Warsaw University of Technology
The Faculty of Automotive
and Construction Machinery Engineering
Institute of Machine Design Fundamentals
Department of Mechanics
http://www.ipbm.simr.pw.edu.pl/

Theory of Machines and Automatic Control

Winter 2017/2018

Lecturer: Sebastian Korczak, PhD, Eng.

Lecture 13

Stability criteria.
Gain margin and phase margin.
System correction.

STABILITY CRITERIA

General stability criterion

Hurwitz criterion

Nyquist stability criterion

General stability criterion

LTI SISO system is assymptotically stable if real part of every pole of the system's transfer function is less than zero.

$$G(s)=\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

$$\mathrm{Re}\, p_1<0 \ \wedge\ \mathrm{Re}\, p_2<0 \ \wedge\ldots\wedge\ \mathrm{Re}\, p_n<0$$

Hurwitz criterion

mathematics

a necessary and sufficient condition whether all the roots of the polynomial are in the left half of the complex plane

Hurwitz criterion

<u>mathematics</u>

a necessary and sufficient condition whether all the roots of the polynomial are in the left half of the complex plane

<u>control theory</u>

a necessary and sufficient condition whether all the poles of transfer function of a linear system have negative real parts

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

is stable if:

Hurwitz criterion

LTI SISO system with a transfer function

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is stable if:

1 $\quad a_n>0,\ \ a_{n-1}>0,\ \ \ldots,\ \ a_1>0,\ \ a_0>0$

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

is stable if:

1 $\quad a_n>0,\ \ a_{n-1}>0,\ \ \ldots,\ \ a_1>0,\ \ a_0>0$

2

$$M_n=\begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Hurwitz criterion

LTI SISO system with a transfer function

$$H(s)=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}=\frac{(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)}$$

is stable if:

1. $a_n>0,\quad a_{n-1}>0,\quad \ldots,\quad a_1>0,\quad a_0>0$

2. $\det\Delta_2>0$

   $\det\Delta_3>0$

   $\ldots$

   $\det\Delta_{n-1}>0$

$\Delta_2 \qquad \Delta_3 \qquad \Delta_{n-1}$

$$M_n=\begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Δ_i - *leading principal minor of order i*

Hurwitz criterion

Hurwitz criterion ≠ Routh criterion
(1895) (1876)

Liénard–Chipart criterion – modification of Hurwitz criterion

Hurwitz criterion

Example 1

$$G(s) = \frac{5s+3}{10s^2+3s+1}$$

Hurwitz criterion

Example 2

$$G(s)=\frac{2s}{2s^3+s+20}$$

Hurwitz criterion

Example 3

$$G(s)=\frac{3s-5}{s^3+4s^2+3s+10}$$

Hurwitz criterion

Example 4

$$G(s)=\frac{1}{3s^4+4s^3+6s^2+4s+5}$$

Hurwitz criterion

Example 5

Choose k parameter to satisfy Hurwitz criterion

$$\frac{k\,s}{4\,s^{3}+3\,s^{2}+k\,s+1}$$

Hurwitz criterion

Example 6

Choose k parameter to satisfy Hurwitz criterion

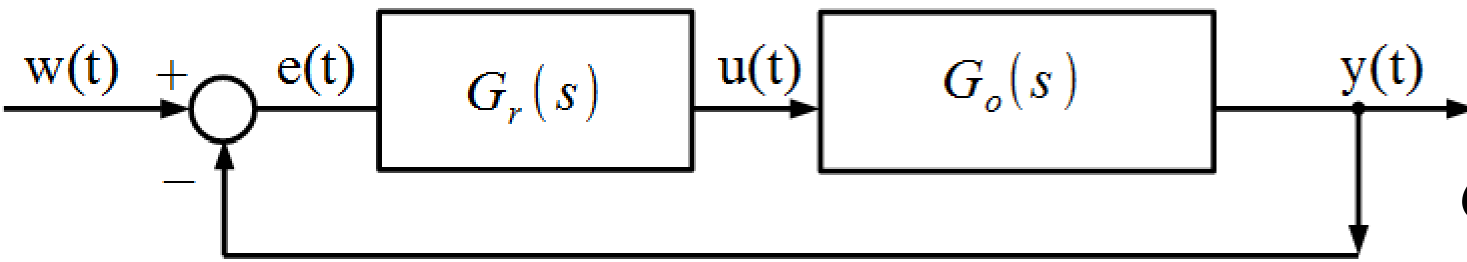
$$\frac{2}{2s^3 + ks^2 + (1+k)s + 3}$$

Homework

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



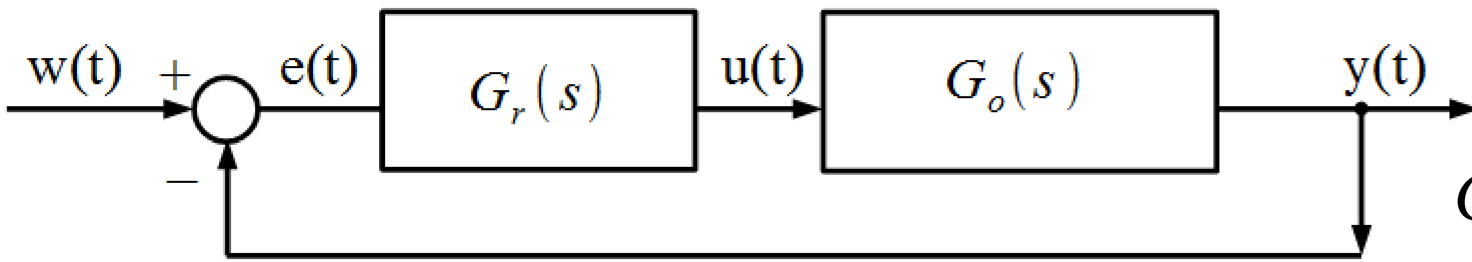
$$G_r(s)=\frac{4s}{Ts+1}$$

$$G_o(s)=\frac{1}{s^3+2s^2+s+1}$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

$$G_o(s)=\frac{1}{s^3+2s^2+s+1}$$

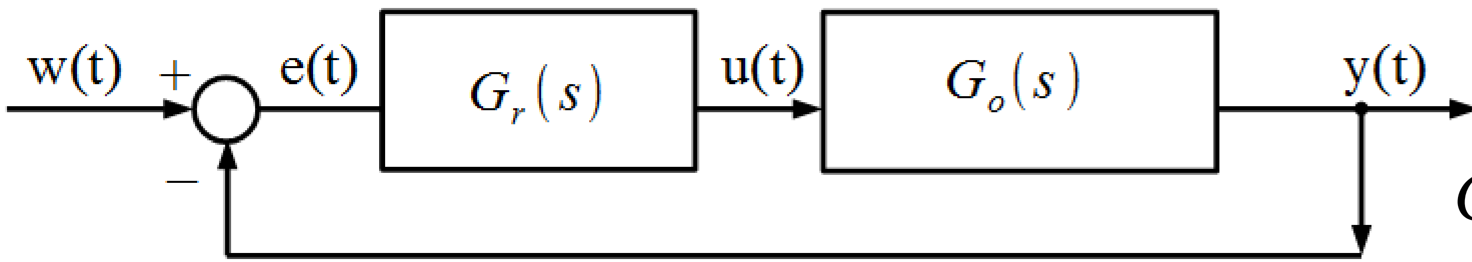
$$G_z(s)=\frac{G_r G_o}{1+G_r G_o G_p}=\frac{4s}{a_4 s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

$$a_4=T\,,\ a_3=2T+1,$$
$$a_2=T+2\,,\ a_1=T+5\,,\ a_0=1$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

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$$G_z(s)=\frac{G_r G_o}{1+G_r G_o G_p}=\frac{4s}{a_4 s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

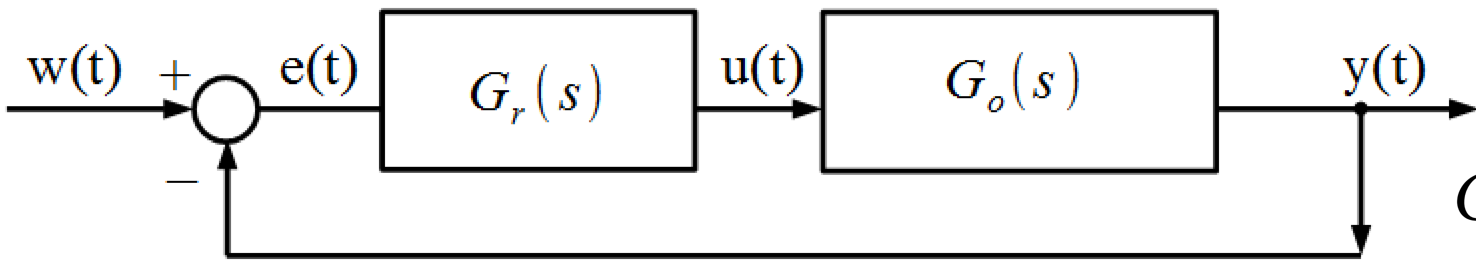
$$a_4=T\,,\ a_3=2T+1,$$
$$a_2=T+2\,,\ a_1=T+5\,,\ a_0=1$$

$$a_4>0\,,\ a_3>0\,,\ a_2>0\,,\ a_1>0\,,\ a_0>0 \rightarrow T>0$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

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$$G_z(s)=\frac{G_r G_o}{1+G_r G_o G_p}=\frac{4s}{a_4 s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

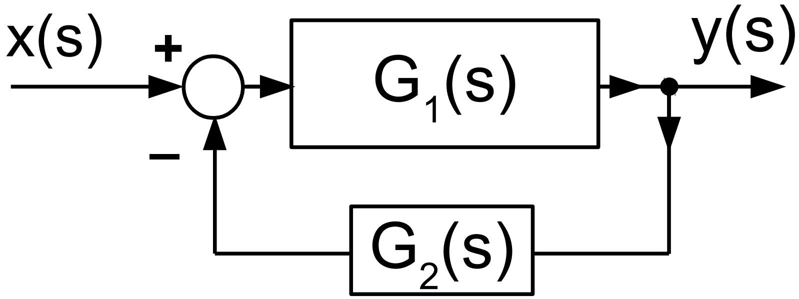
$$a_4=T\,,\ a_3=2T+1,$$
$$a_2=T+2\,,\ a_1=T+5\,,\ a_0=1$$

$$a_4>0\,,\ a_3>0\,,\ a_2>0\,,\ a_1>0\,,\ a_0>0\ \rightarrow\ T>0$$

$$\Delta_2=\begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \end{bmatrix}=T^2+2>0\quad T\in\mathbb{R}$$

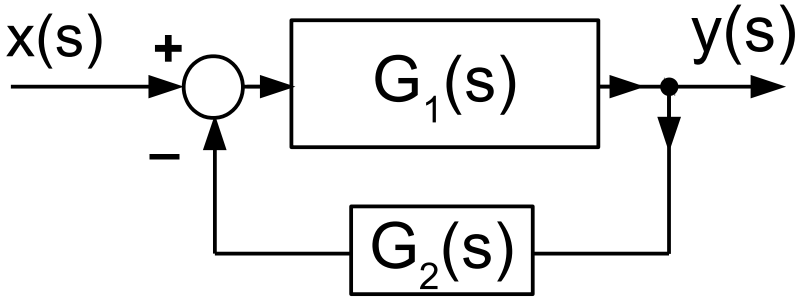
$$\Delta_3=\begin{bmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{bmatrix}=T^3+T^2-2T+9>0\ \rightarrow\ T>2.83$$

Nyquist stability criterion



$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

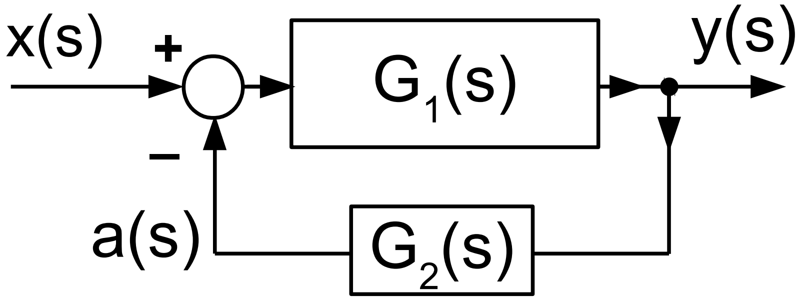
Nyquist stability criterion



$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

Unstable if: $G_1 G_2 = -1$

Nyquist stability criterion



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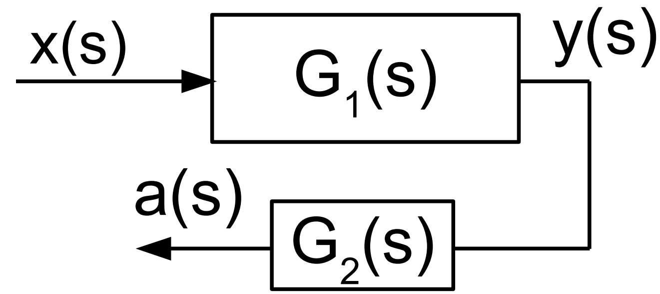
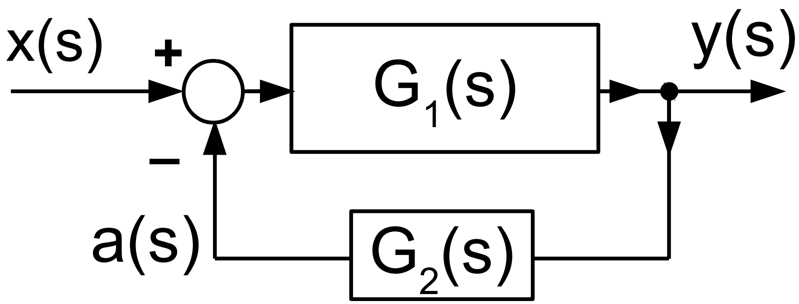
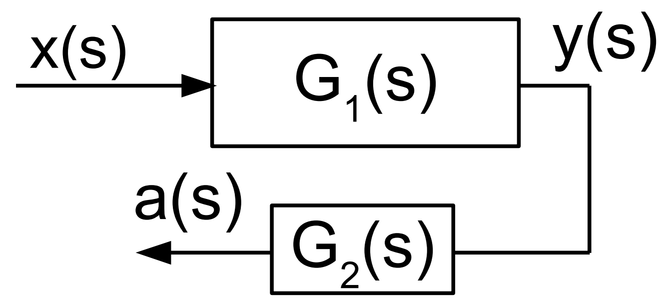


$$G_{open}(s) = \frac{a(s)}{x(s)} = G_1(s)G_2(s)$$

Unstable if: $\boxed{G_1 G_2} = -1$

Nyquist stability criterion





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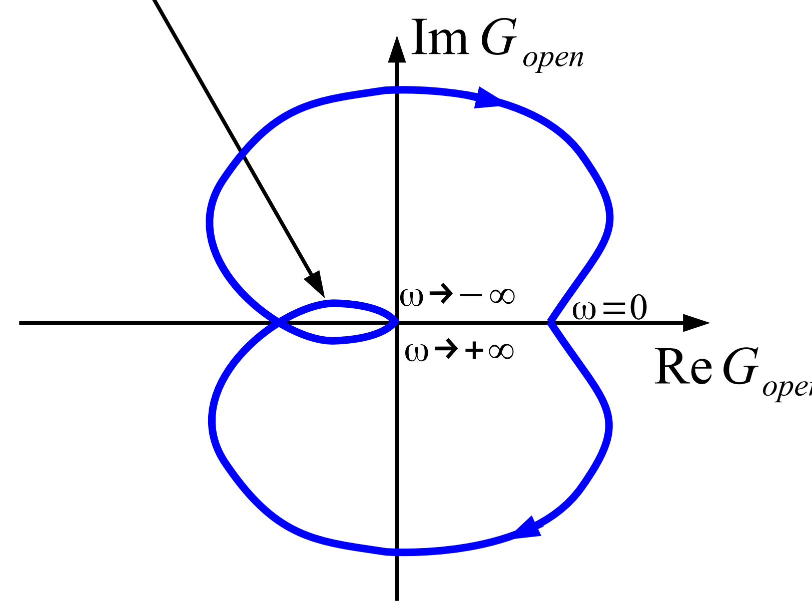
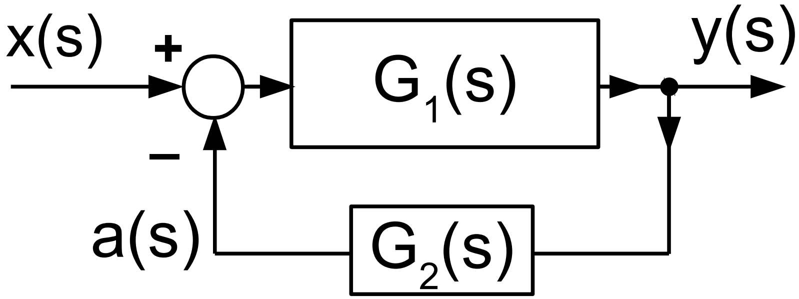


