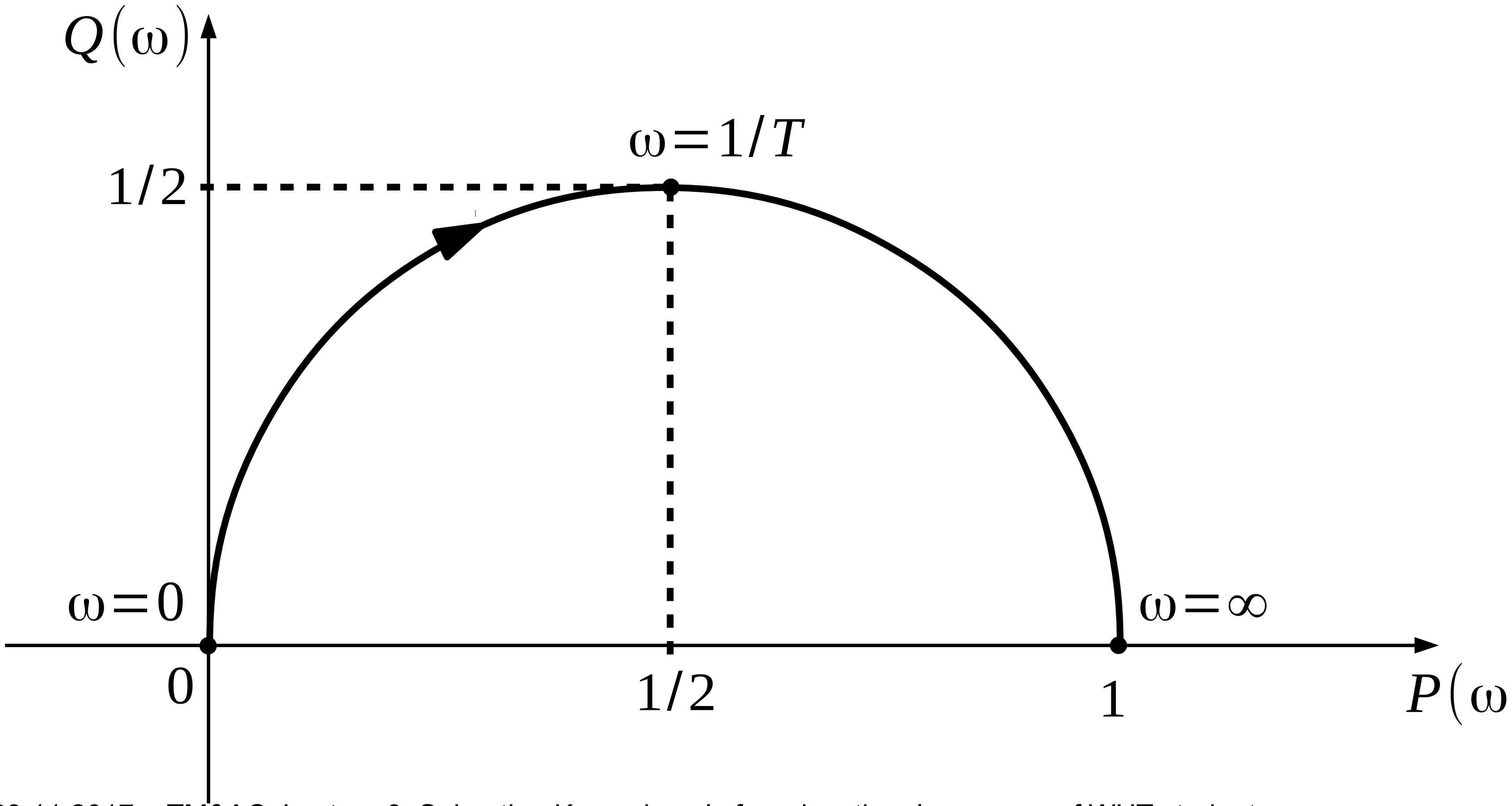
$$P(\omega)=\frac{T^2\omega^2}{1+T^2\omega^2} \qquad Q(\omega)=\frac{T\omega}{1+T^2\omega^2}$$

Transfer function – frequency response

RC circuit example

Nyquist plot

$$P(\omega)=\frac{T^2\omega^2}{1+T^2\omega^2} \quad Q(\omega)=\frac{T\omega}{1+T^2\omega^2}$$



Transfer function – frequency response

RC circuit example

$$A(\omega)=\frac{T\,\omega}{\sqrt{T^2\,\omega^2+1}}$$

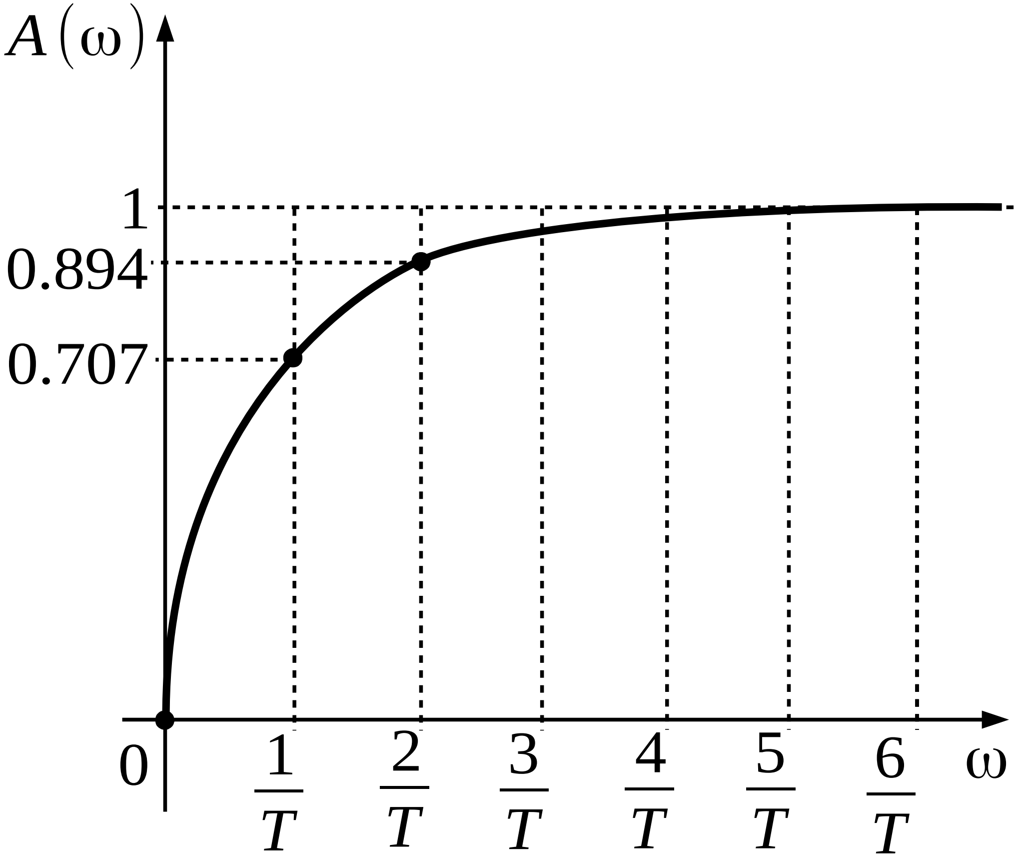
$$\varphi(\omega)=\arctan\left(\frac{1}{T\,\omega}\right)$$

Transfer function – frequency response

RC circuit example

$$A(\omega) = \frac{T\omega}{\sqrt{T^2\omega^2+1}}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T\omega}\right)$$



Transfer function – frequency response

RC circuit example

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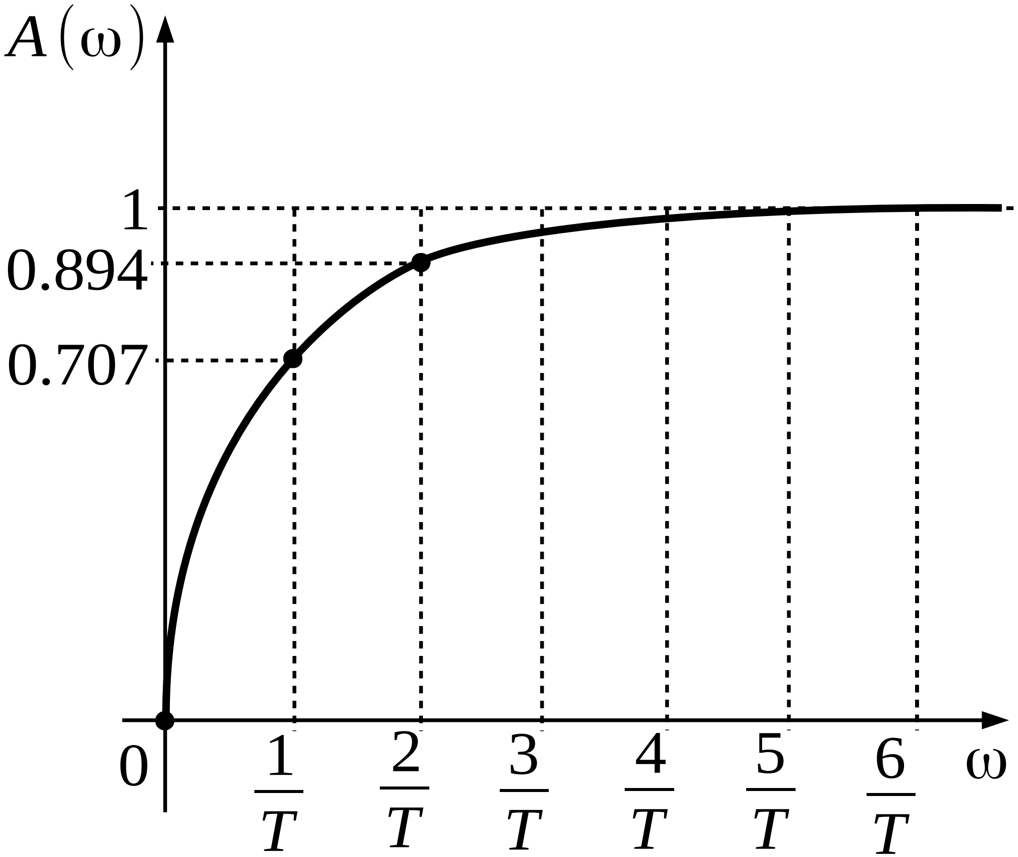

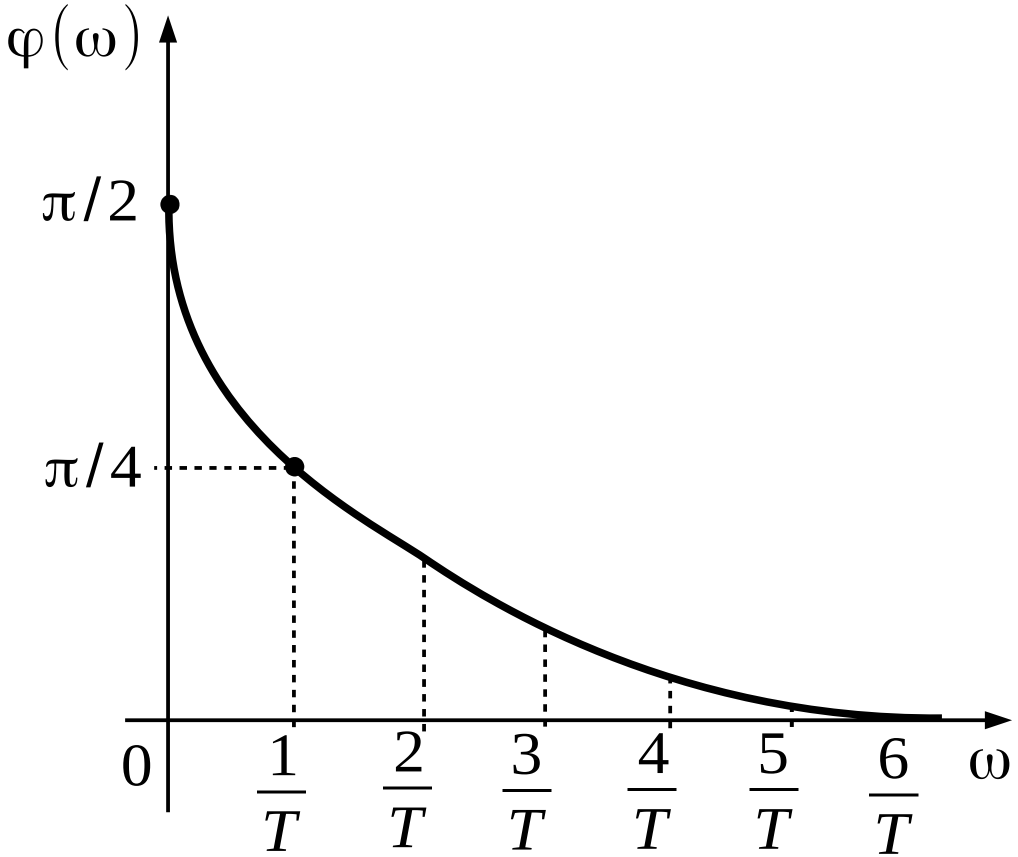

Transfer function – frequency response

RC circuit example

$$L(\omega) = 20\log T\omega - 20\log\sqrt{T^2\omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T\omega}\right)$$