Nyquist stability criterion



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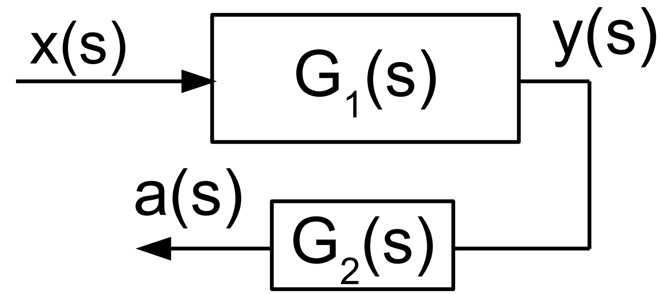

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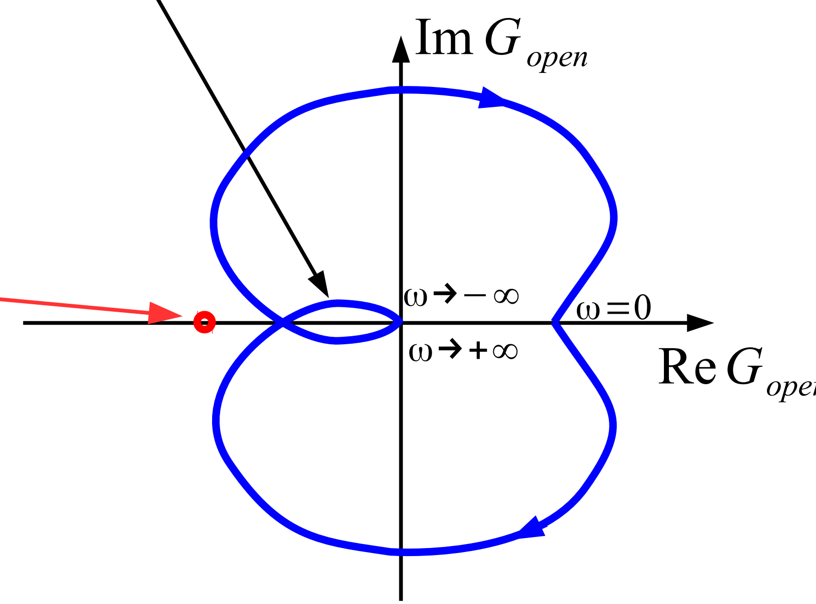
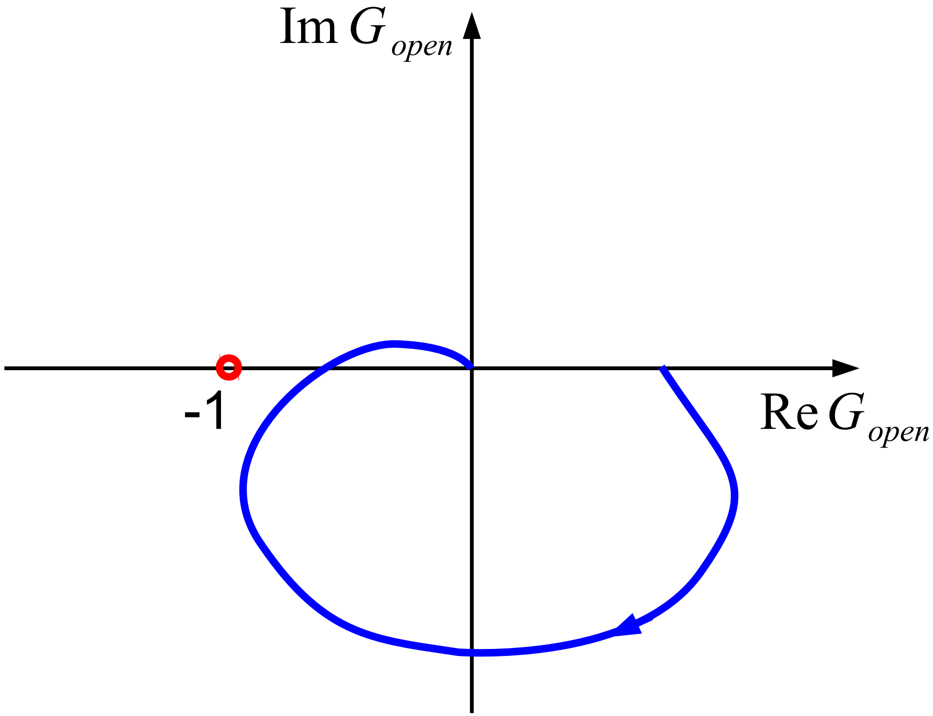
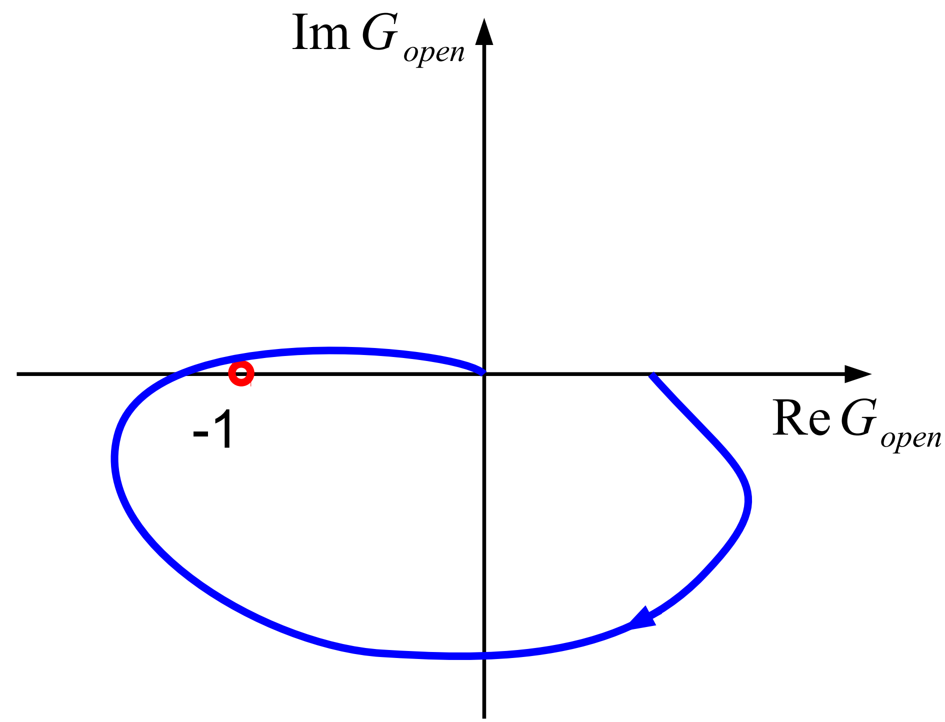


Nyquist stability criterion (particular)

The closed-loop system is stable if:
1) open-loop transfer function is stable AND
2) open-loop transfer function not enclosing the point (-1,j0).



stable

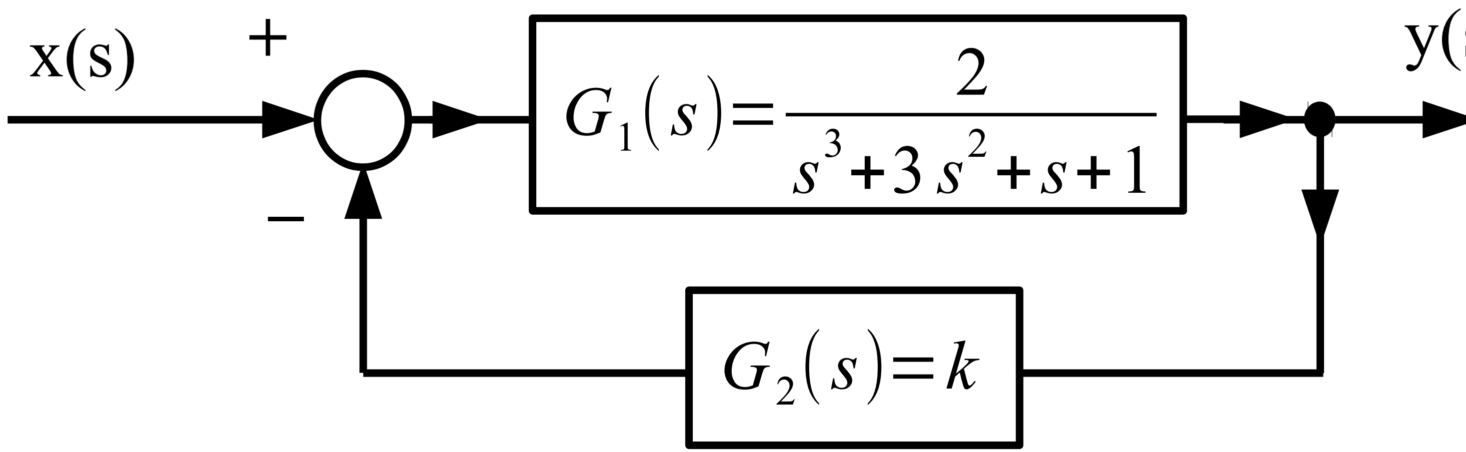


unstable

Nyquist criterion

Example 8

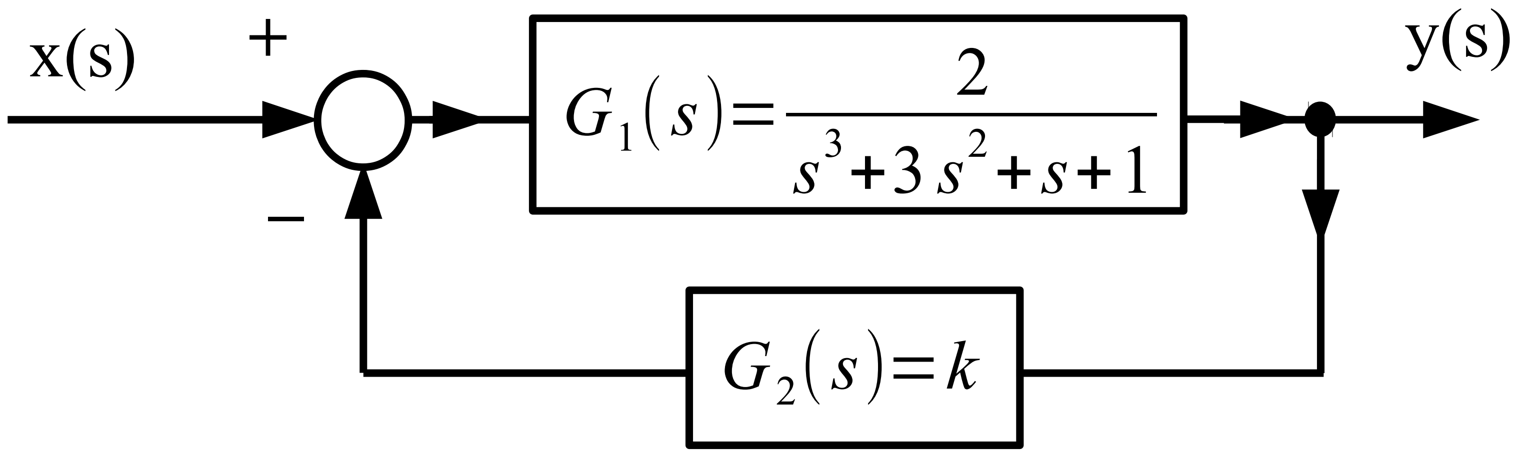
Choose k parameter to satisfy Nyquist criterion



Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion

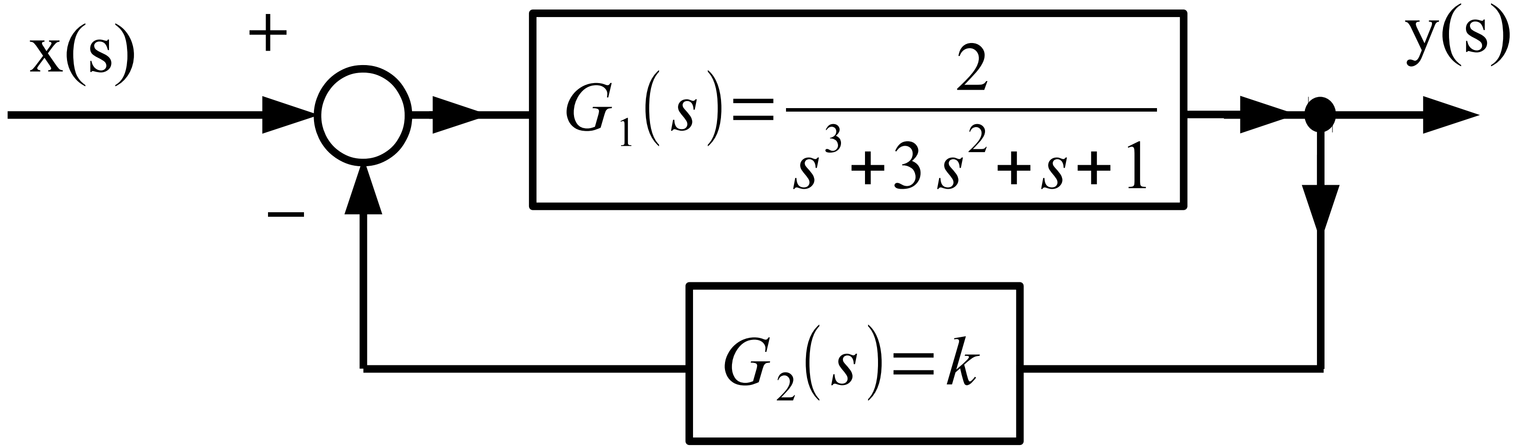


$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$ - stable from Hurwitz

Nyquist criterion

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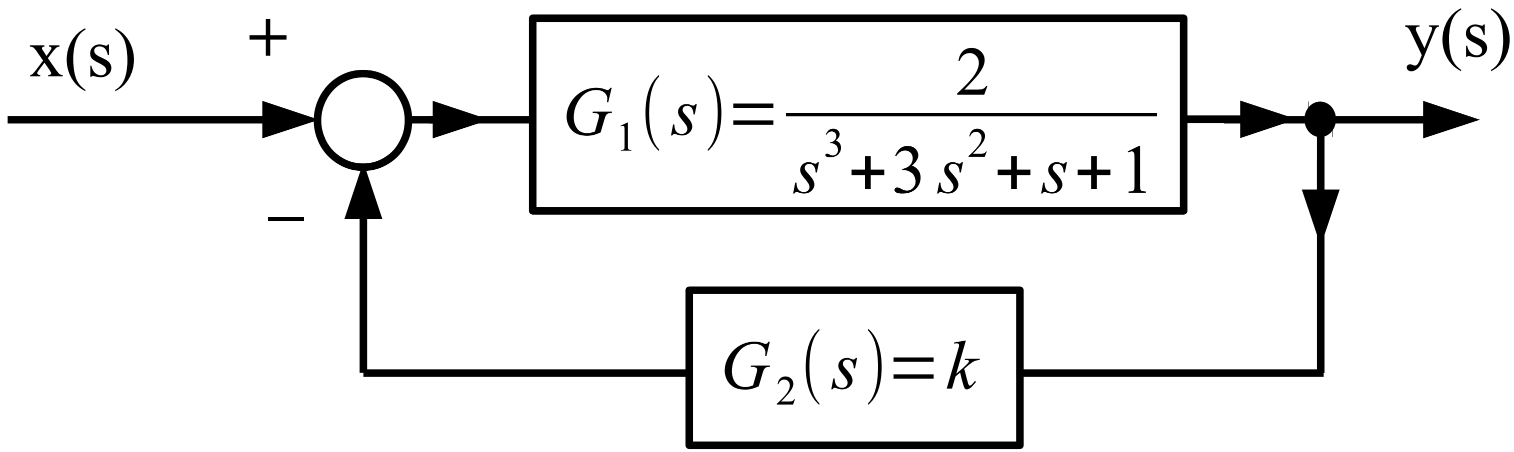