Transfer function – frequency response

RC circuit example

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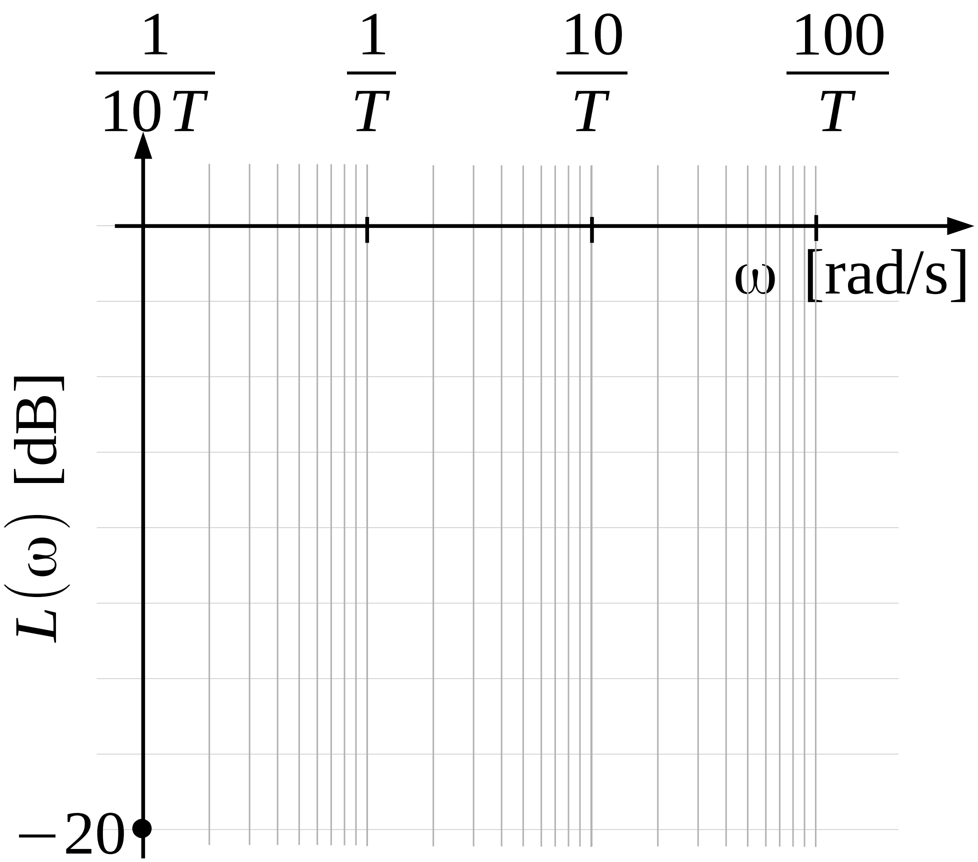

Transfer function – frequency response

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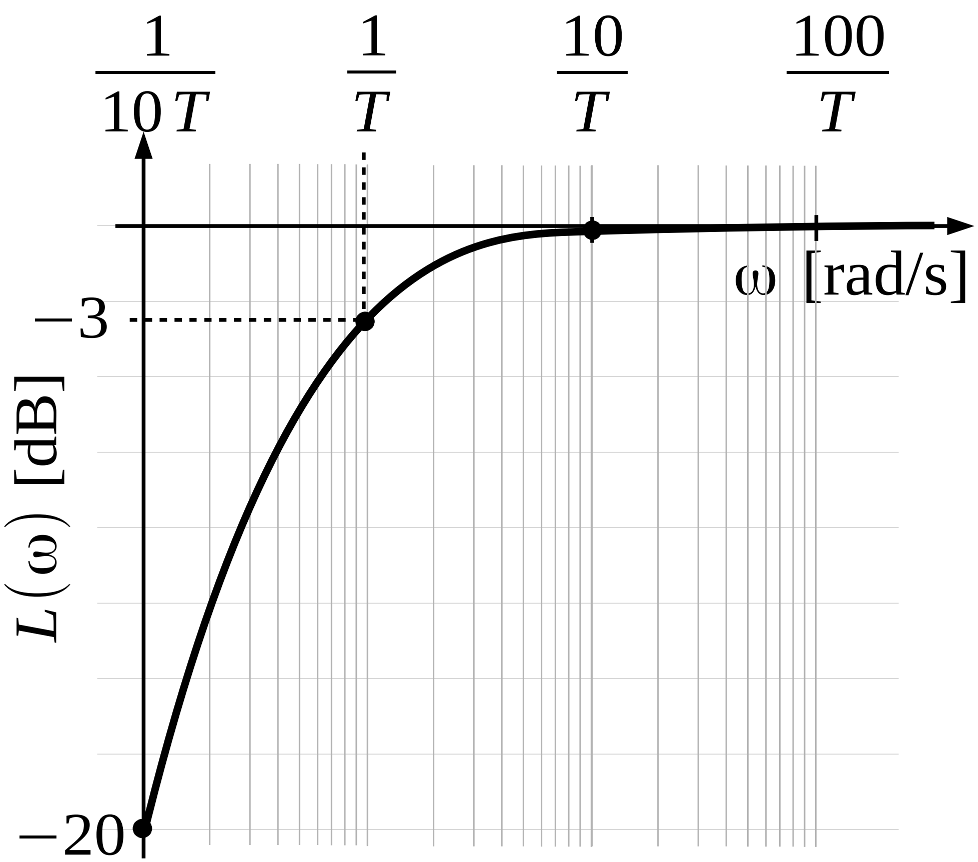

Transfer function – frequency response

RC circuit example

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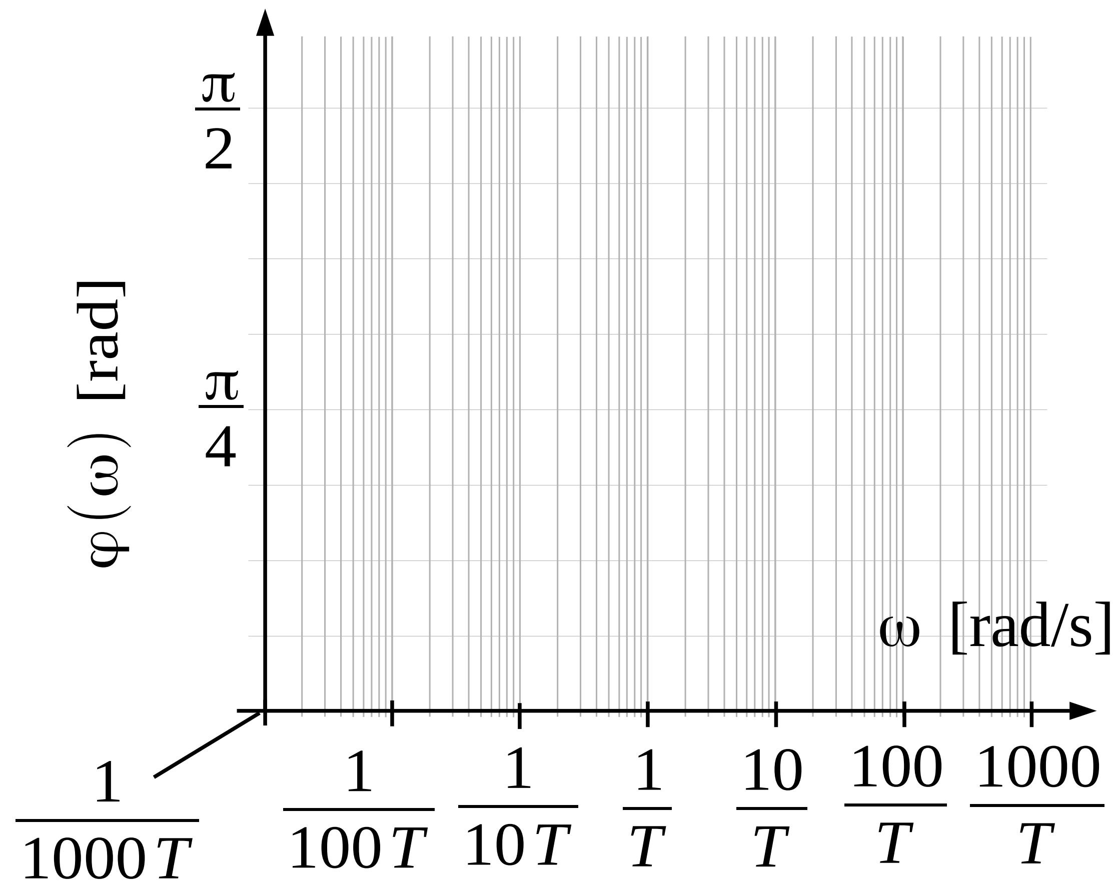

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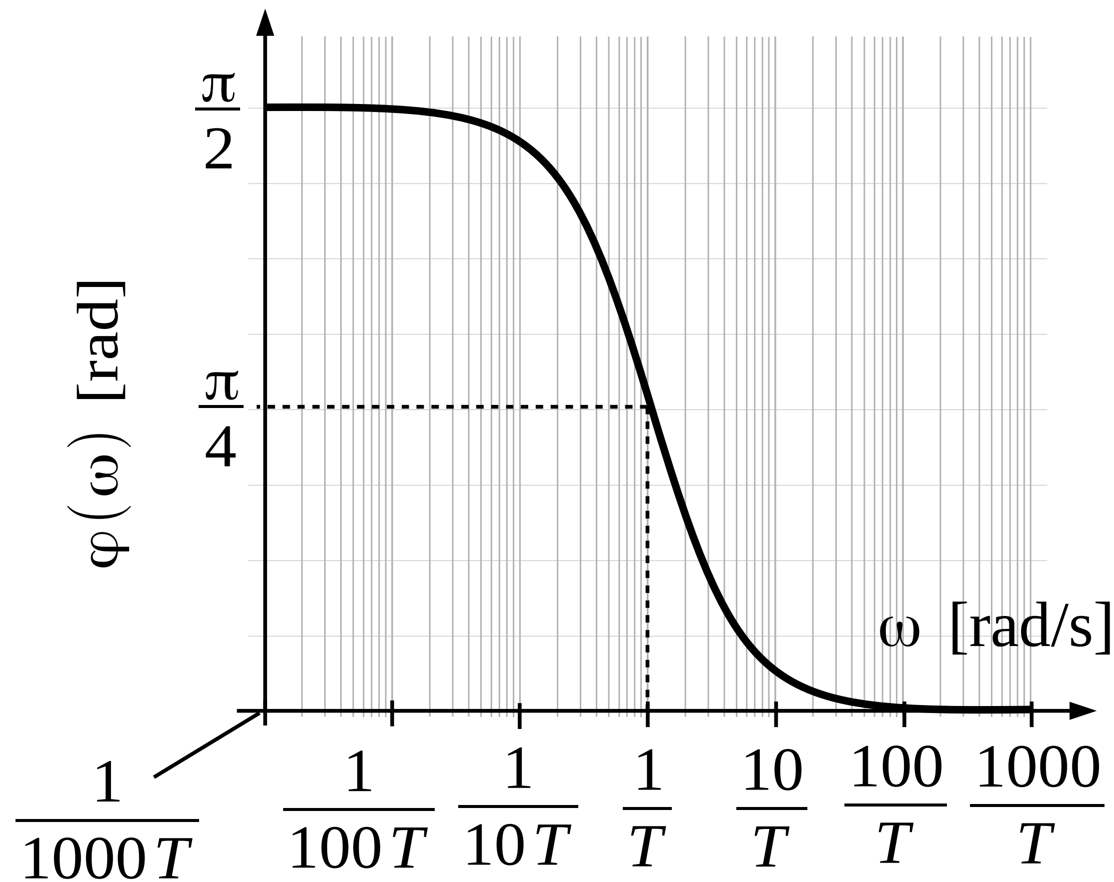