$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$ - stable from Hurwitz

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$

Nyquist criterion

Example 8

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$

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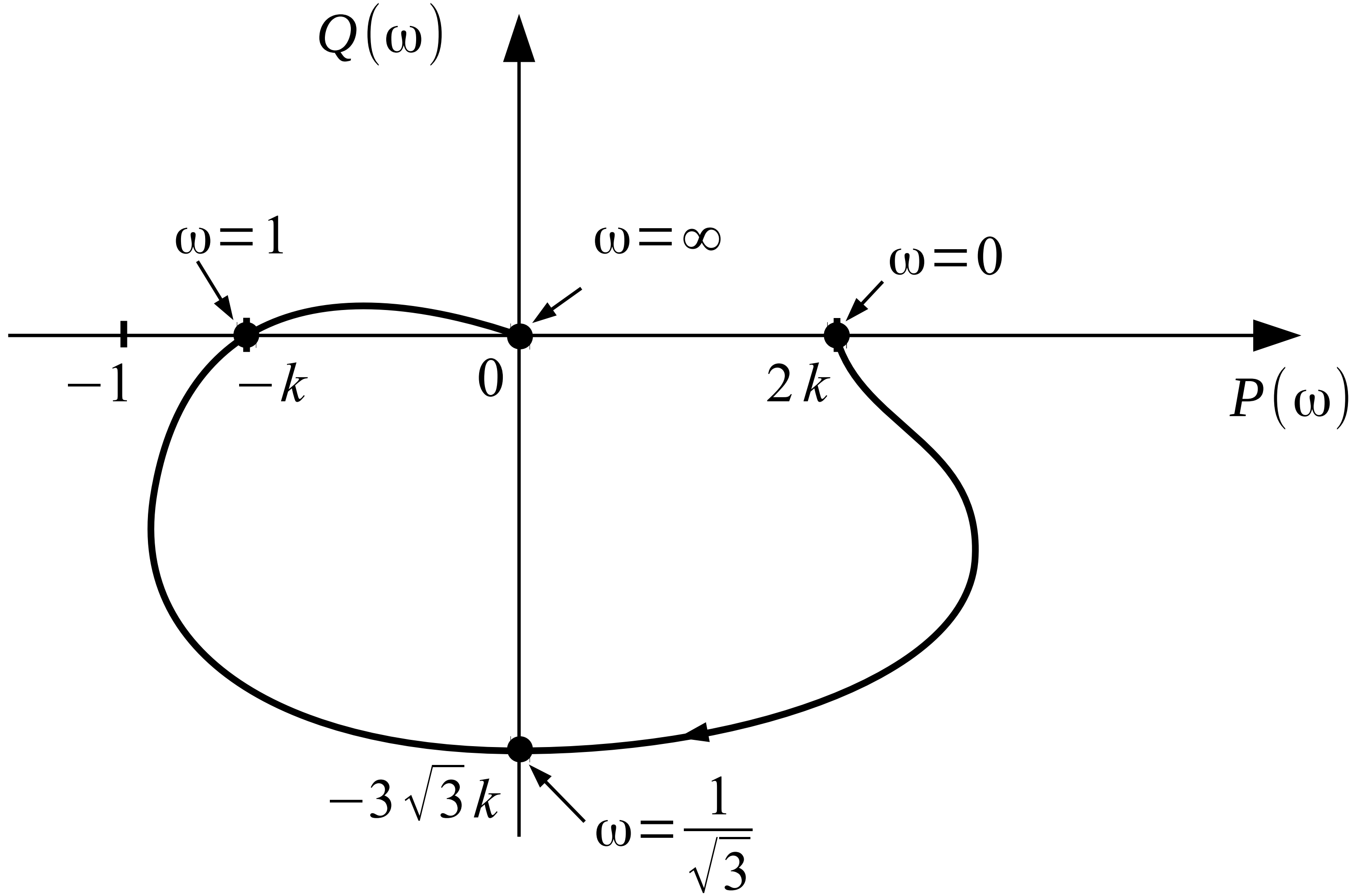

Nyquist criterion

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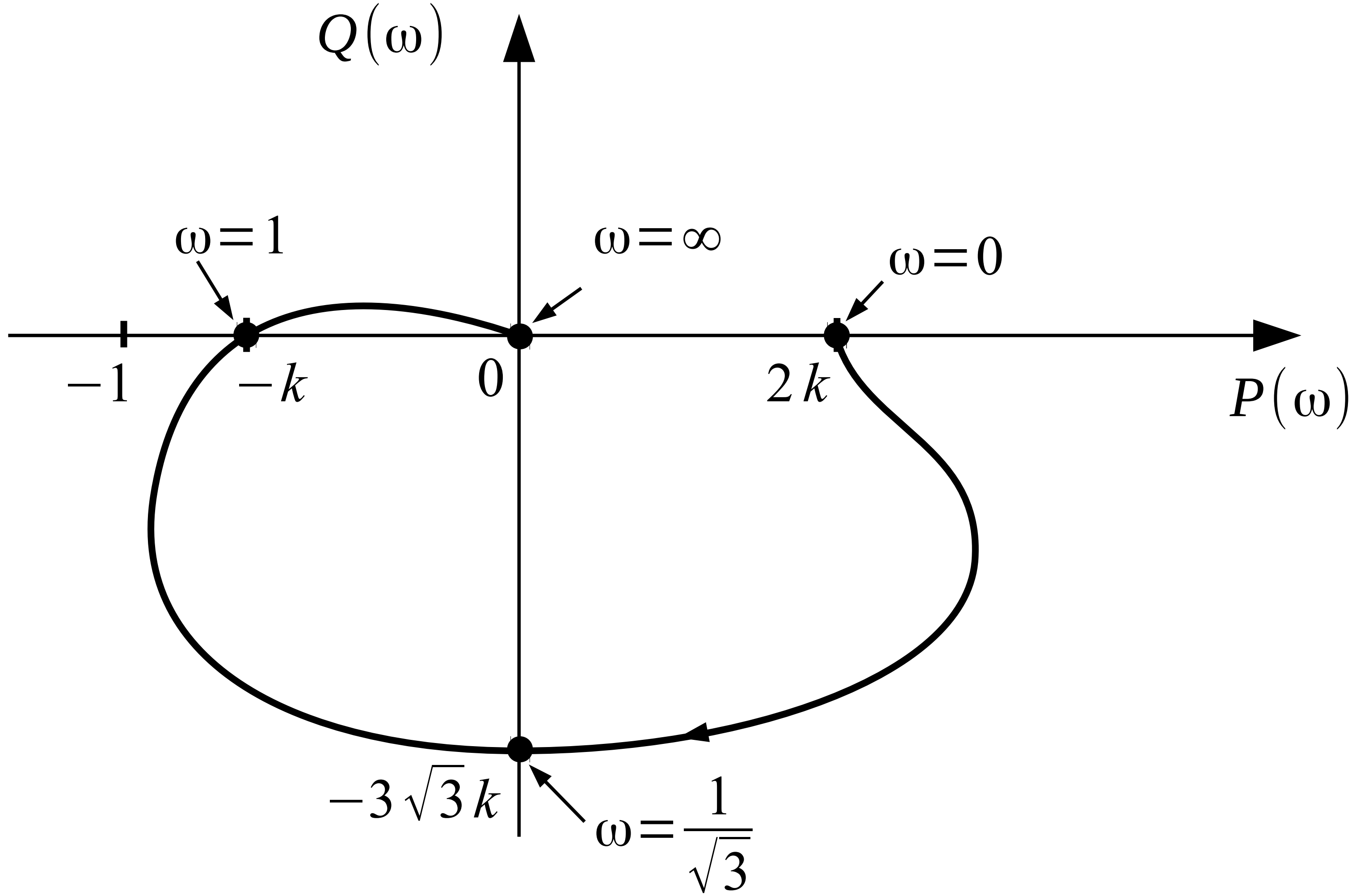
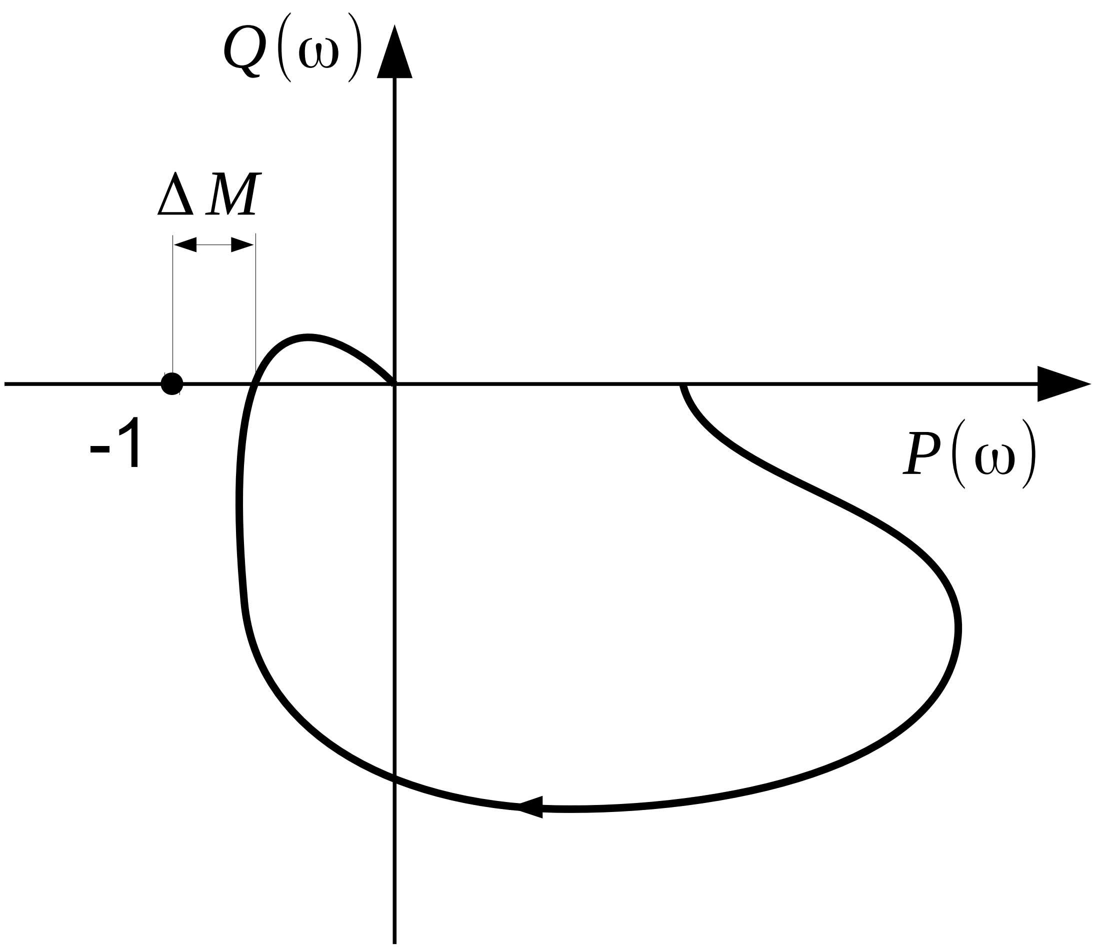
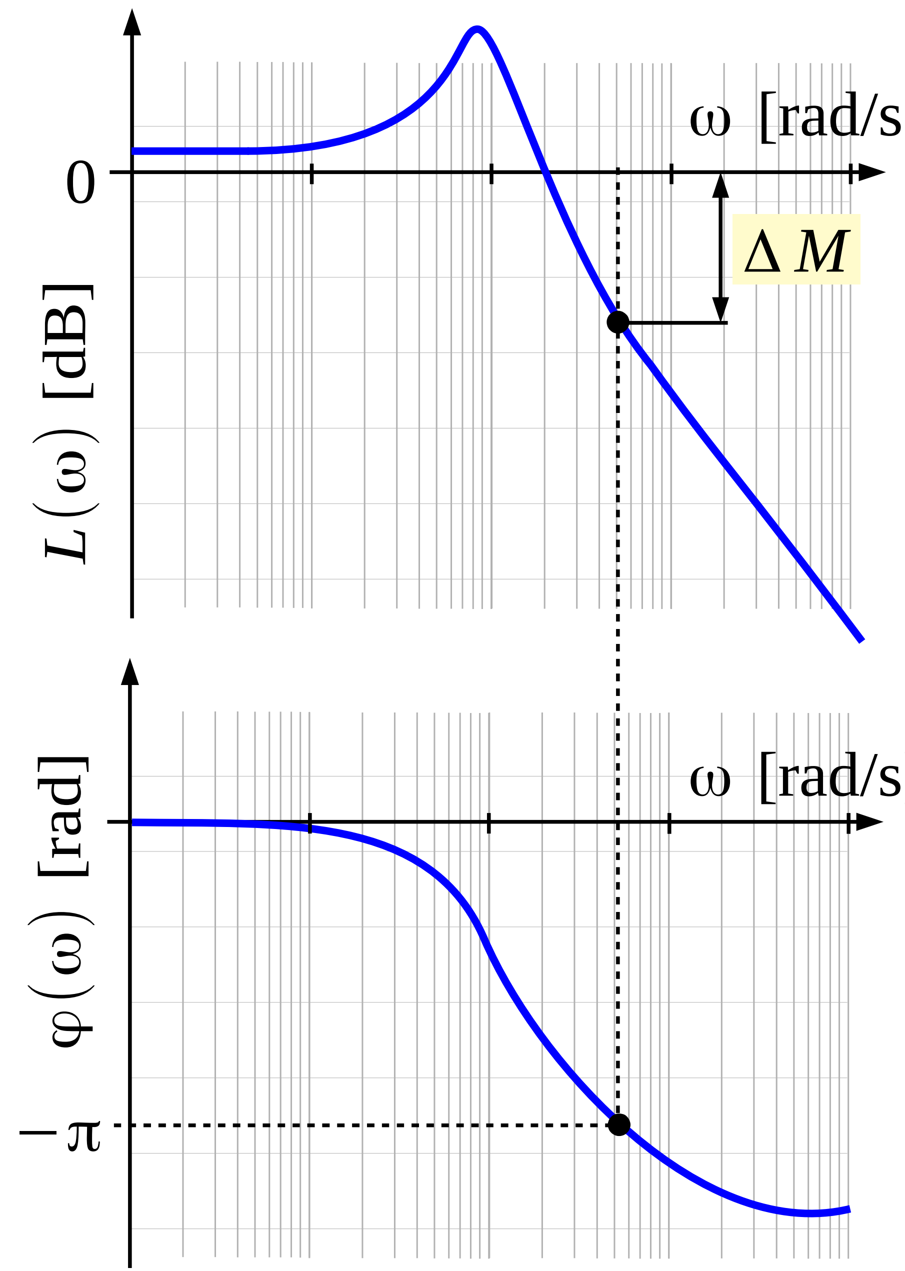


closed-loop system stable for 0 < k <1

Gain margin

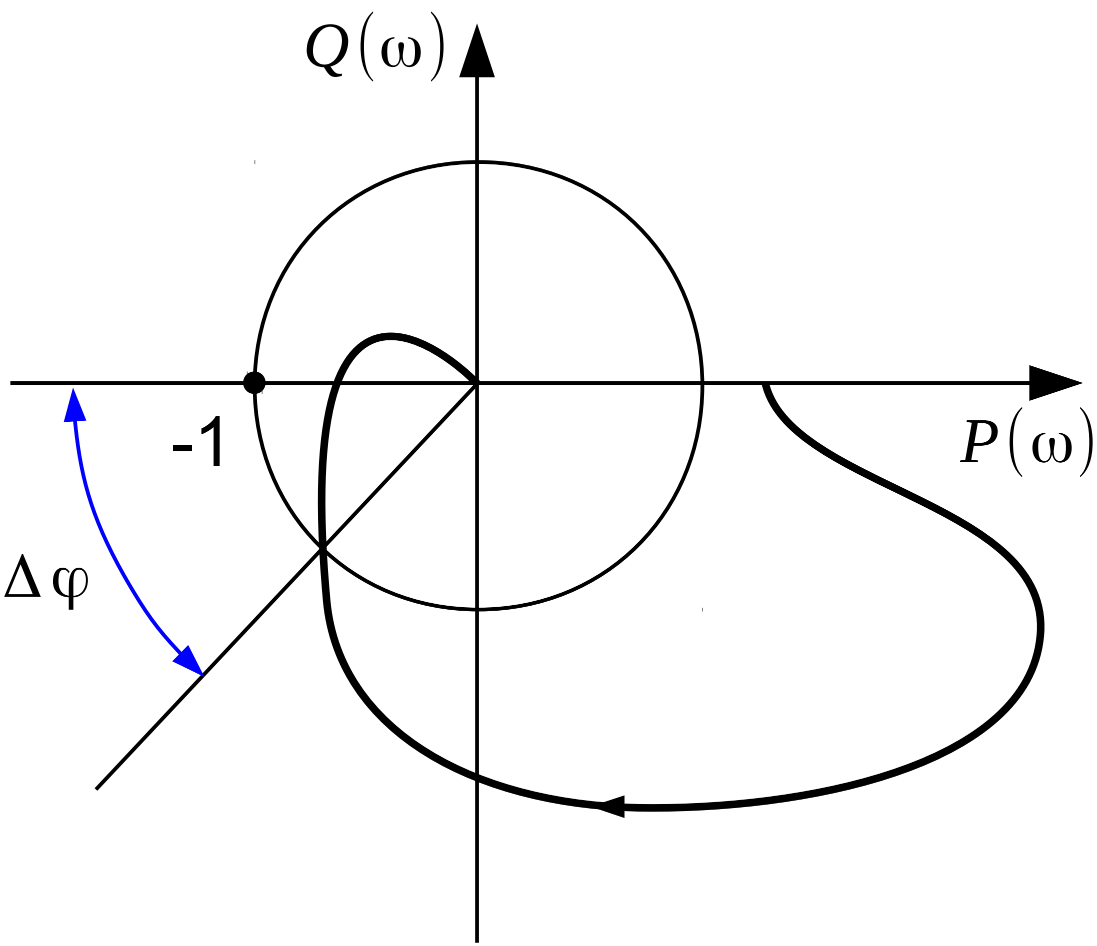
Closed-loop system will loose its stability if we add additional gain (in serial) greater or equals to gain margin.

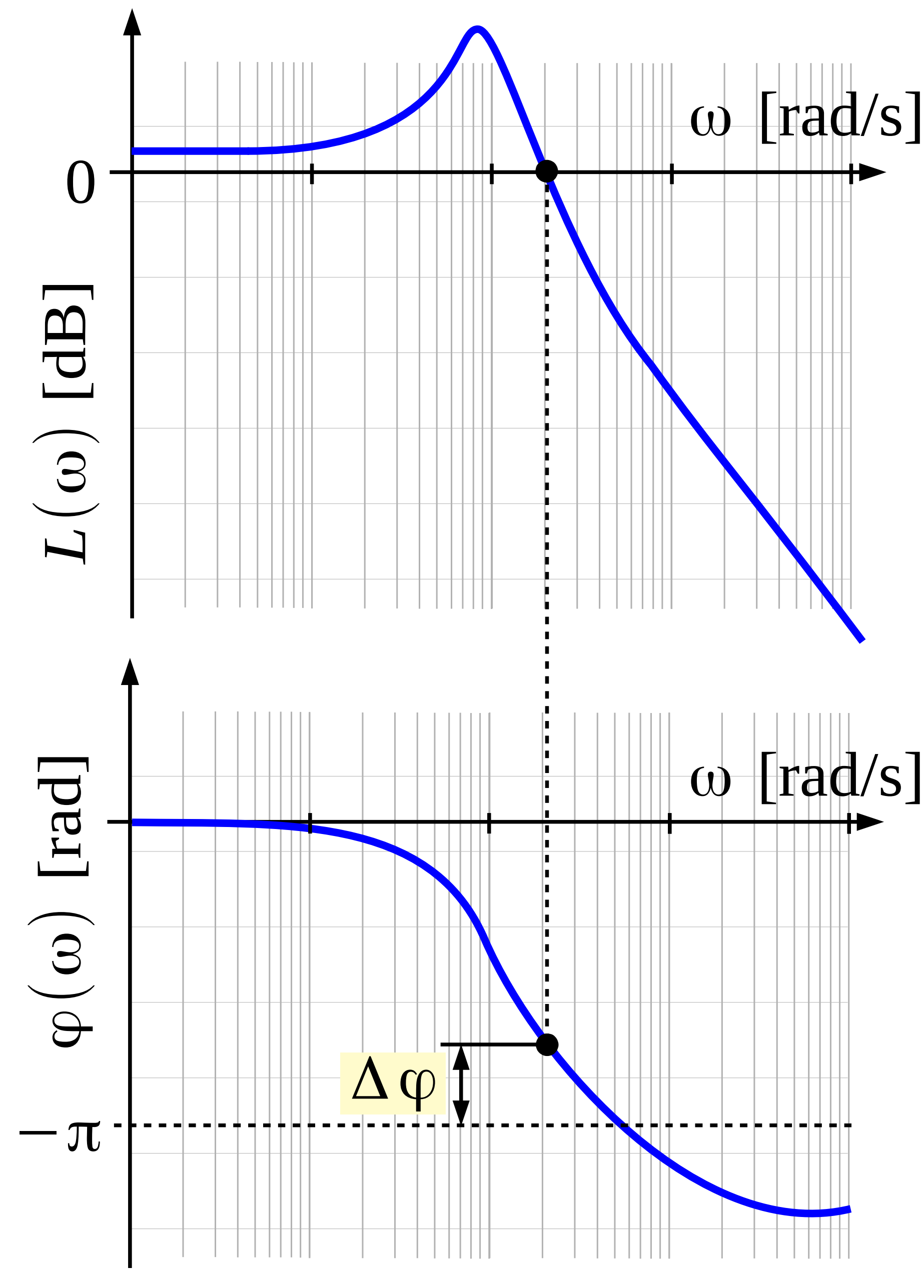




Phase margin

Closed-loop system will loose its stability if we add additional delay (in serial) greater or equals to phase margin.



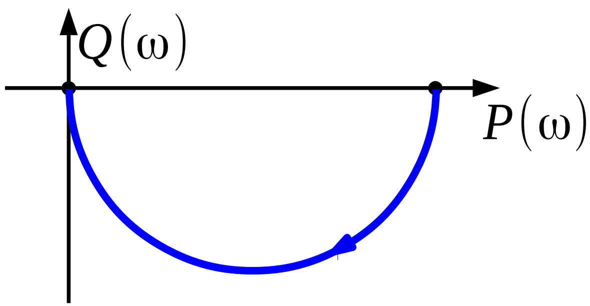


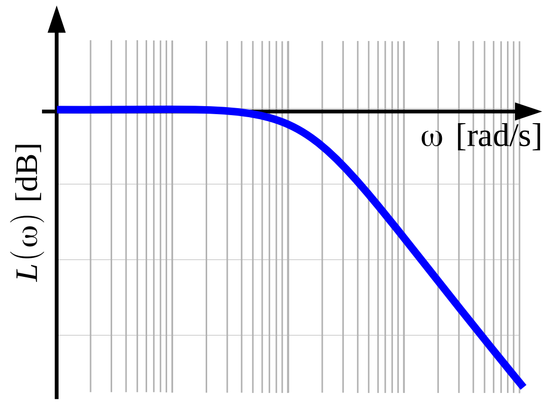
Stability vs Bode plot

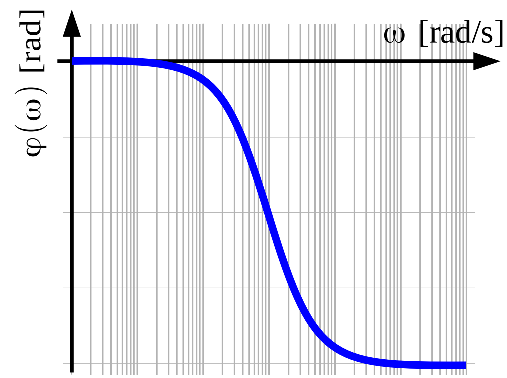
<u>Bodego plot (gain + delay) has no physical meaning if the system is unstable!</u>

Example:

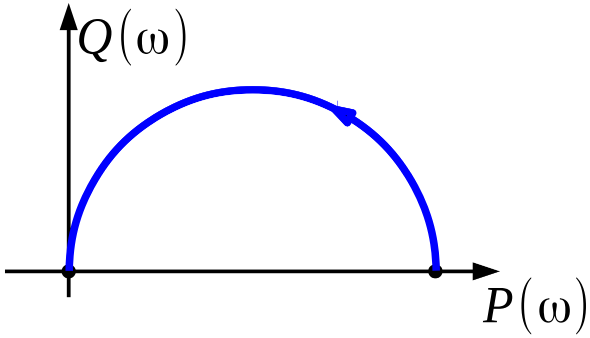
$$G(s)=\frac{1}{s+1}$$

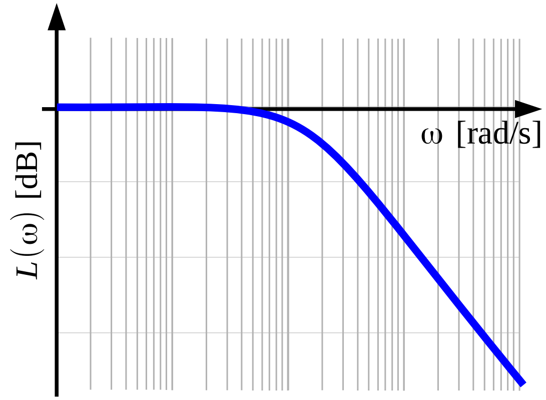


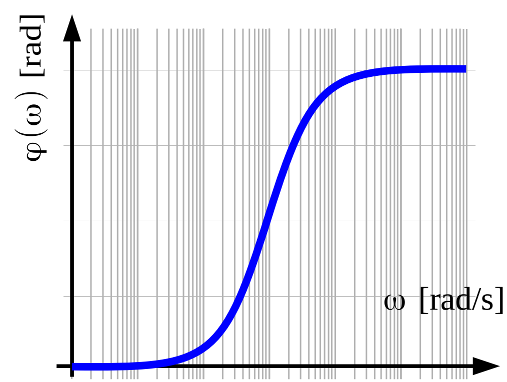




$$G(s)=\frac{1}{s-1}$$







Summing of Bode plots – example

$$G(s) = \frac{10}{s^2+s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$

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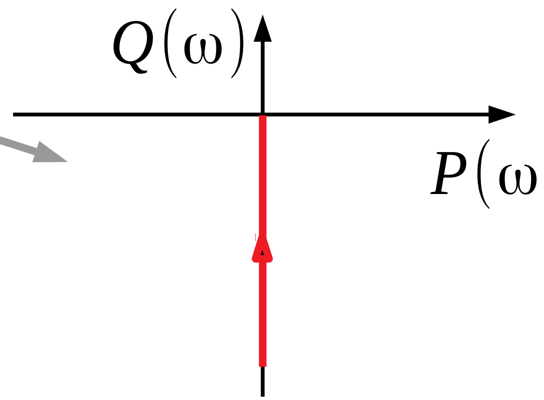