Transfer function – frequency response

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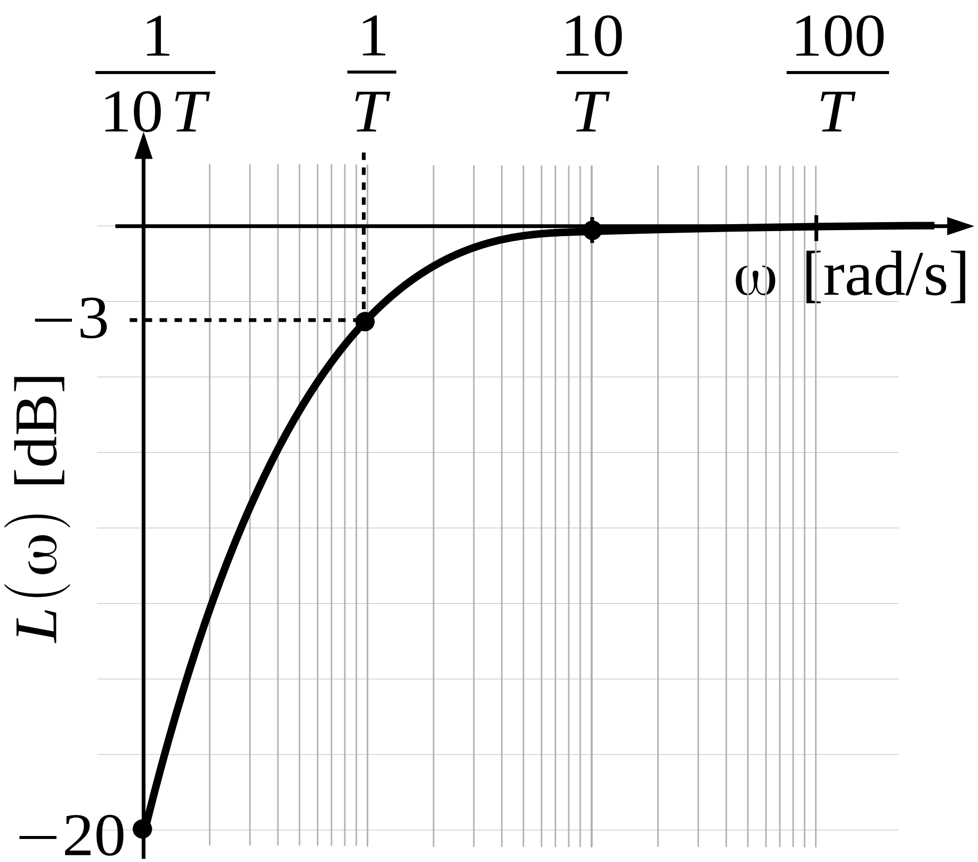
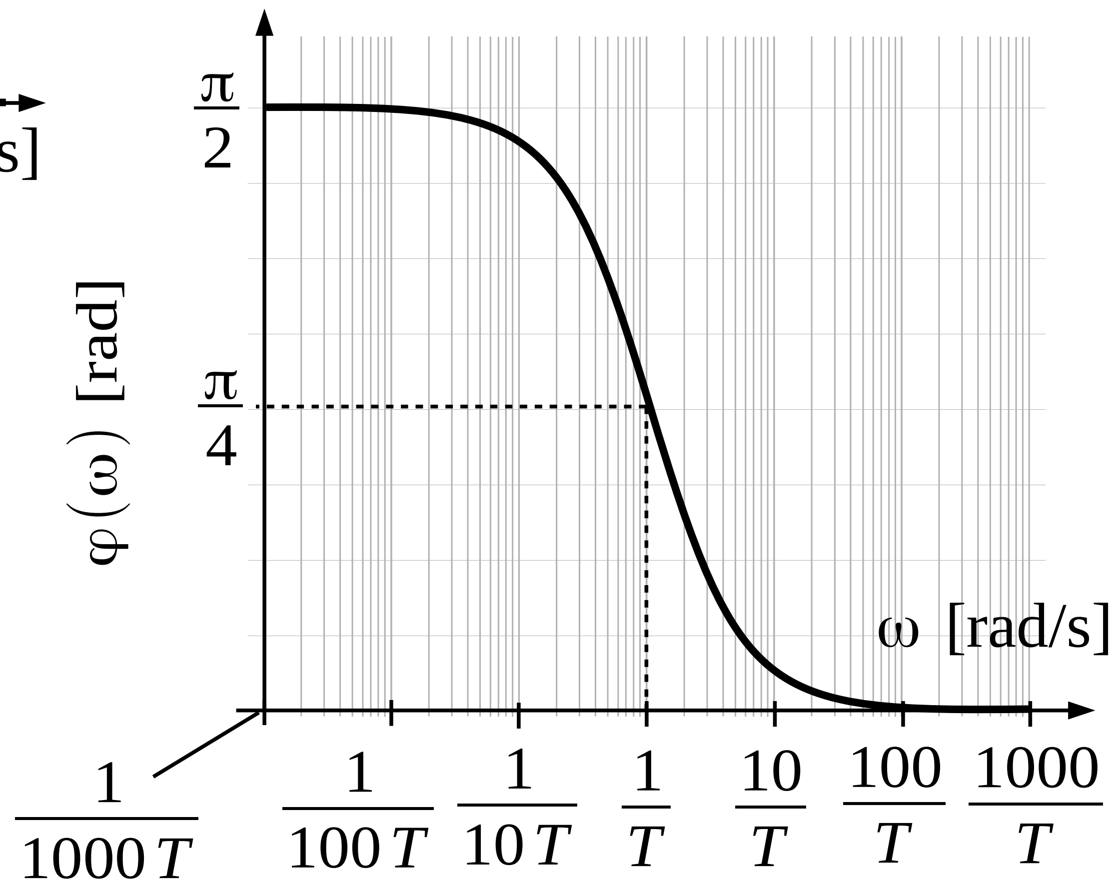
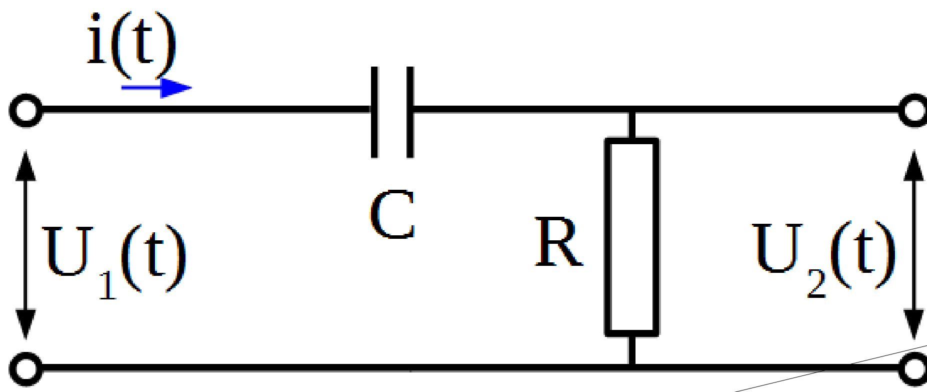




Transfer function – frequency response

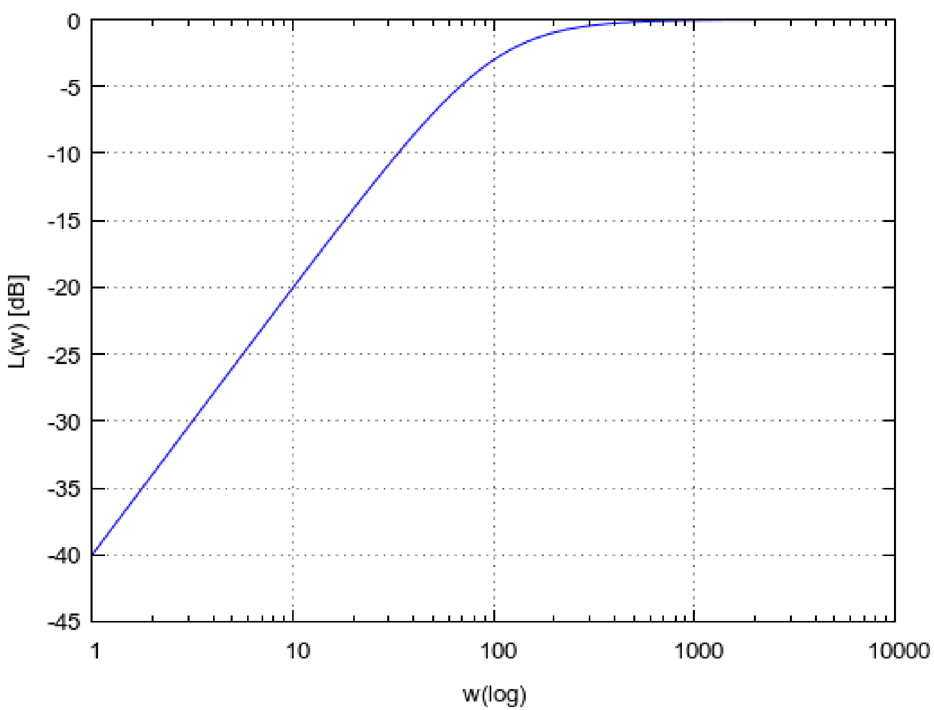
RC circuit example

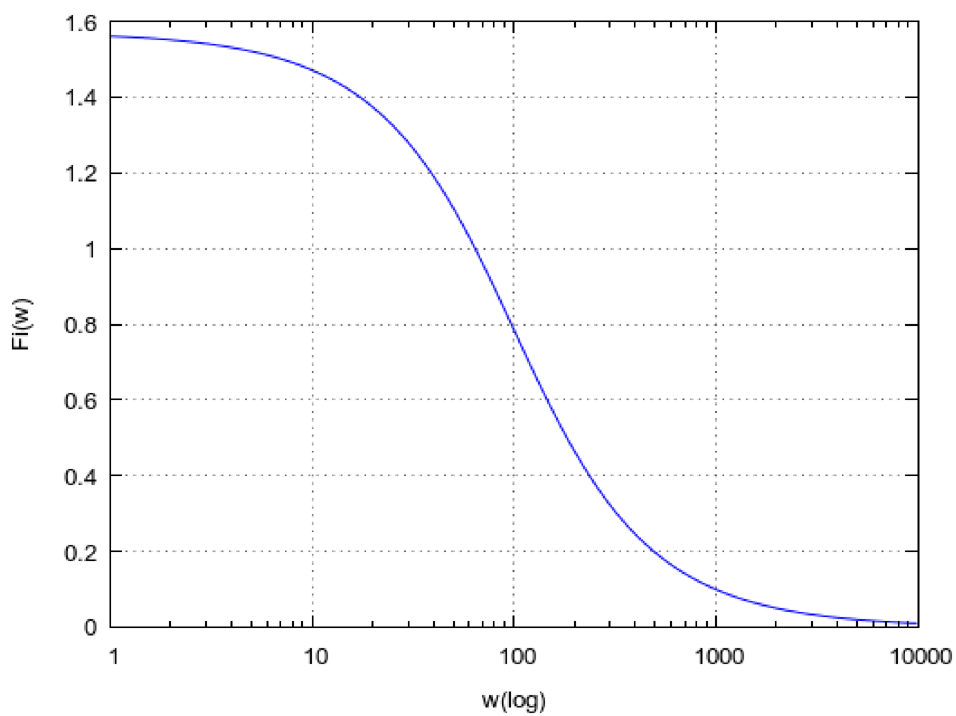
$u_1(t) = \sin(\omega t)$



$u_2(t) = A \sin(\omega t + \varphi)$

Example:
$R = 1\,k\Omega,\ C = 10\,\mu F$





Transfer function – frequency response

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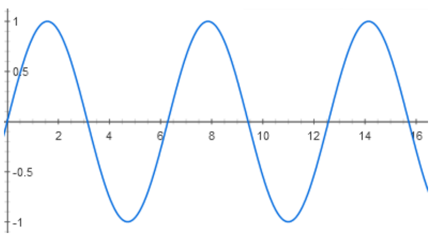
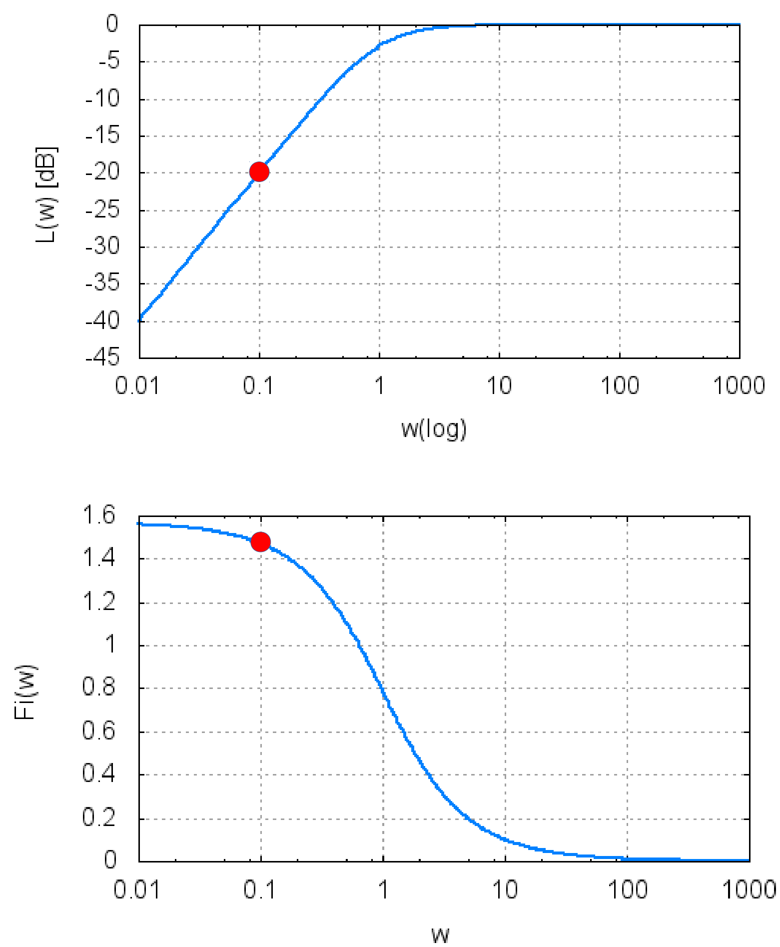


RC filter



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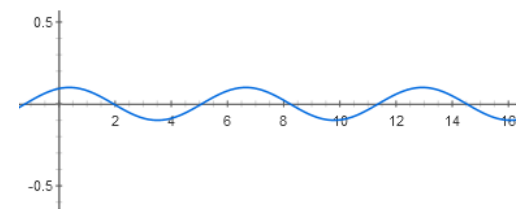

Transfer function – frequency response

RC circuit example

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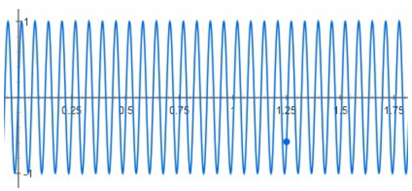
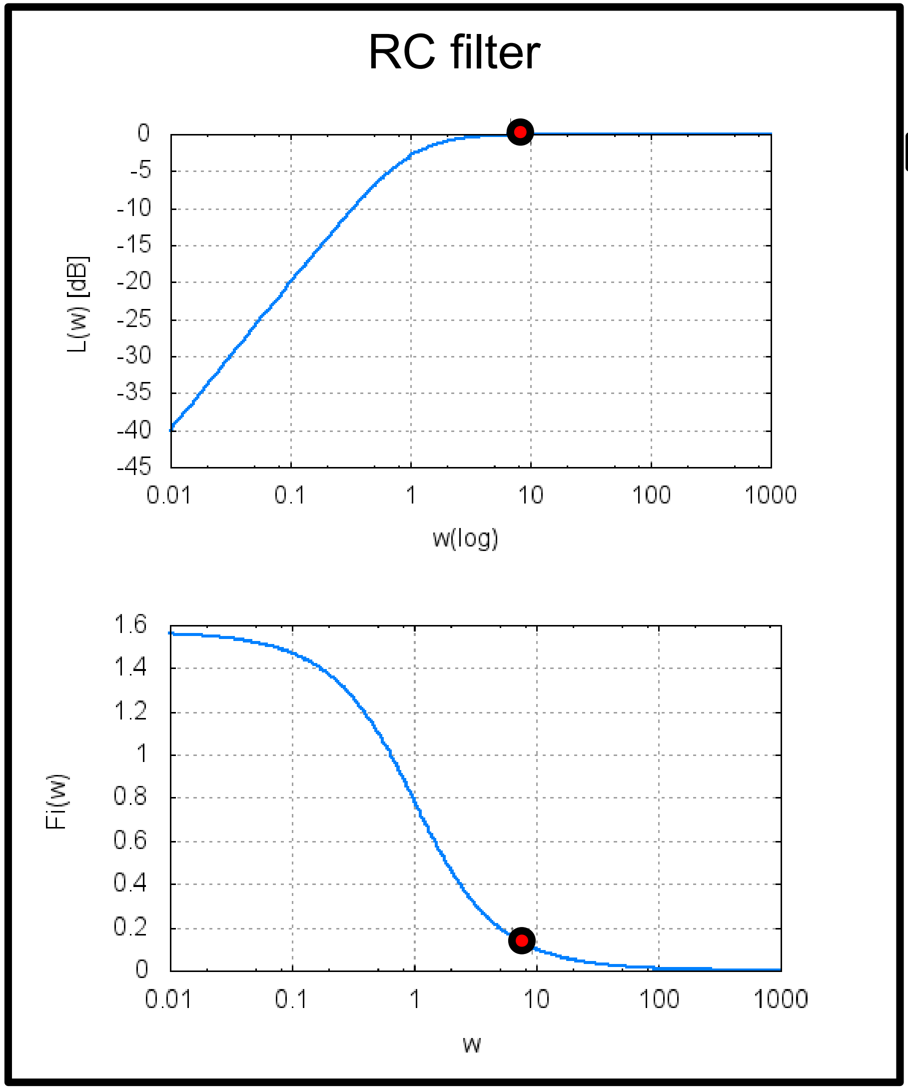
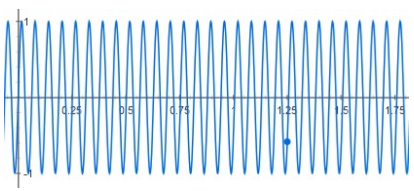
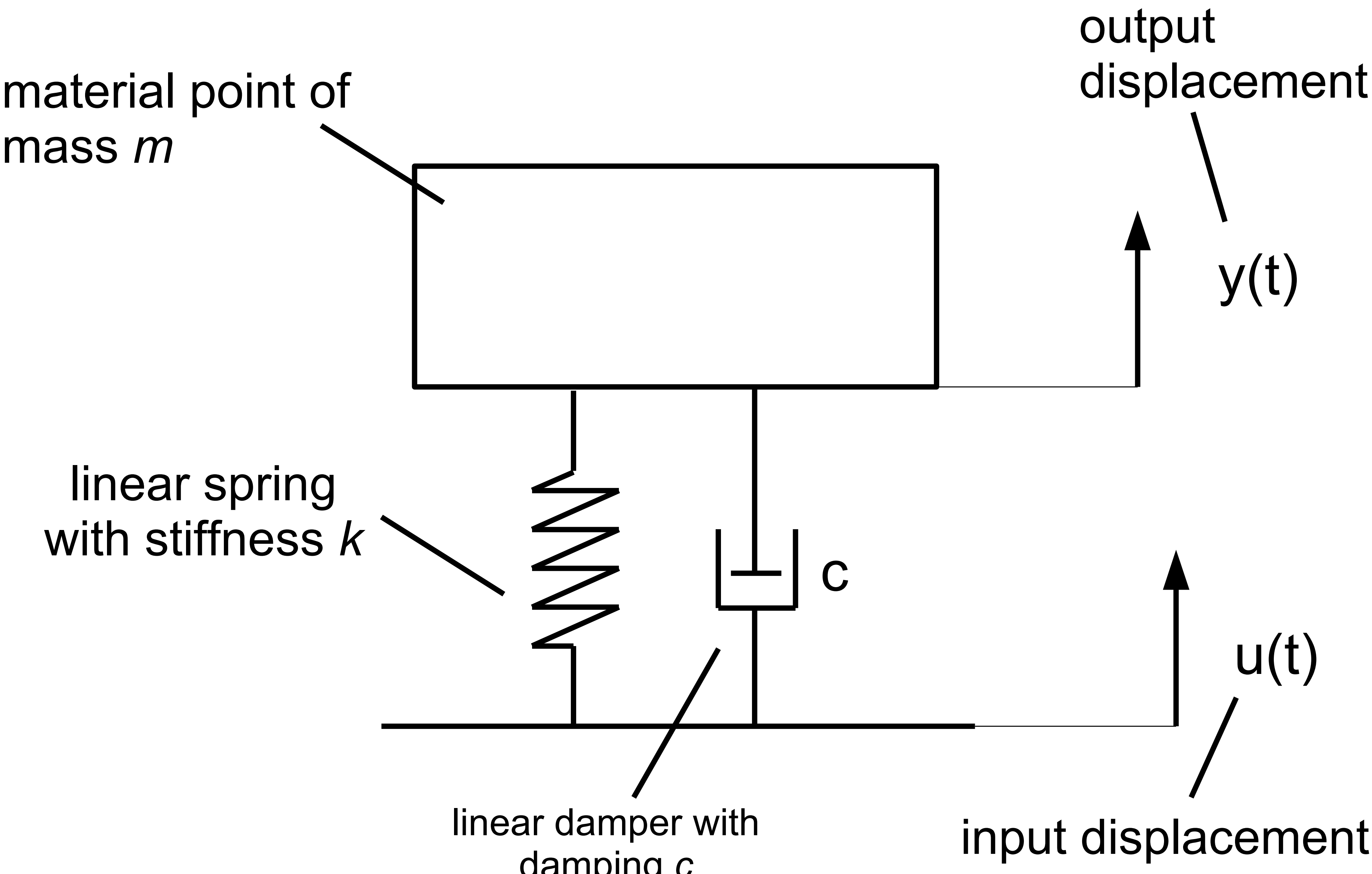






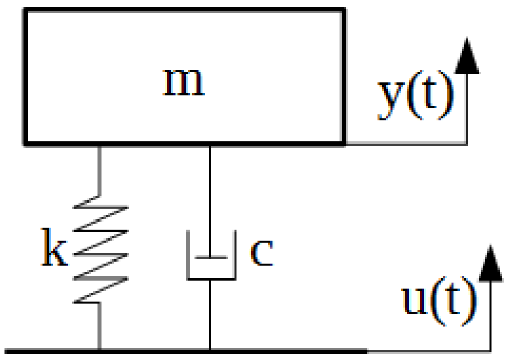
Transfer function – frequency response

Example 2 - vibrating system



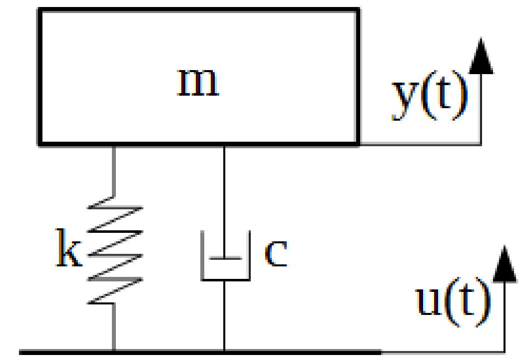
Transfer function – frequency response

Example 2 - vibrating system



Transfer function – frequency response

Example 2 - vibrating system



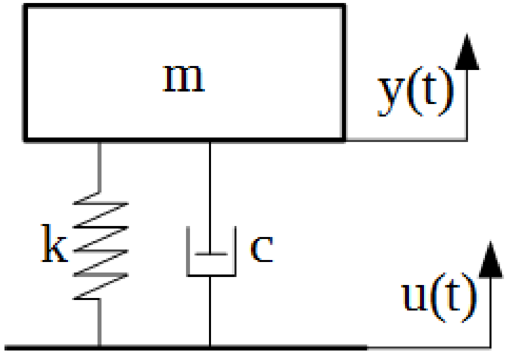
$$m\,\ddot{y}(t)+c\,\dot{y}(t)+k\,y(t)=c\,\dot{u}(t)+k\,u(t)$$

$$H(s)=\frac{c\,s+k}{m\,s^2+c\,s+k}$$

$$P(\omega)=\frac{k^2+c^2\omega^2-km\omega^2}{\left(k-m\omega^2\right)^2+c^2\omega^2}\,,\;Q(\omega)=\frac{-cm\omega^3}{\left(k-m\omega^2\right)^2+c^2\omega^2}$$