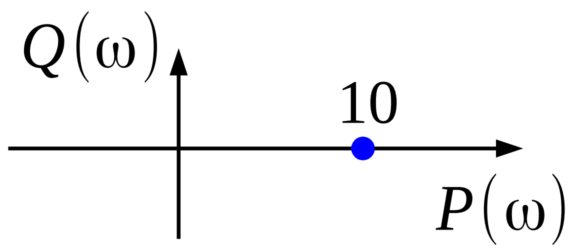

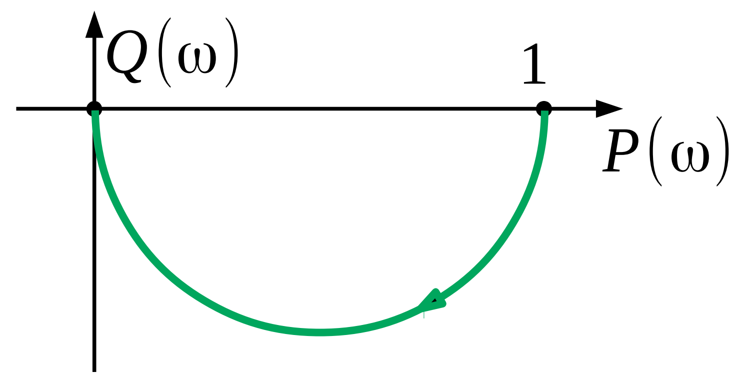

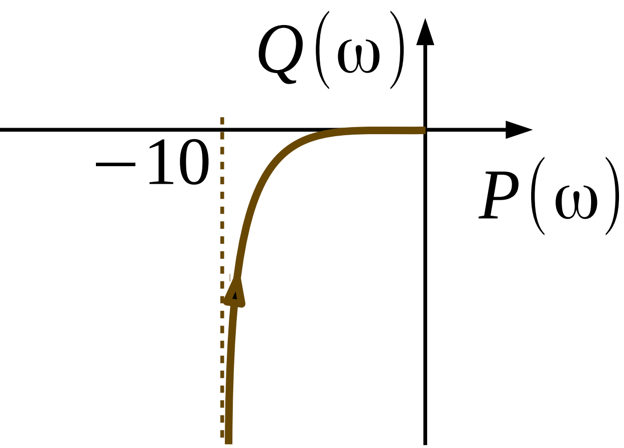

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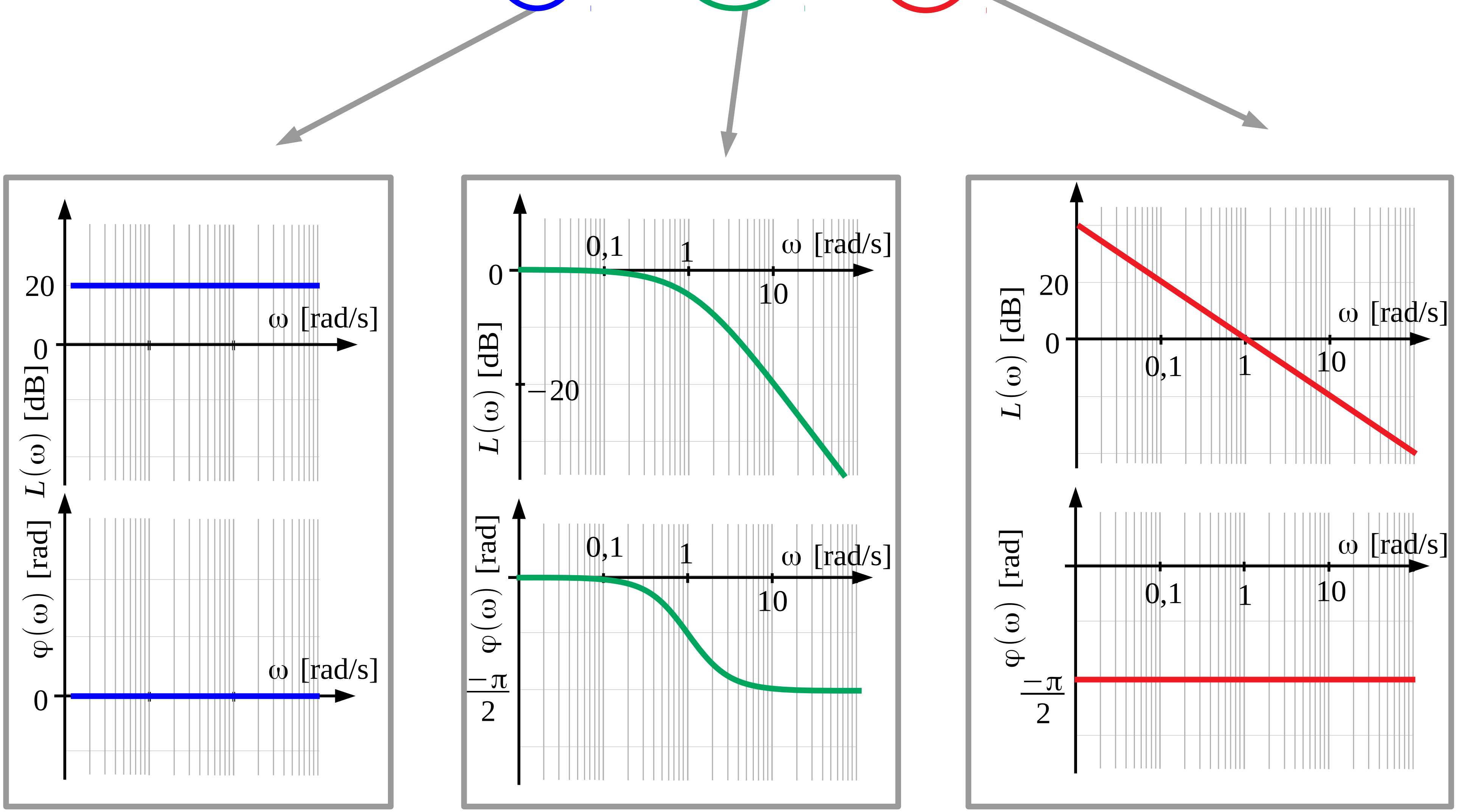

Summing of Bode plots – example

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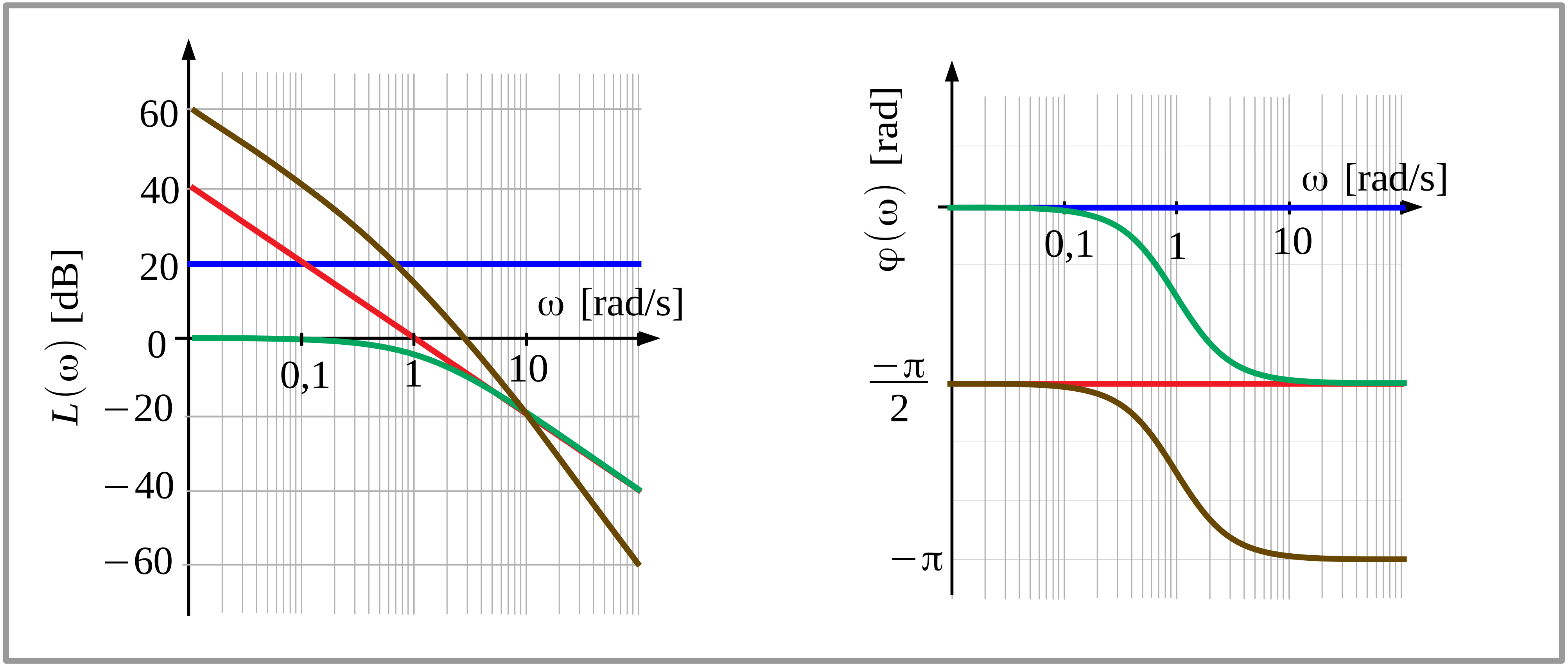
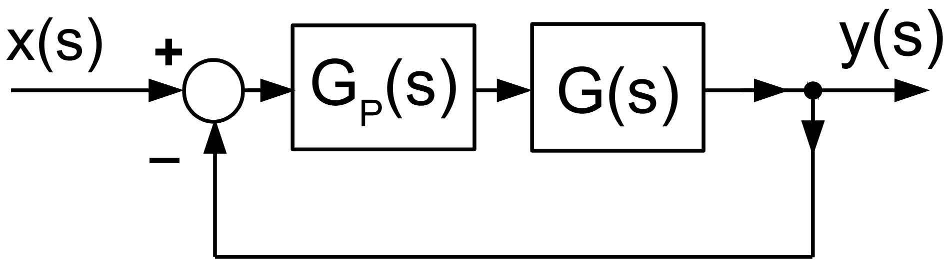


Nyquist stability criterion

control loop with P controller



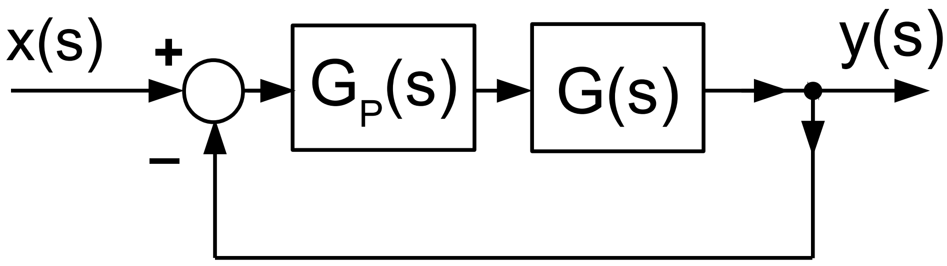
$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

Nyquist stability criterion

control loop with P controller

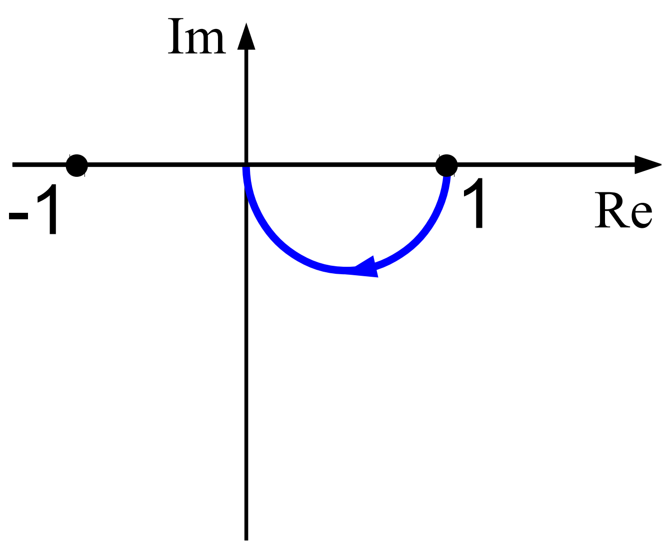


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

$$G_{opened}(s) = G_P(s)G(s)$$

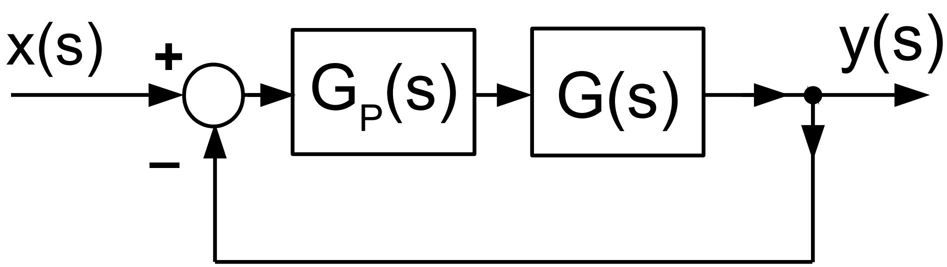
$$G_P(s) = k_P$$

$$G(s) = \frac{1}{Ts+1}$$



Nyquist stability criterion

control loop with P controller

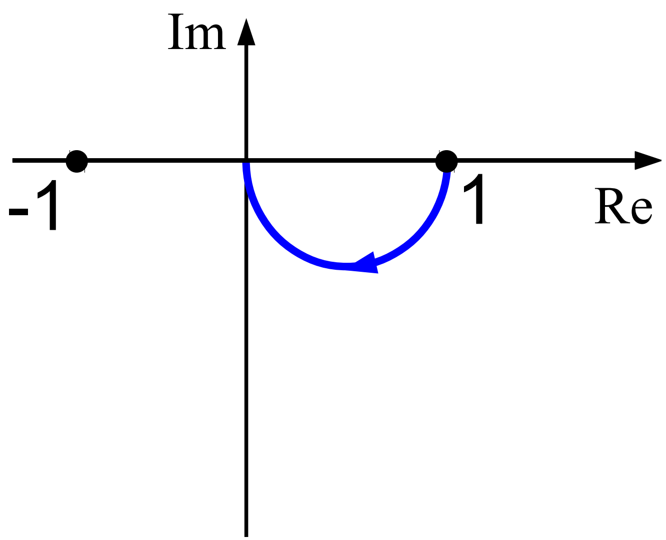


$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

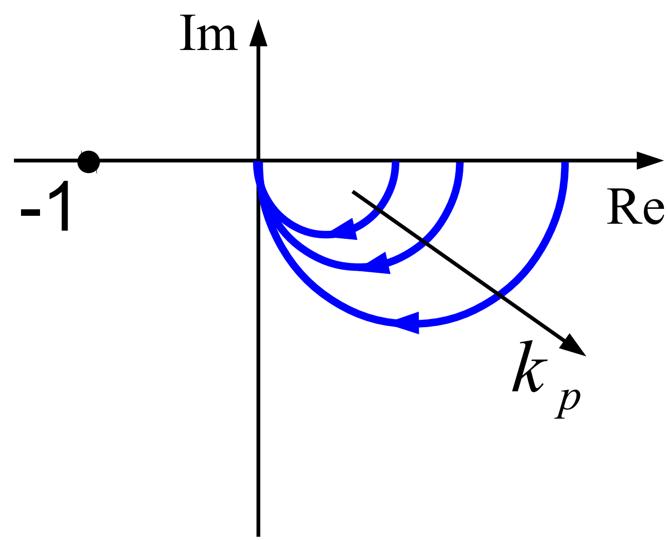
$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

$$G(s)=\frac{1}{Ts+1}$$

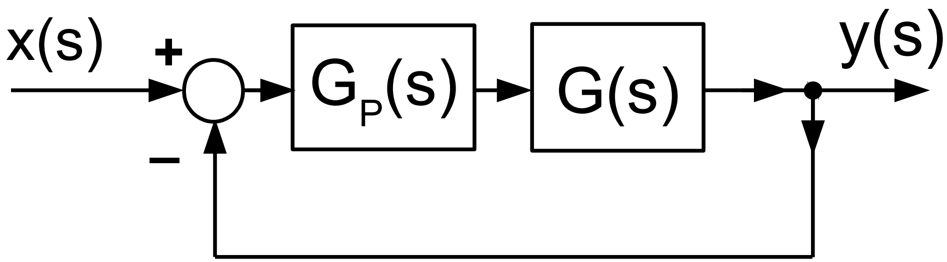


$$G_{opened}(s)=k_p\frac{1}{Ts+1}$$



Nyquist stability criterion

control loop with P controller



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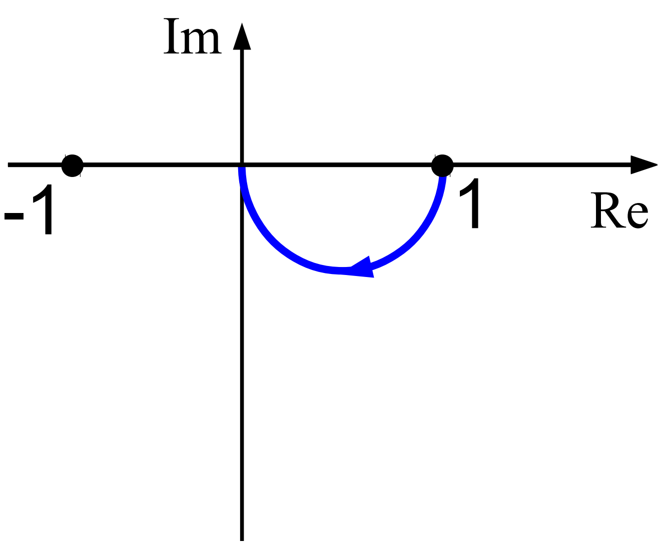