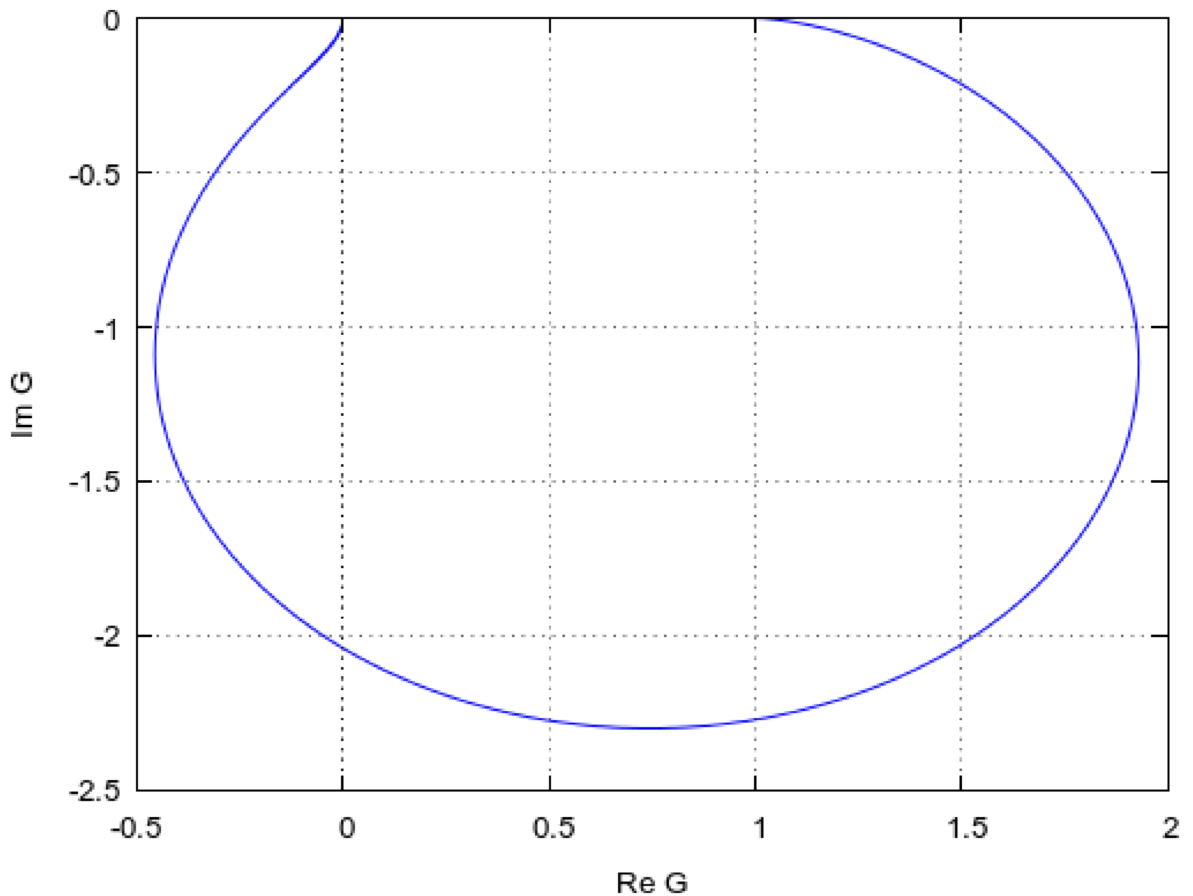
Transfer function – frequency response

Example 2 - vibrating system



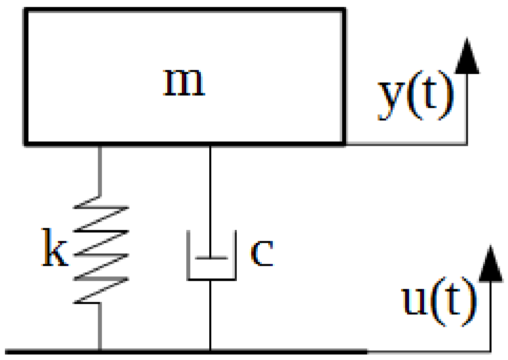
Plots for: $m = 300\,\text{kg},\ c = 800\,\frac{\text{Ns}}{\text{m}},\ k = 11000\,\frac{\text{N}}{\text{m}}$