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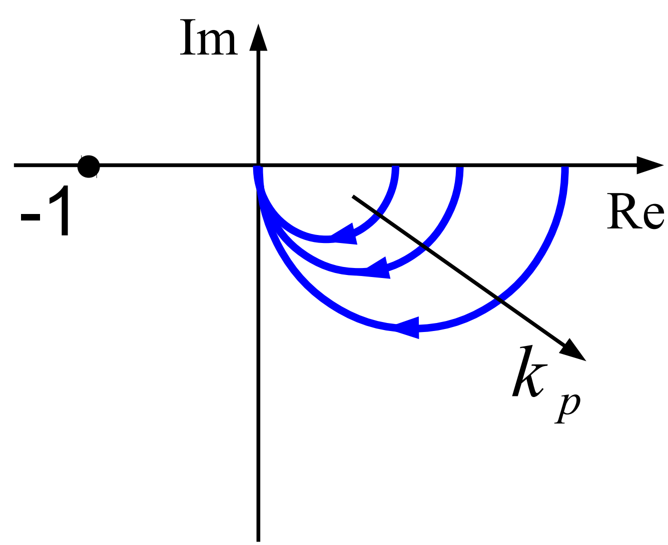


G_{opened} is always stable

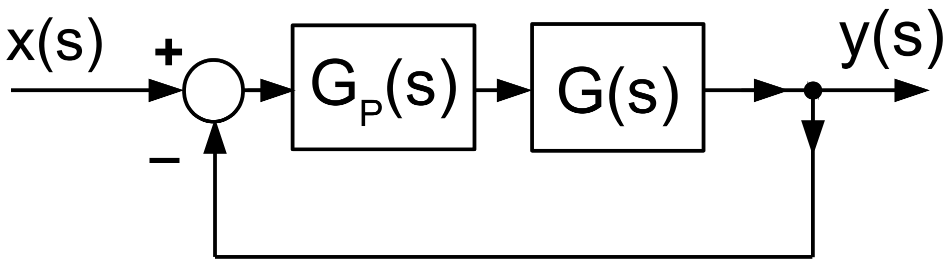
G_{closed} is always stable

steady state error ratio:

$$\frac{k_P}{k_P+1}$$

Nyquist stability criterion

control loop with P controller



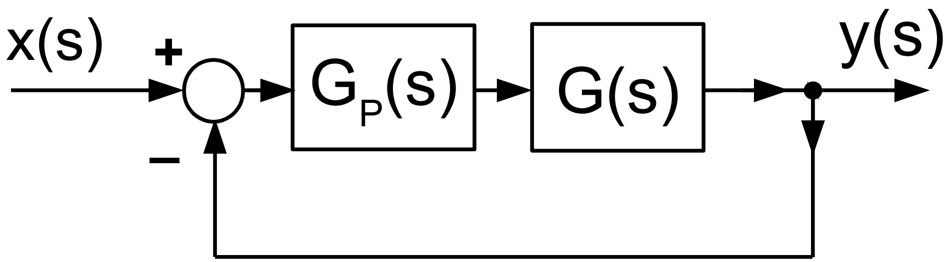
$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

Nyquist stability criterion

control loop with P controller

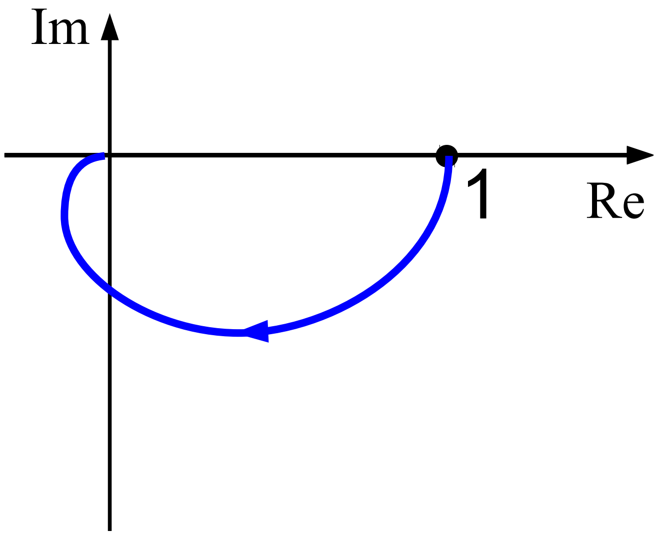


$$G_{closed}(s)=\frac{G_P(s)\,G(s)}{1+G_P(s)\,G(s)}$$

$$G_{opened}(s)=G_P(s)\,G(s)$$

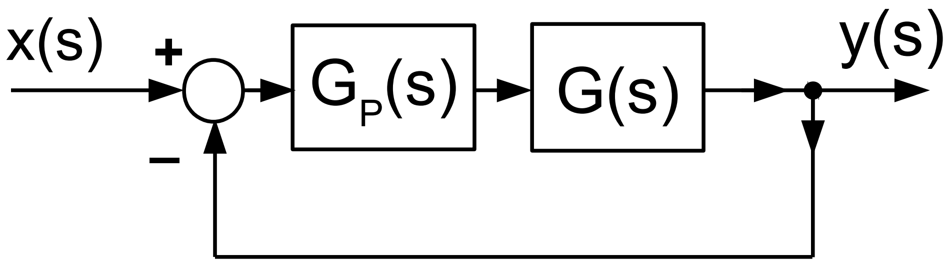
$$G_P(s)=k_P$$

$$G(s)=\frac{1}{T_1^2 s^2+T_2 s+1}$$



Nyquist stability criterion

control loop with P controller



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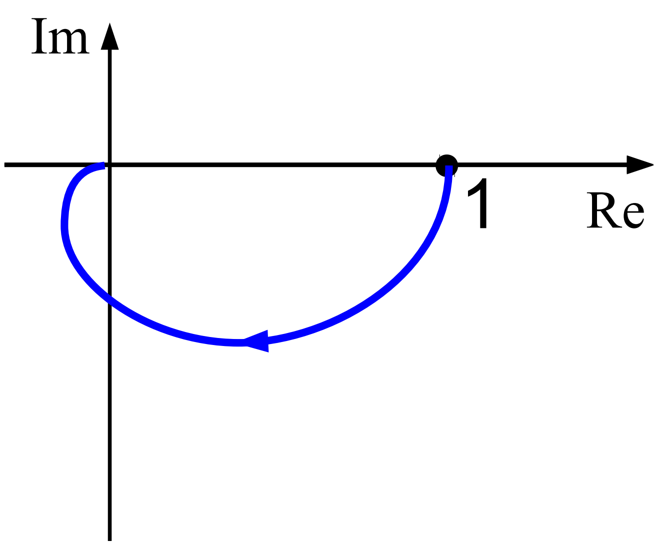
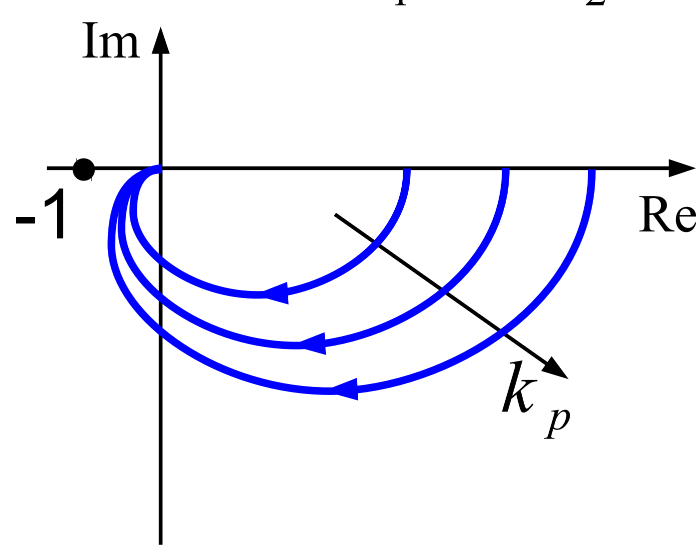
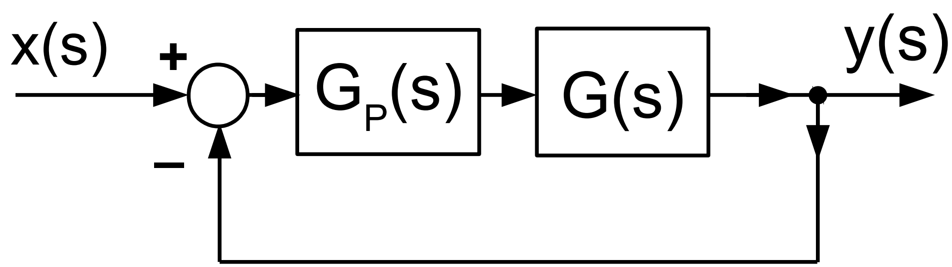


$$G_{opened}(s)=\frac{k_p}{T_1^2 s^2+T_2 s+1}$$



Nyquist stability criterion

control loop with P controller

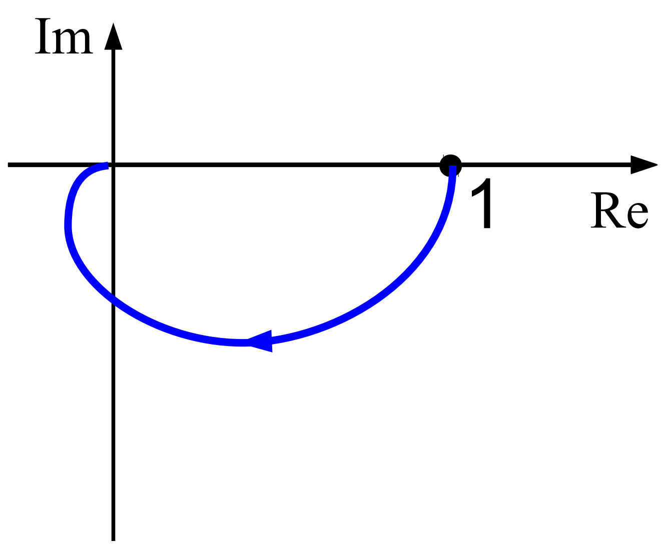


$$G_{closed}(s) = \frac{G_P(s)G(s)}{1 + G_P(s)G(s)}$$

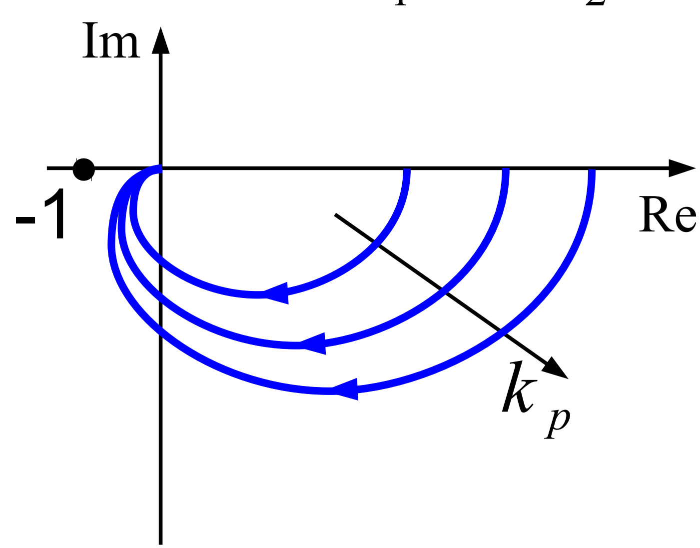
$$G_{opened}(s) = G_P(s)G(s)$$

$$G_P(s) = k_P$$

$$G(s) = \frac{1}{T_1^2 s^2 + T_2 s + 1}$$



$$G_{opened}(s) = \frac{k_p}{T_1^2 s^2 + T_2 s + 1}$$



G_{opened} is always stable

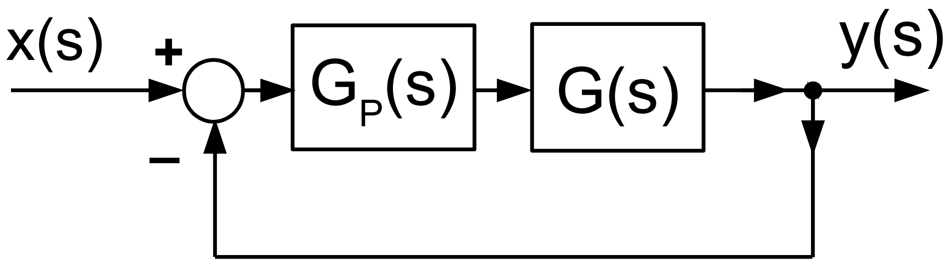
G_{closed} is always stable

steady state error ratio:

$$\frac{k_P}{k_P + 1}$$

Nyquist stability criterion

control loop with P controller



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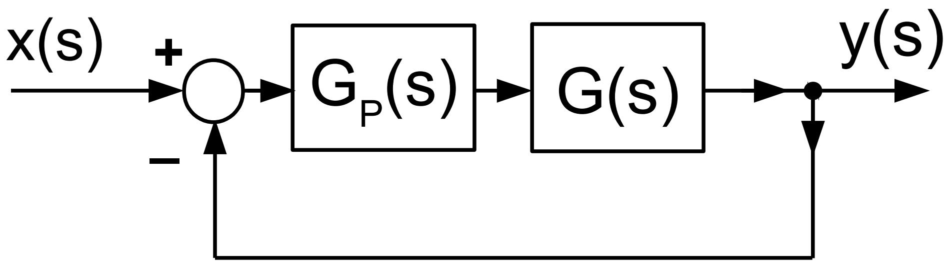
$$G_P(s)=k_P$$

Nyquist stability criterion

control loop with P controller



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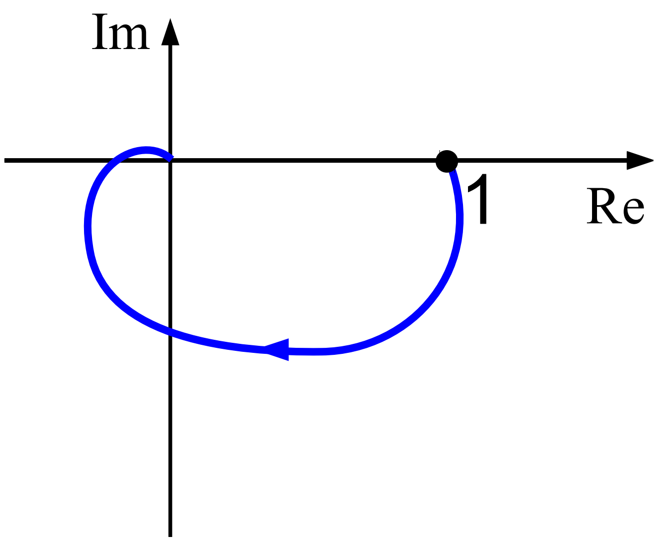
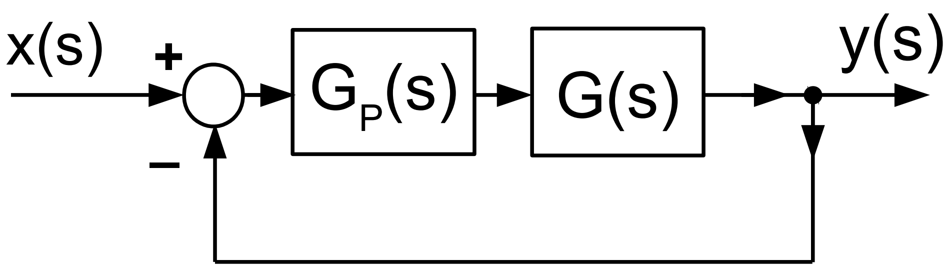