Nyquist plot

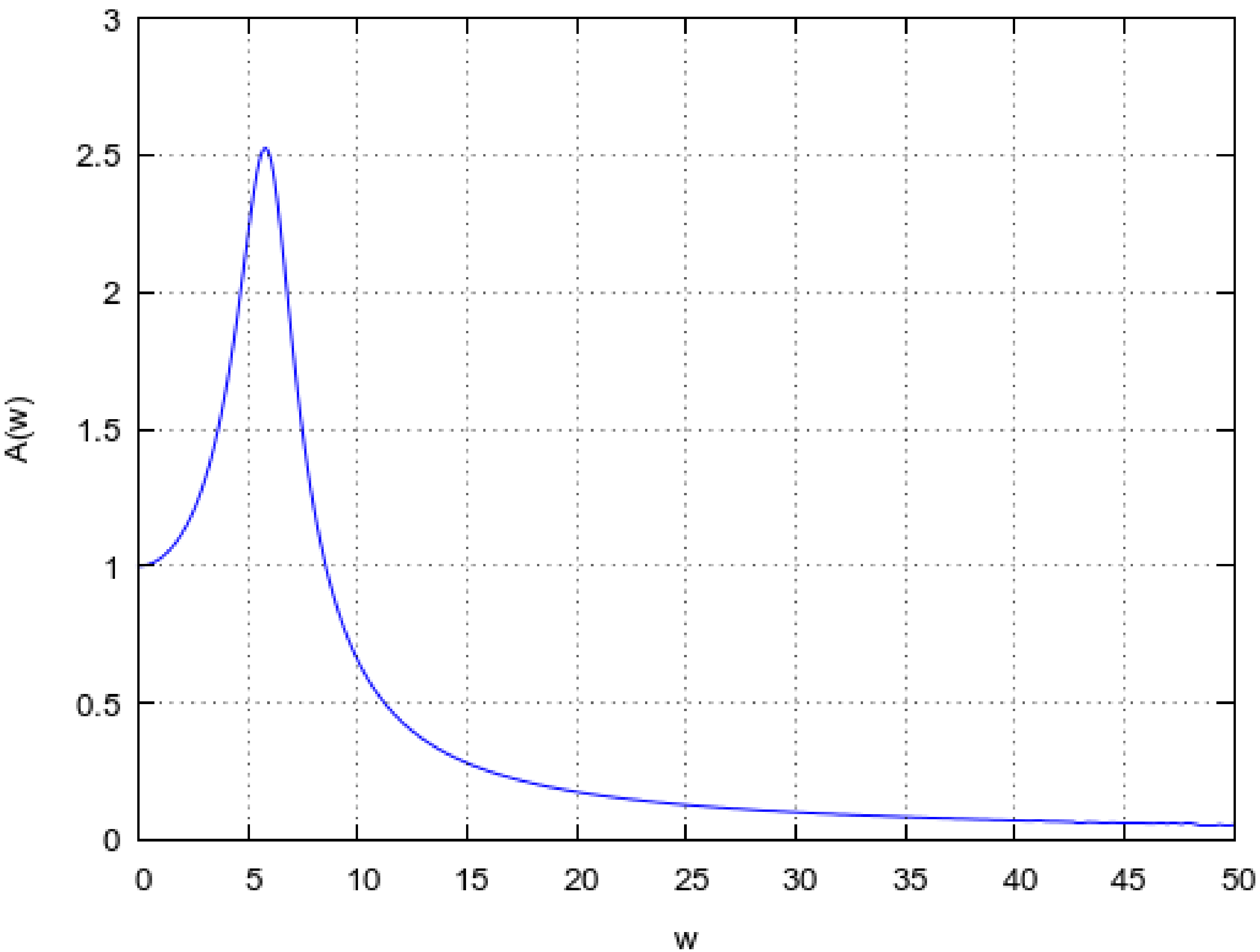


Transfer function – frequency response

Example 2 - vibrating system

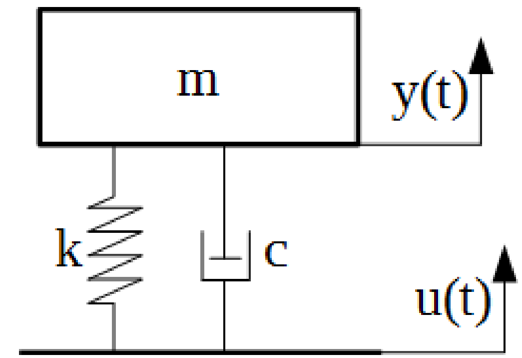


gain plot

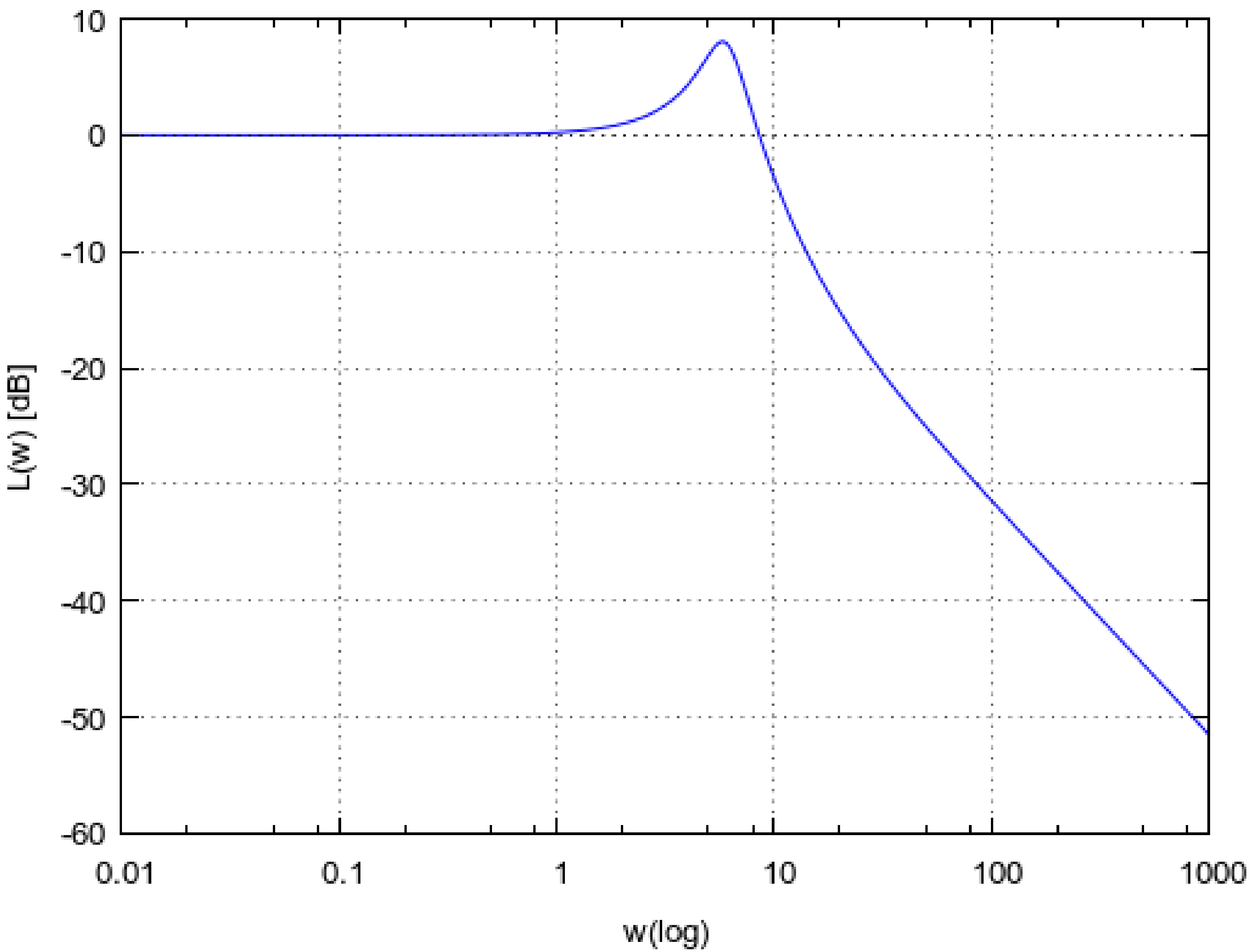


Transfer function – frequency response

Example 2 - vibrating system

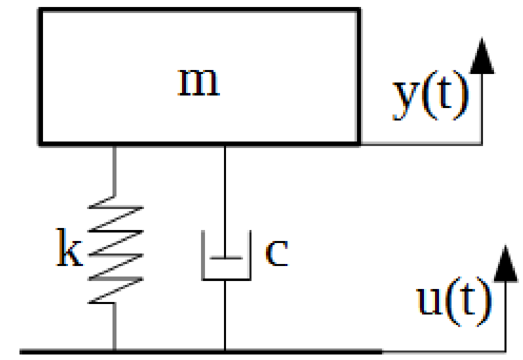


gain plot

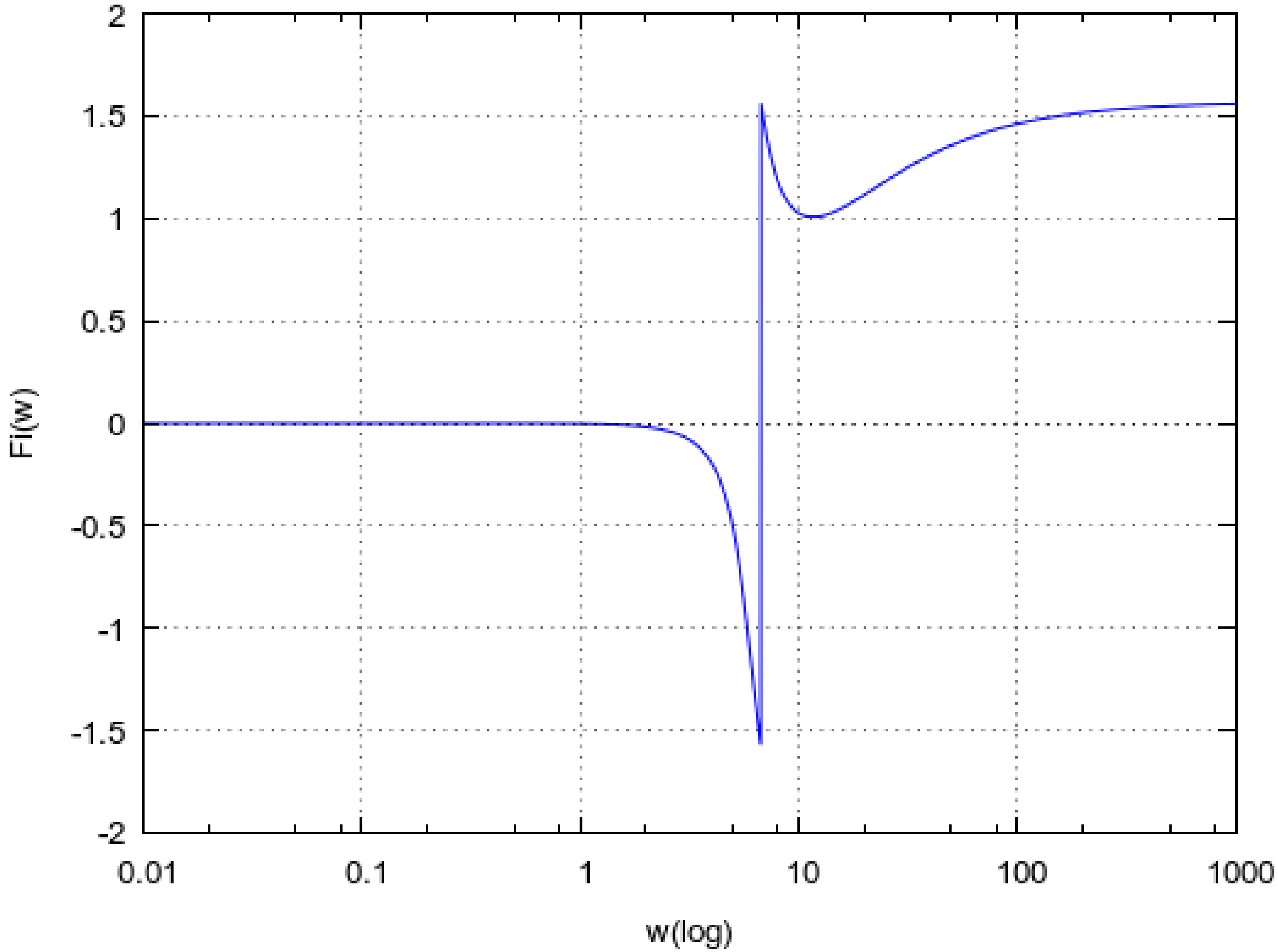


Transfer function – frequency response

Example 2 - vibrating system

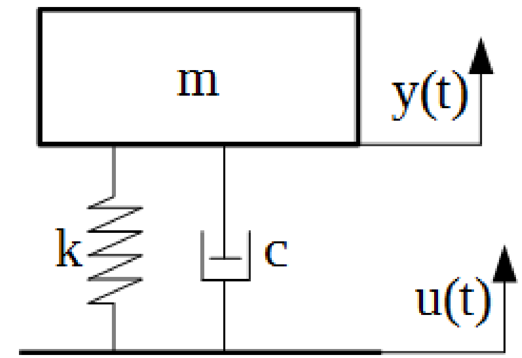


phase plot with „atan”

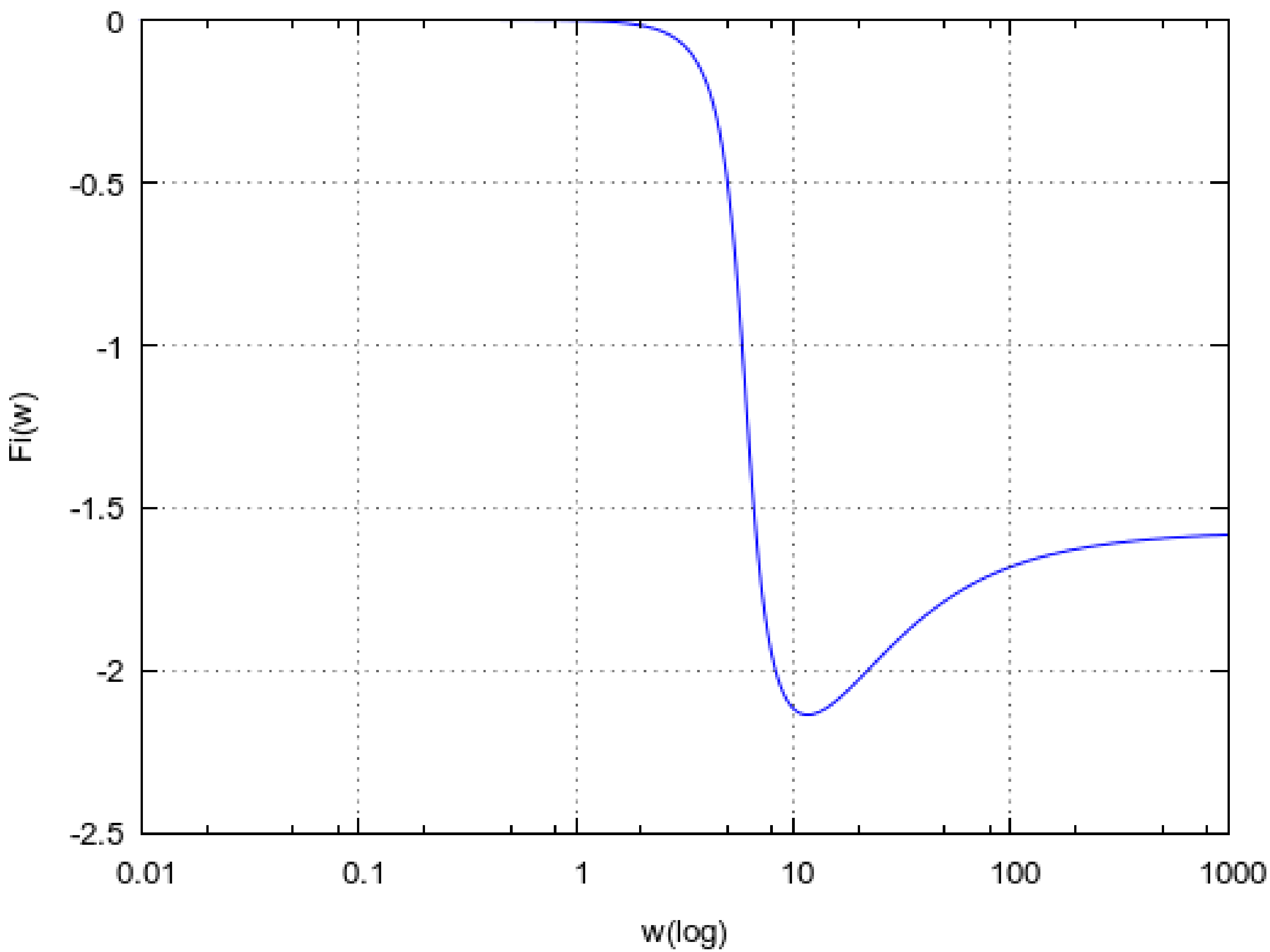


Transfer function – frequency response

Example 2 - vibrating system

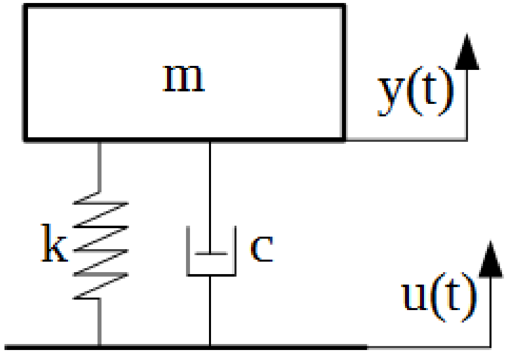


phase plot with „atan2"

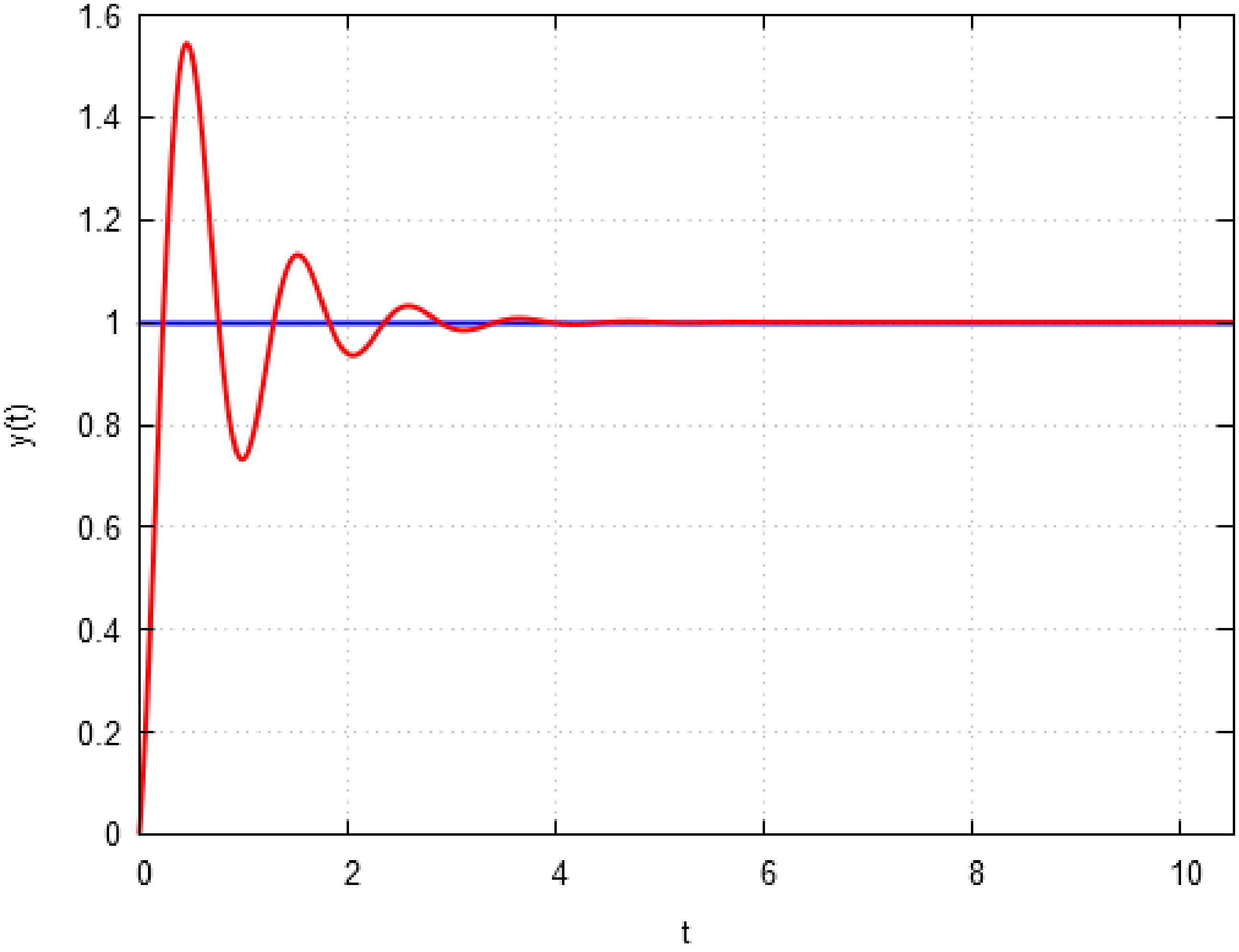


Transfer function – frequency response

Example 2 - vibrating system



Step response



Classification of basic automatic systems

Element name	Equation	Transfer function
proportional	$y(t)=ku(t)$	k
first order (inertial)	$T\frac{dy(t)}{dt}+y(t)=ku(t)$	$\frac{k}{Ts+1}$
integrator	$y(t)=k\int_0^t u(t)dt$ or $\frac{dy(t)}{dt}=ku(t)$	$\frac{k}{s}$

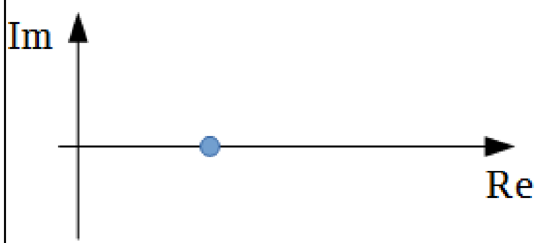
Classification of basic automatic systems

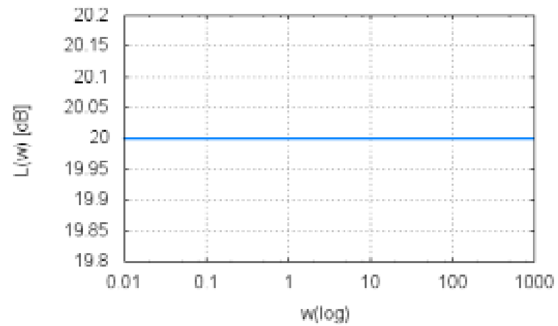
Element name	Equation	Transfer function
derivative	$y(t)=k\frac{du(t)}{dt}$	ks
derivative with inertia	$T\frac{dy(t)}{dt}+y(t)=k\frac{du(t)}{dt}$	$\frac{ks}{Ts+1}$