Nyquist stability criterion

control loop with P controller



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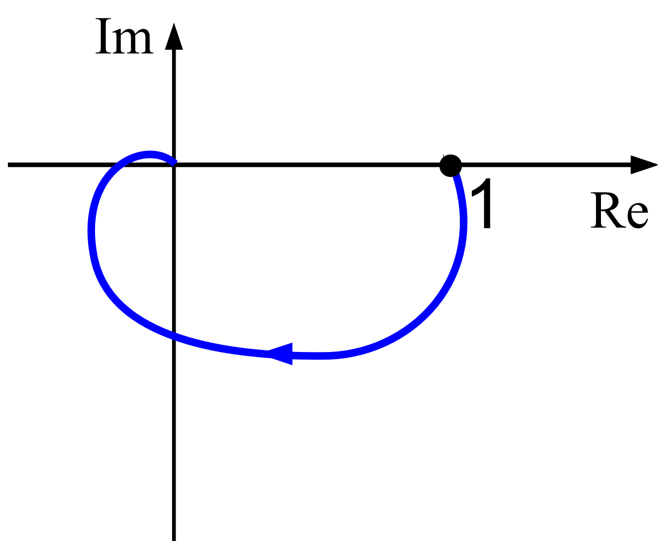
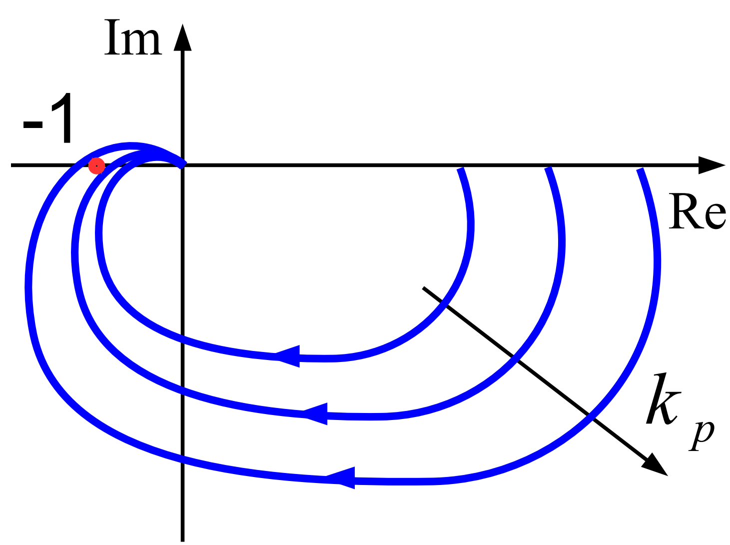
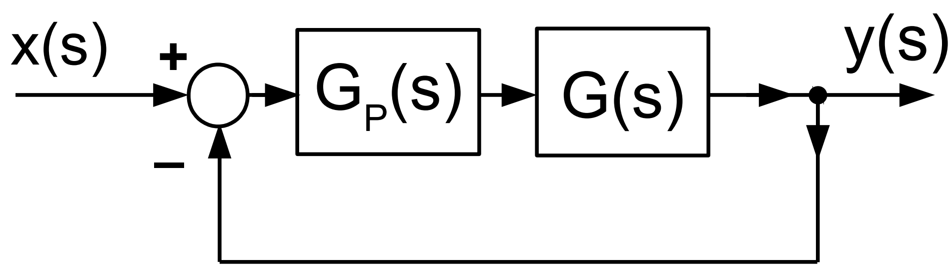


$$G_{opened}(s)=\frac{k_p}{T_3^2s^3+T_2^2s^2+T_1s+1}$$



Nyquist stability criterion

control loop with P controller



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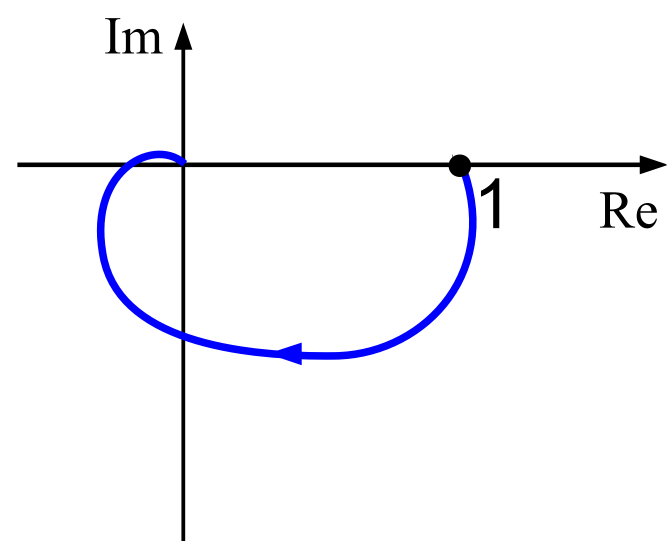
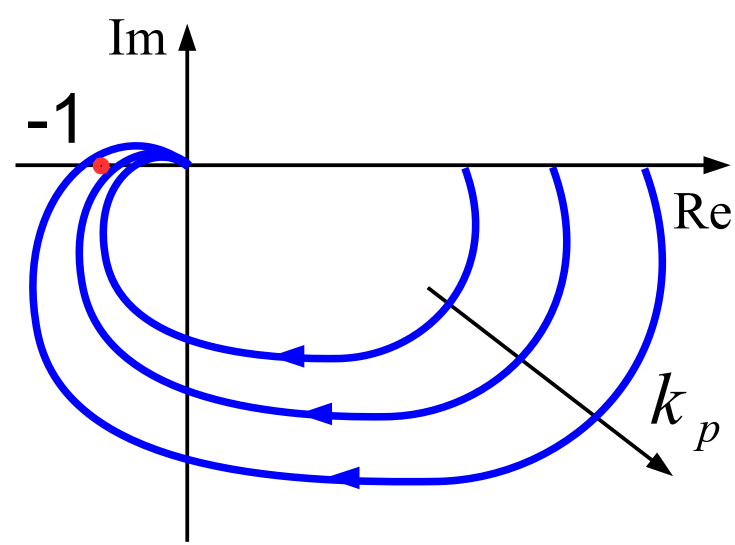


$$G_{opened}(s)=\frac{k_p}{T_3^2 s^3+T_2^2 s^2+T_1 s+1}$$



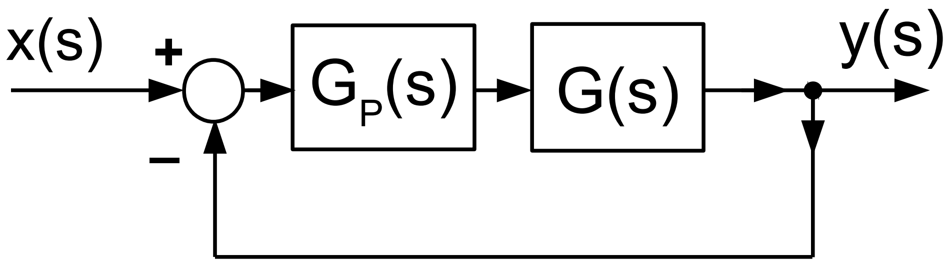
G_{closed} is not always stable

steady state error ratio:

$$\frac{k_P}{k_P+1}$$

Nyquist stability criterion

control loop with P controller



$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

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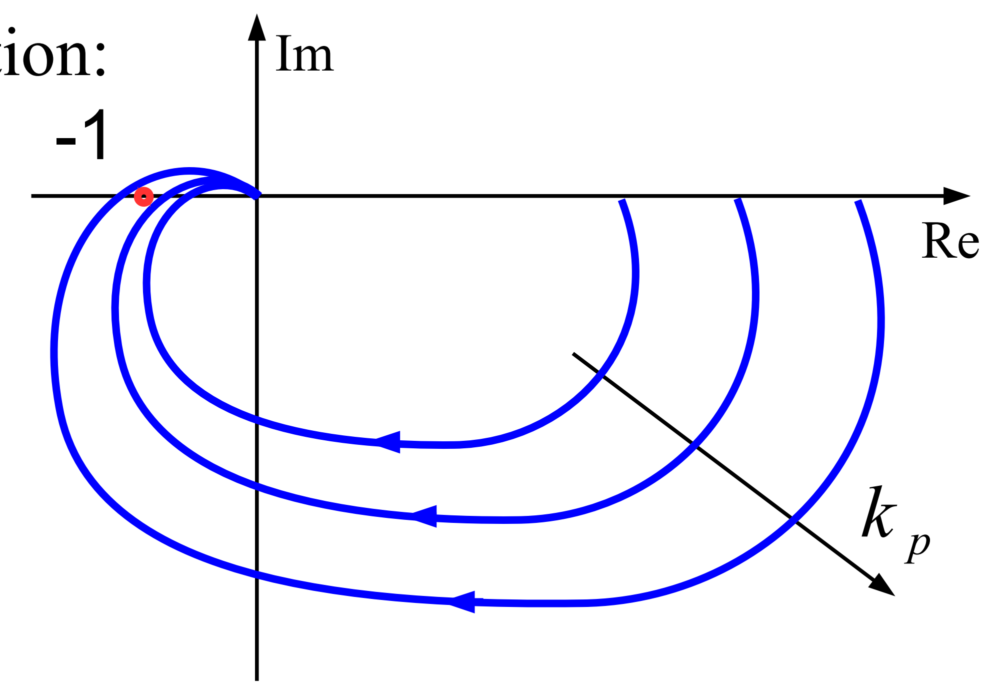
$$G_P(s)=k_P$$

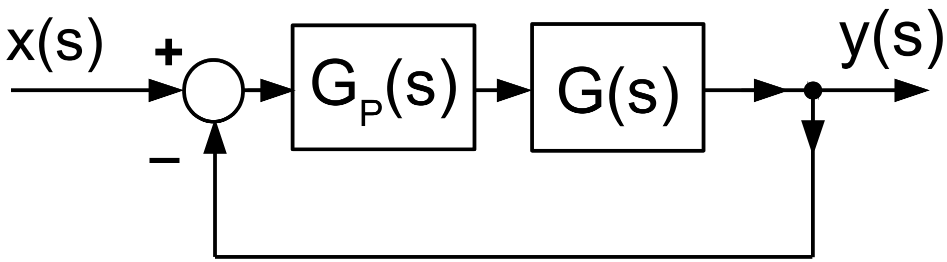
$$G(s)=\frac{1}{T_3^2 s^3+T_2^2 s^2+T_1 s+1}$$

conclusion for open-loop transfer function:
higher kp → lower steady state error
lower kp → better stability
(higher gain margin)