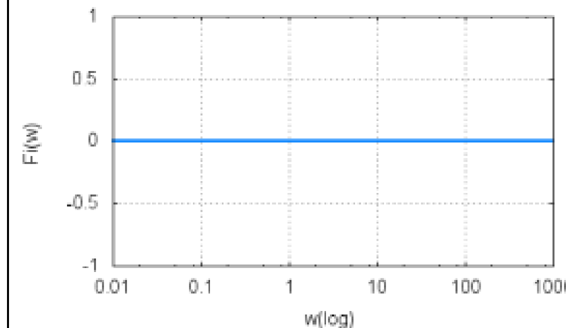
Classification of basic automatic systems

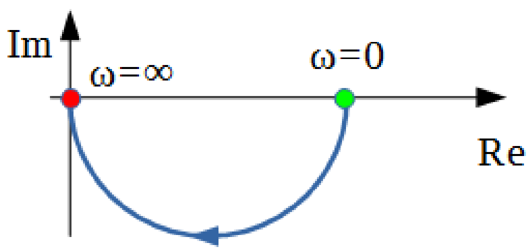
Element name	Equation	Transfer function
delay	$y(t)=u(t-\tau)$	$e^{-\tau s}$
second order (oscillator)	$T_1^2 \frac{d^2 y(t)}{dt^2}+T_2 \frac{dy(t)}{dt}+$ $+y(t)=ku(t)$	$\frac{k}{T_1^2 s^2+T_2 s+1}$

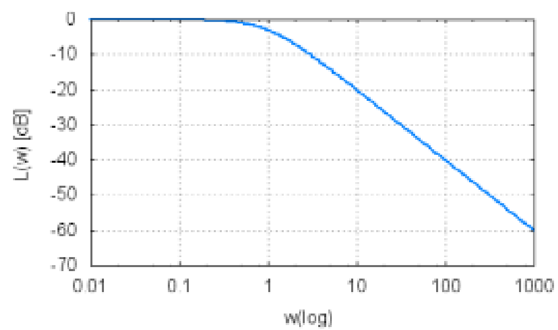
Nyquist plot	Bode Plot (example)

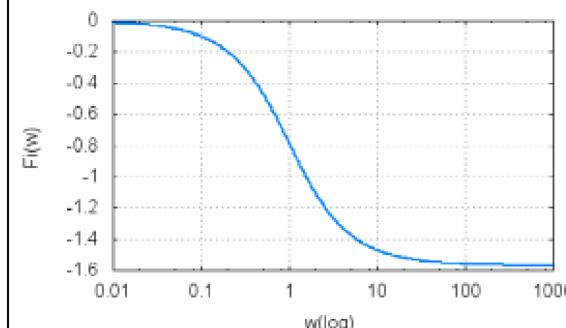


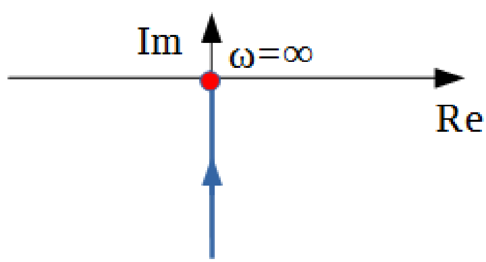


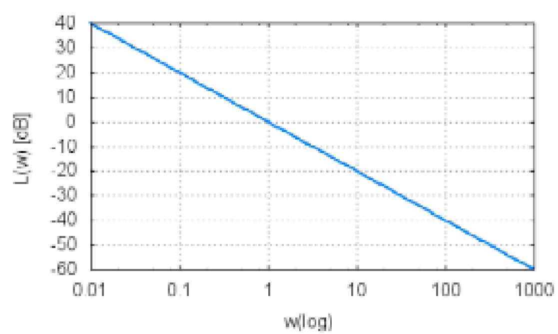


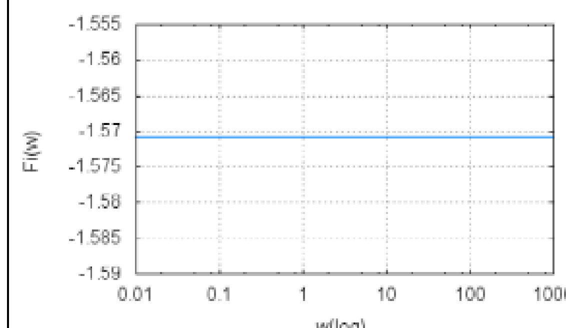


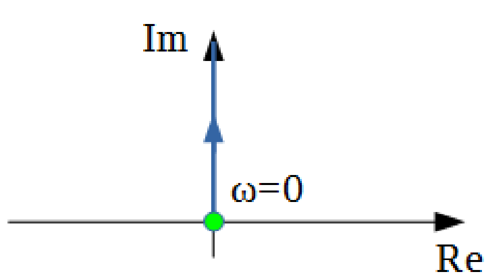


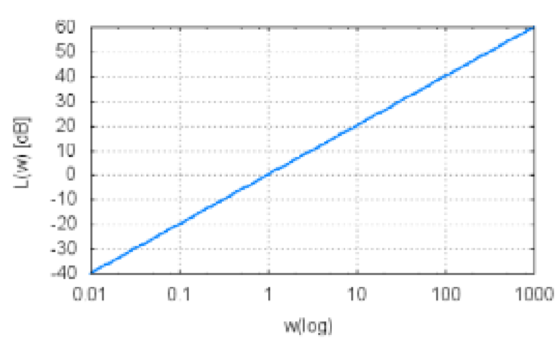


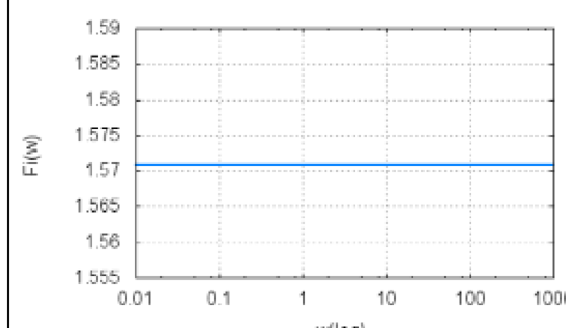




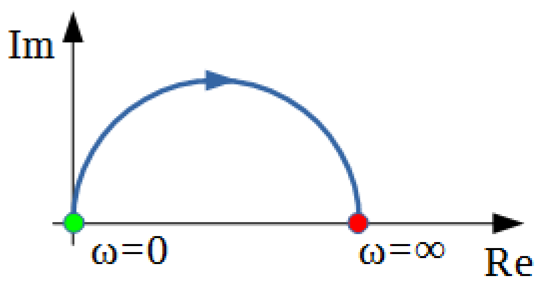


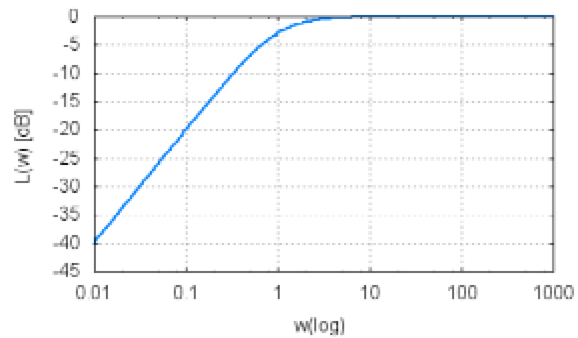


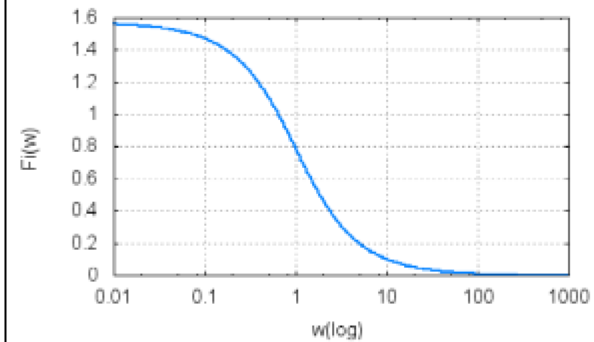


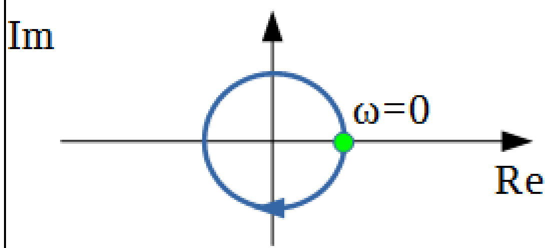


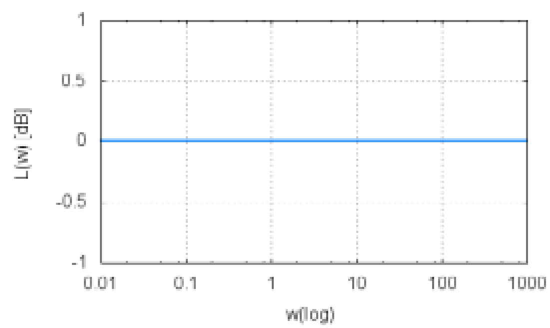
Nyquist plot	Bode Plot (example)	

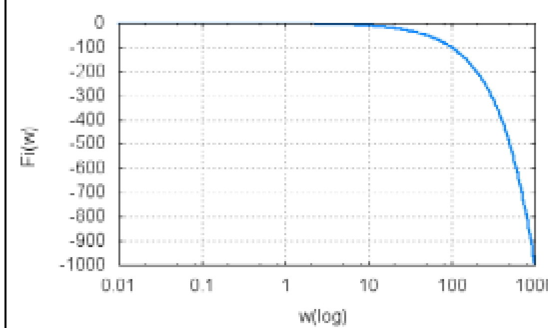


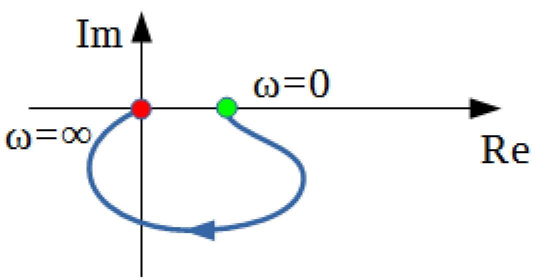


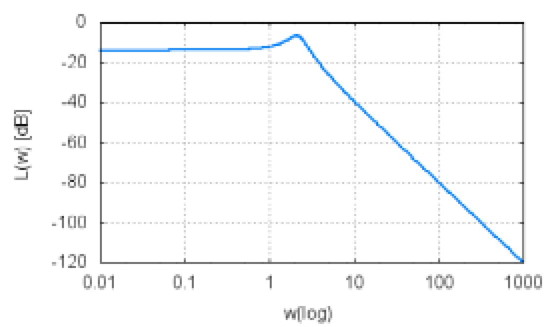


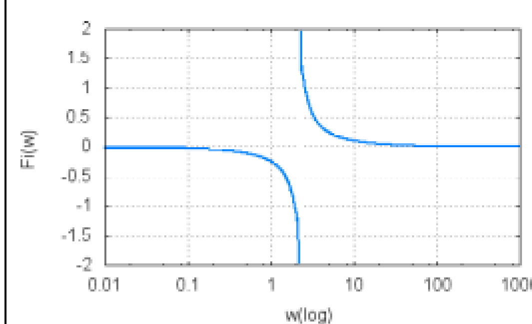




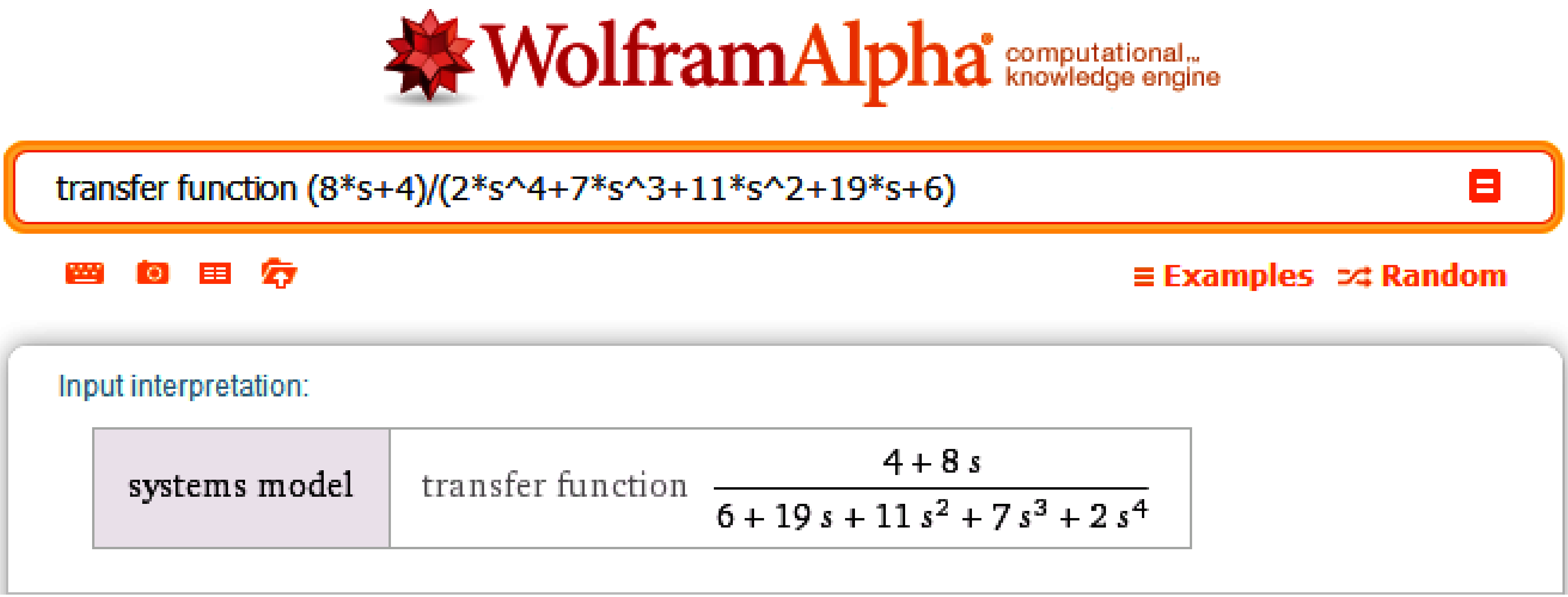




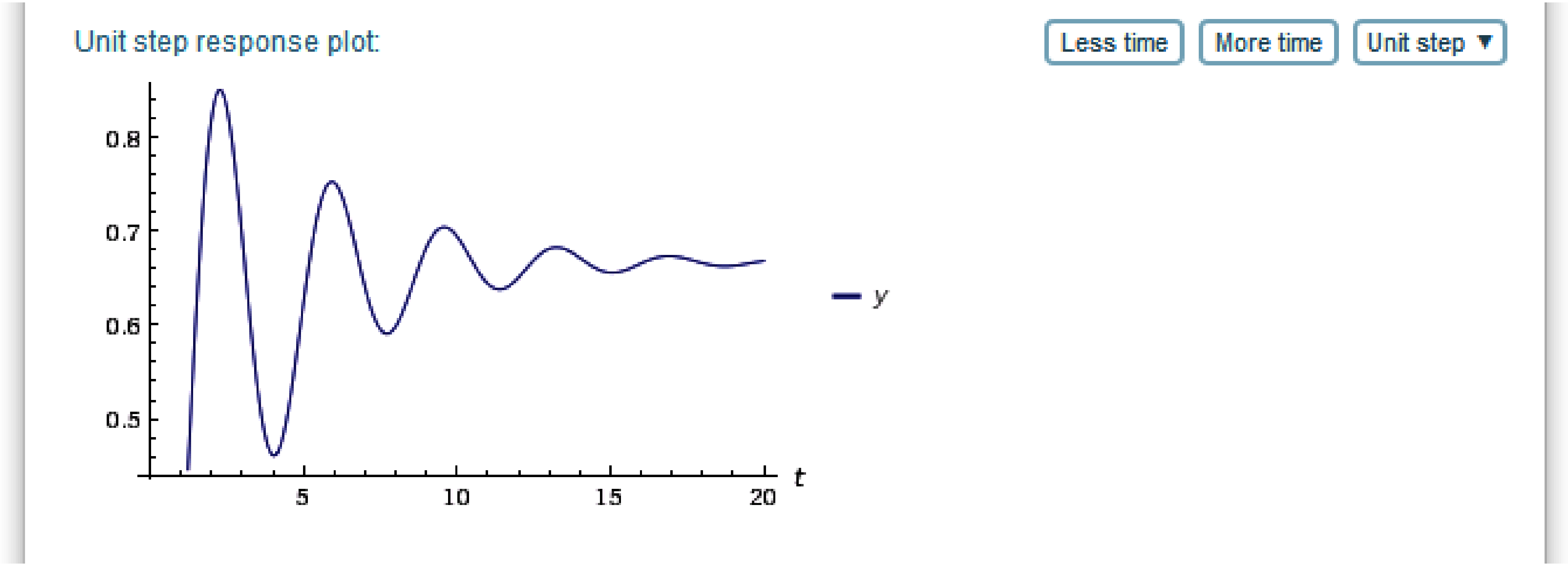




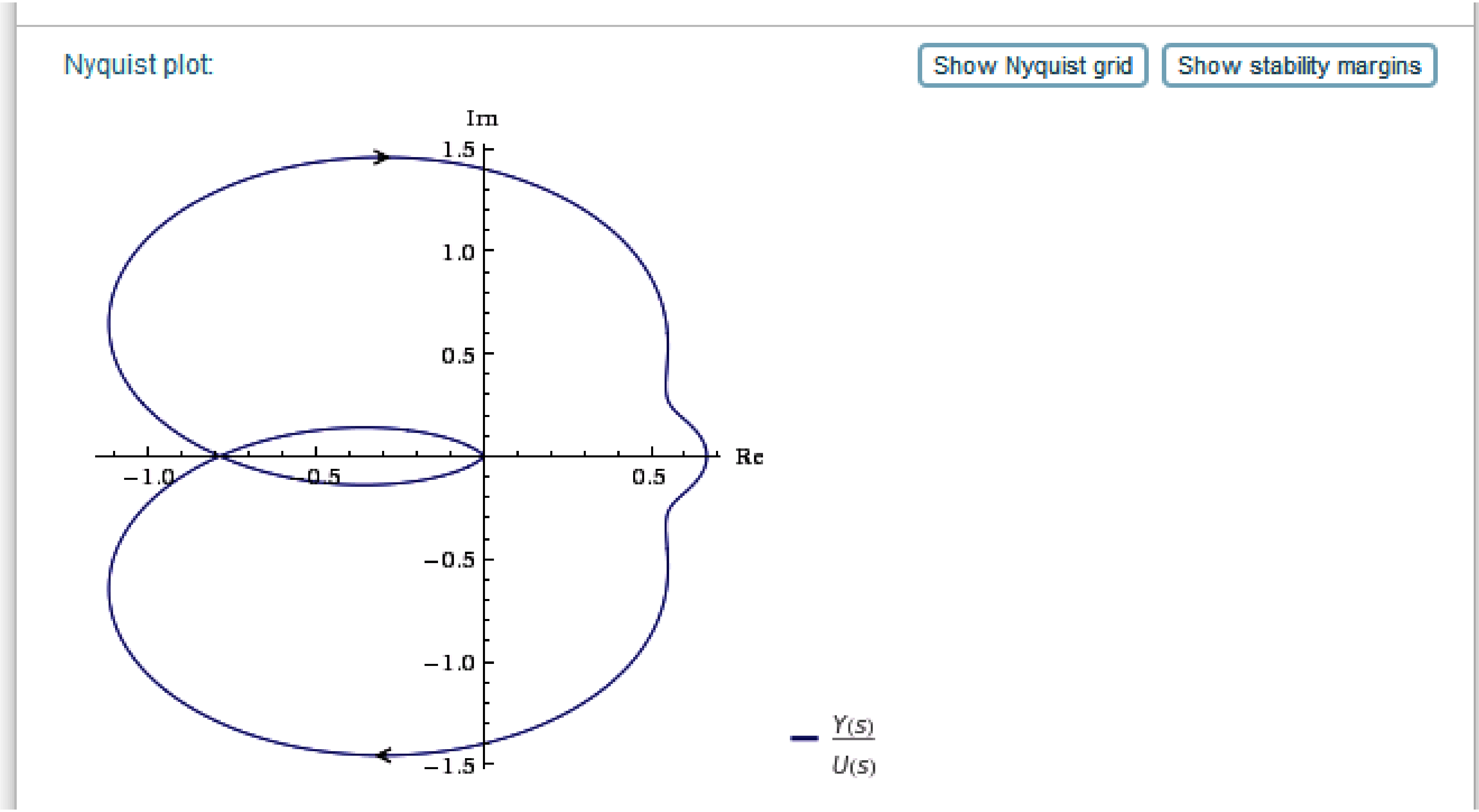
WolframAlpha



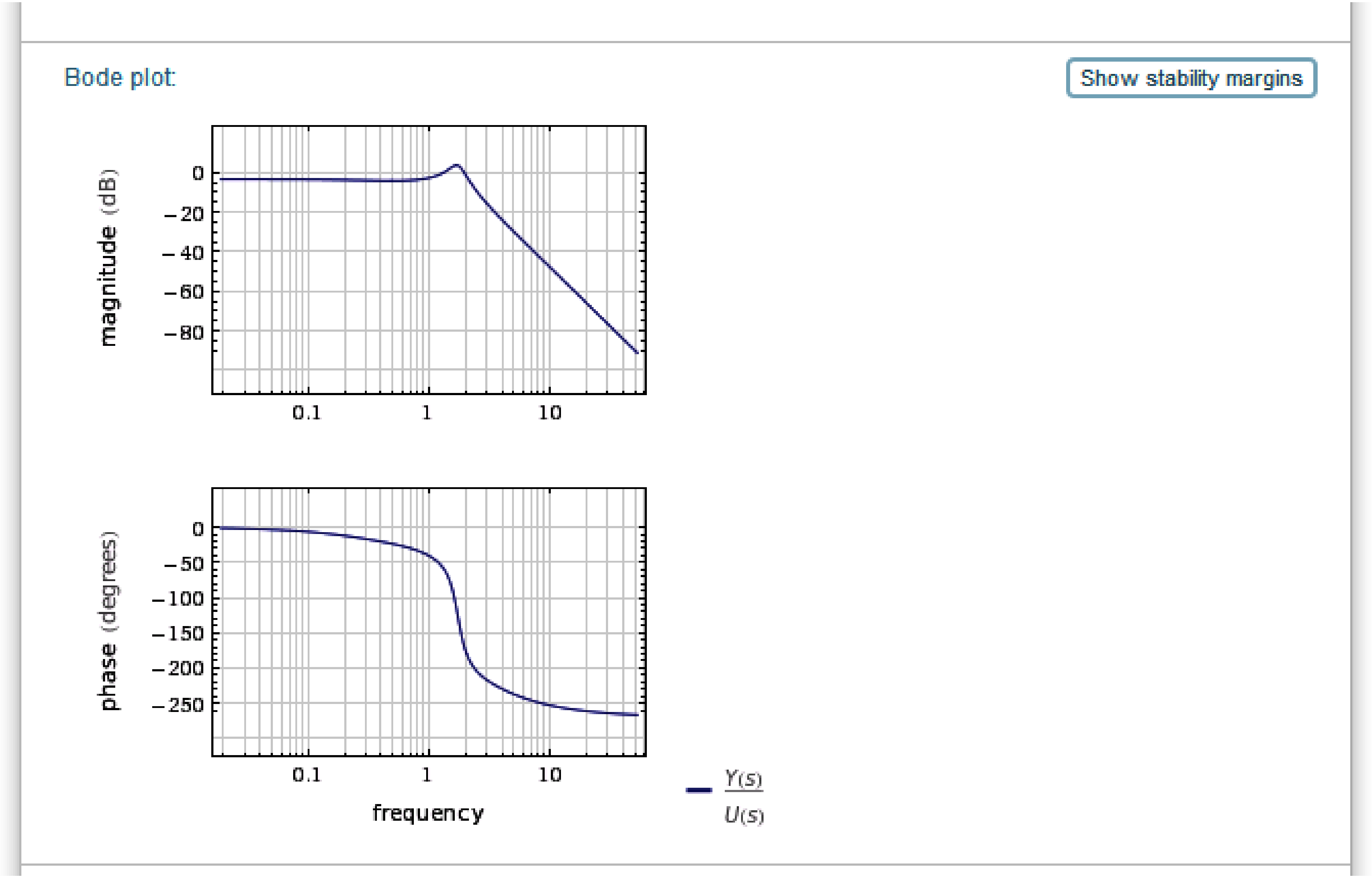
WolframAlpha



WolframAlpha



WolframAlpha



WolframAlpha

WolframAlpha computational knowledge engine

partial fraction decomposition s/(s^3+4*s^2+5*s+2)

Examples Random

Assuming "s" is a variable | Use as a unit instead

Input:

partial fractions	$\frac{s}{s^3+4s^2+5s+2}$

Result: Step-by-step solution

$$\frac{s}{s^3+4s^2+5s+2} = -\frac{2}{s+2} + \frac{2}{s+1} - \frac{1}{(s+1)^2}$$

WolframAlpha

WolframAlpha computational... knowledge engine

inverse laplace transform s/(s^3+ 4*s^2+5*s+2)

Examples Random

Assuming "s" is a variable | Use as a unit instead

Input:

$$\mathcal{L}_s^{-1}\left[\frac{s}{s^3+4\,s^2+5\,s+2}\right](t)$$

$\mathcal{L}_s^{-1}[f(s)](t)$ is the inverse Laplace transform of $f(s)$ with real variable t

Result:

$$-e^{-2t}\left(e^t\,t-2\,e^t+2\right)$$