Nyquist stability criterion

control loop with PI controller

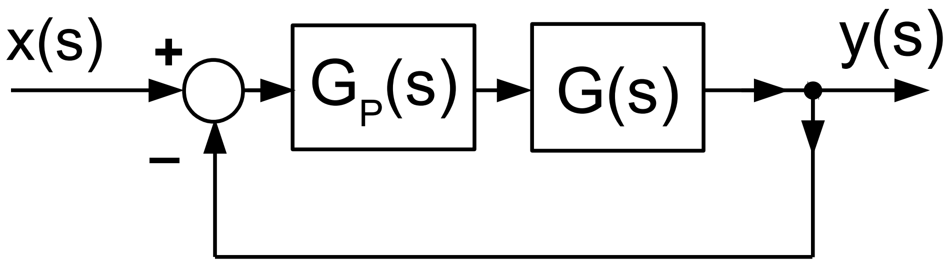


$$G_P(s) = k_P\left(1 + \frac{1}{T_i s}\right)$$

$$G_{opened}(s) = G_P(s)\,G(s)$$

Nyquist stability criterion

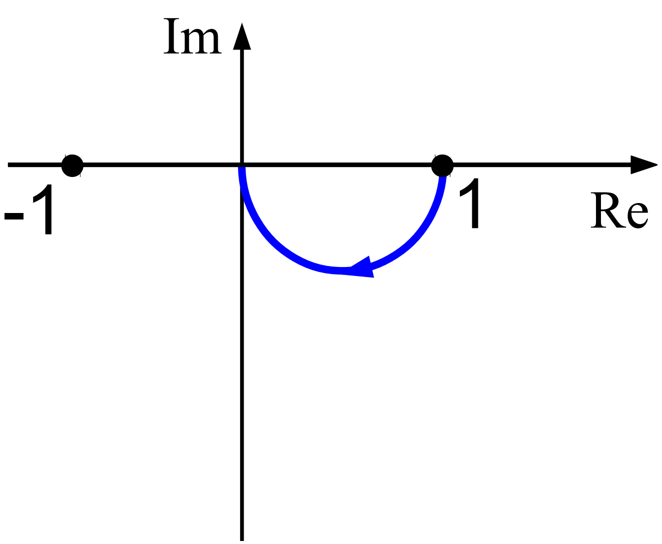
control loop with PI controller



$$G_P(s)=k_P\left(1+\frac{1}{T_i s}\right)$$

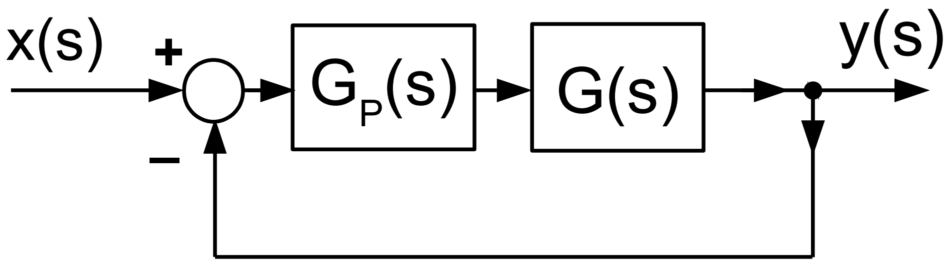
$$G_{opened}(s)=G_P(s)G(s)$$

$$G(s)=\frac{1}{Ts+1}$$



Nyquist stability criterion

control loop with PI controller



$$G_P(s)=k_P\left(1+\frac{1}{T_i s}\right)$$

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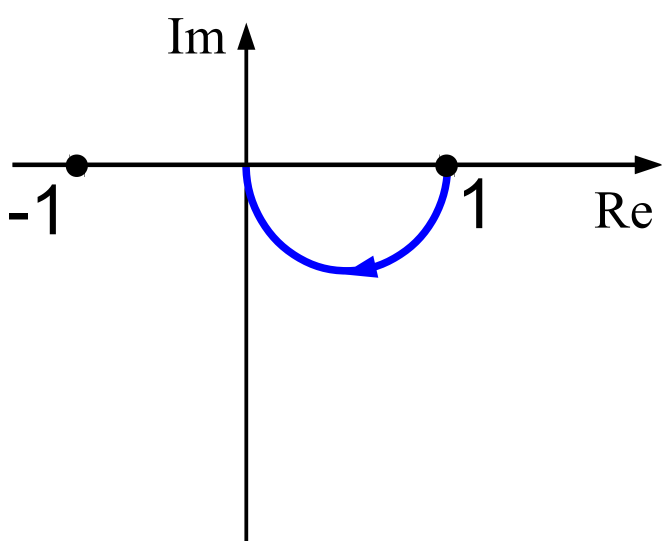
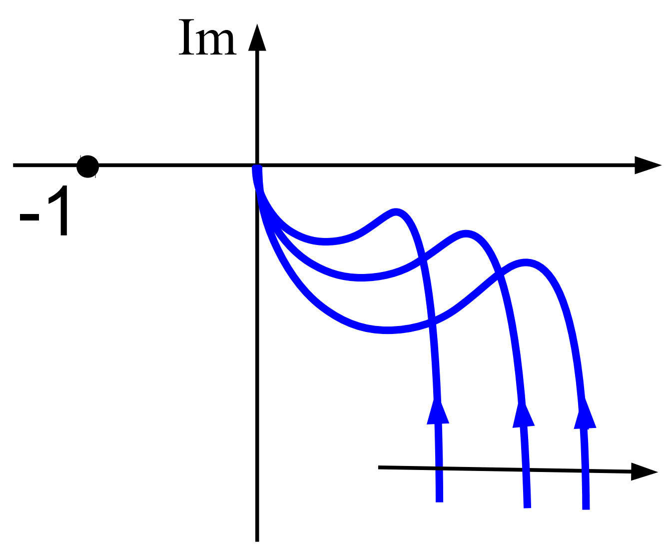
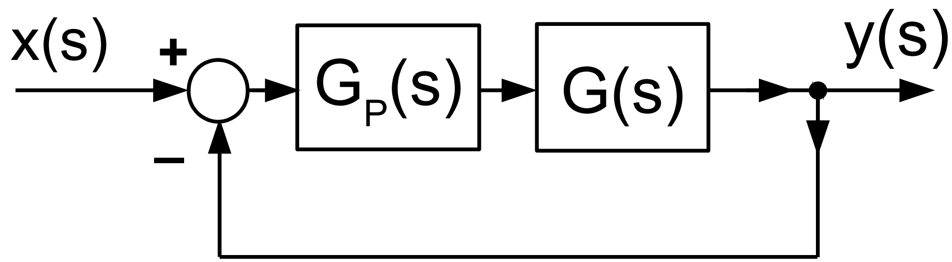


$$G_{opened}(s)=k_p^2\frac{sT_i^2+2T_i}{T_i^3 T s^2+T_i^2 s}$$



Nyquist stability criterion

control loop with PI controller



$$G_P(s) = k_P\left(1 + \frac{1}{T_i s}\right)$$

$$G_{opened}(s) = G_P(s)\,G(s)$$

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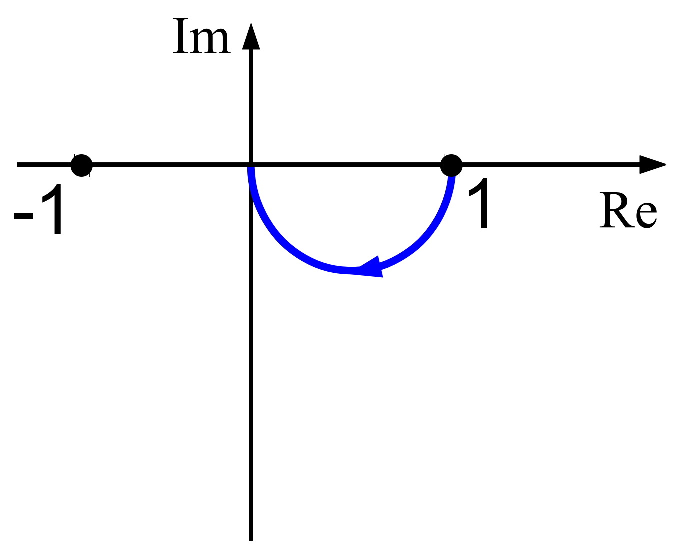
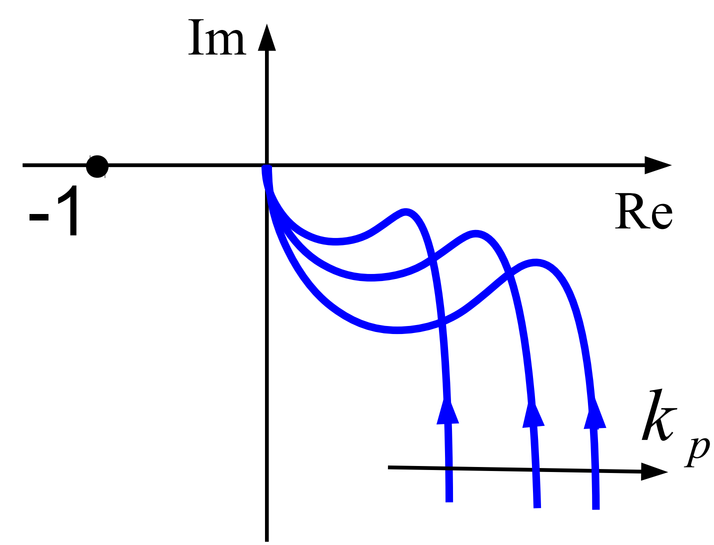


$$G_{opened}(s) = k_p^2 \frac{s\,T_i^2 + 2\,T_i}{T_i^3\,T\,s^2 + T_i^2\,s}$$



G_{opened} is stable,

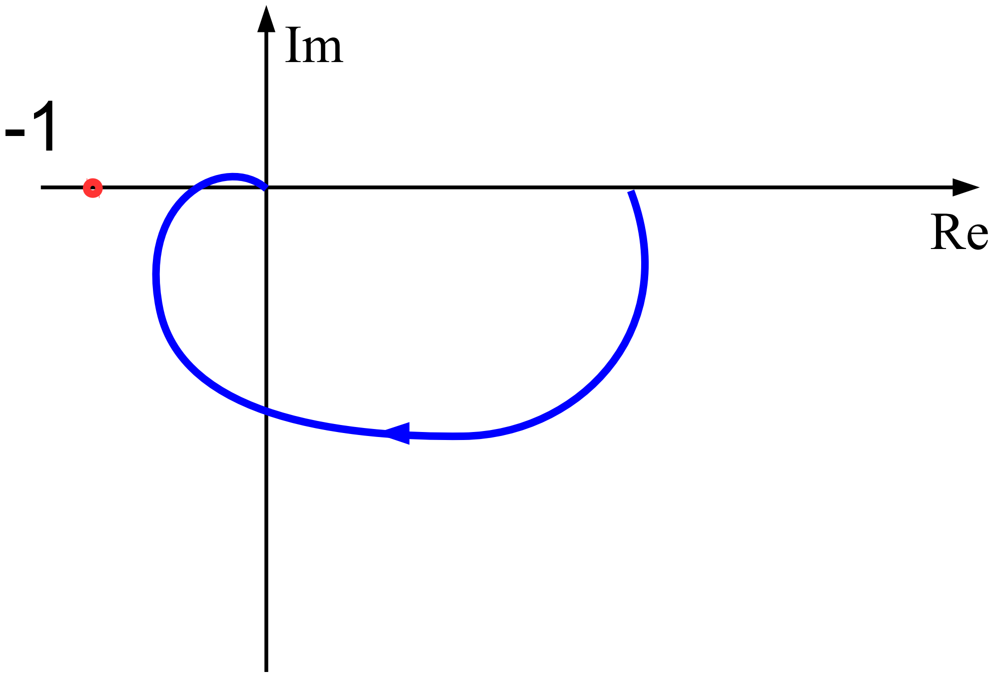
so G_{closed} is stable

$G_{opened}\,(\omega = 0) \rightarrow \infty$
so steady state
error $\rightarrow 0$

Correction of the system

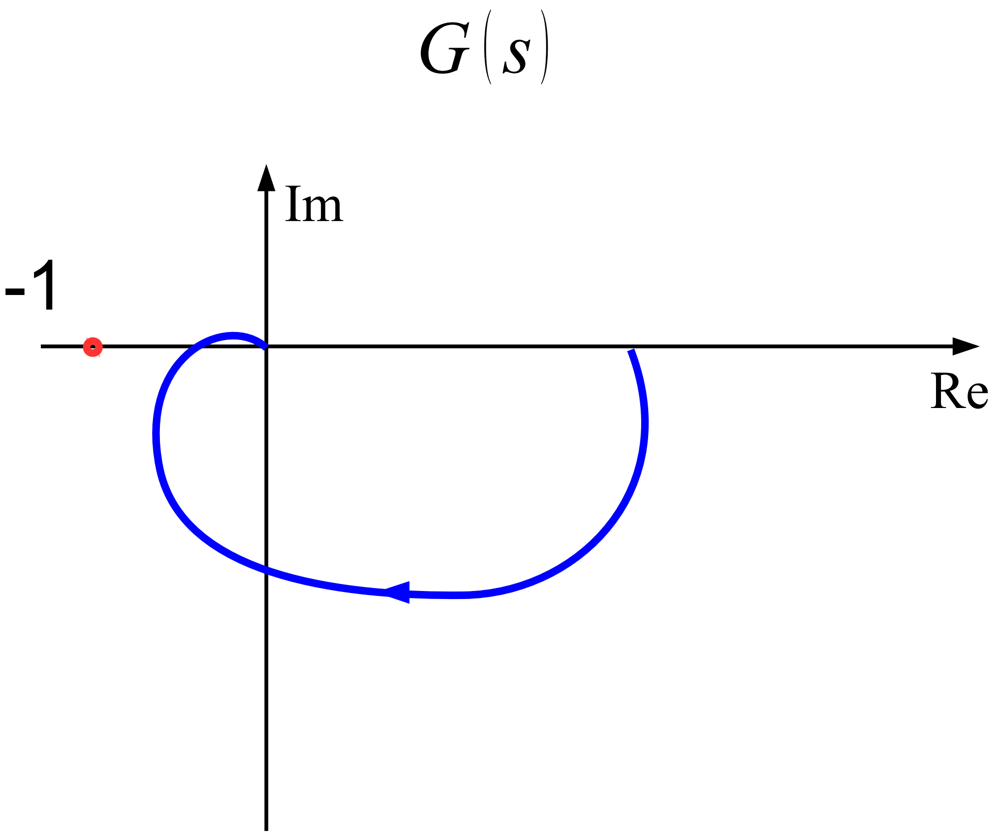
Correction by proportional term

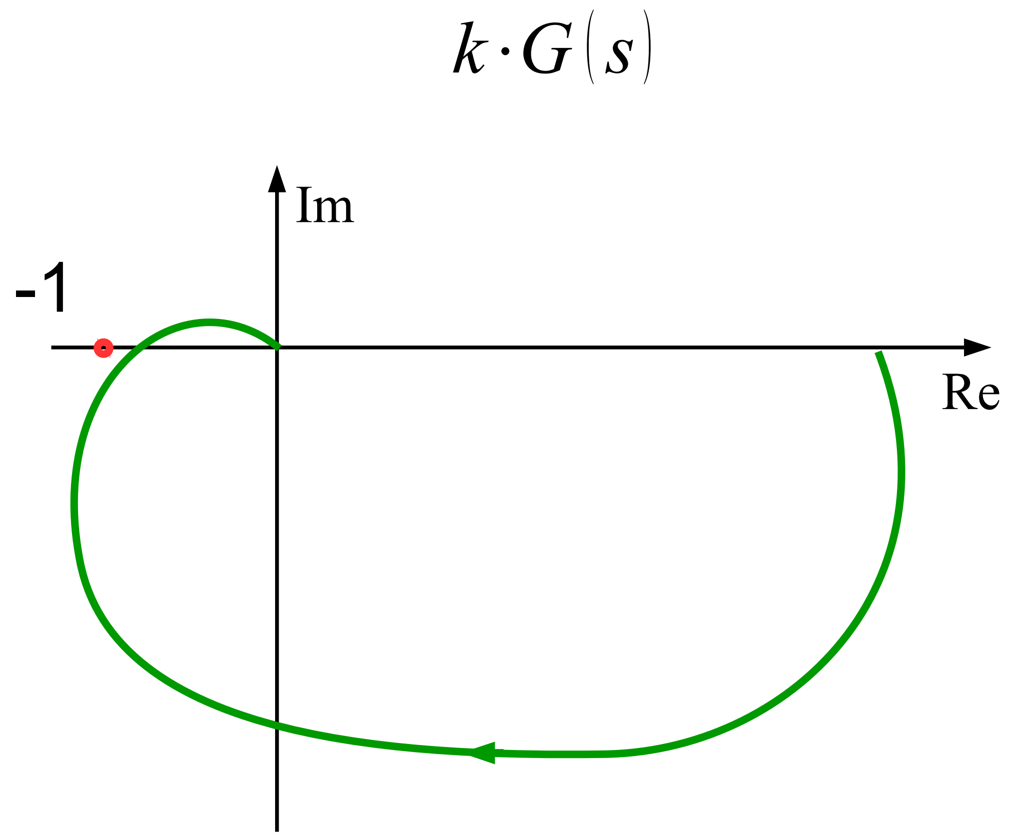
$G(s)$



Correction of the system

Correction by proportional term





Correction of the system

Correction by proportional term

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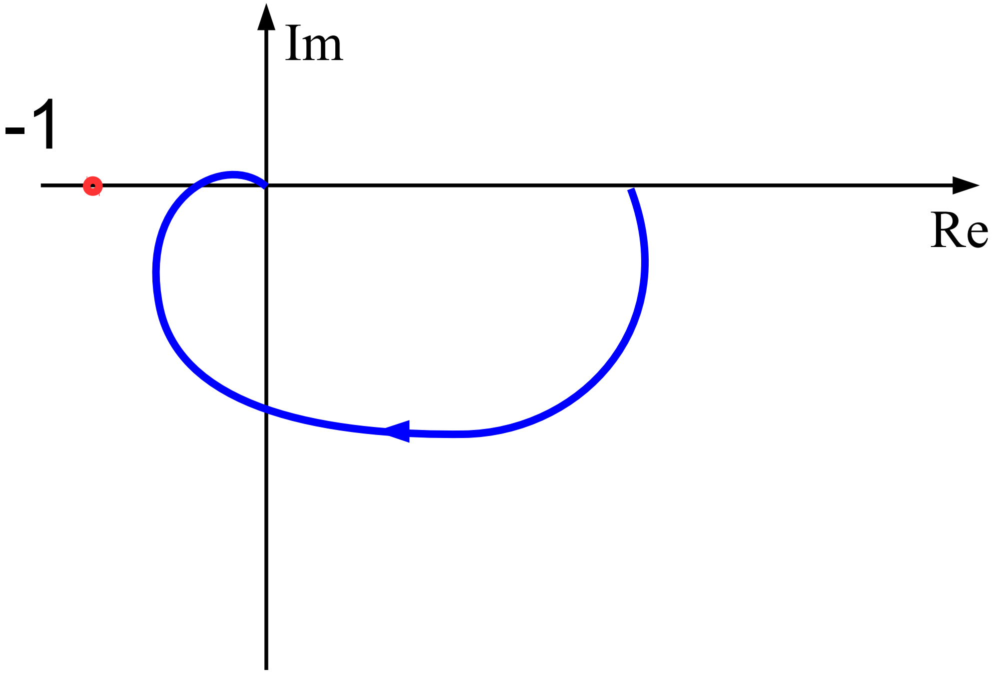
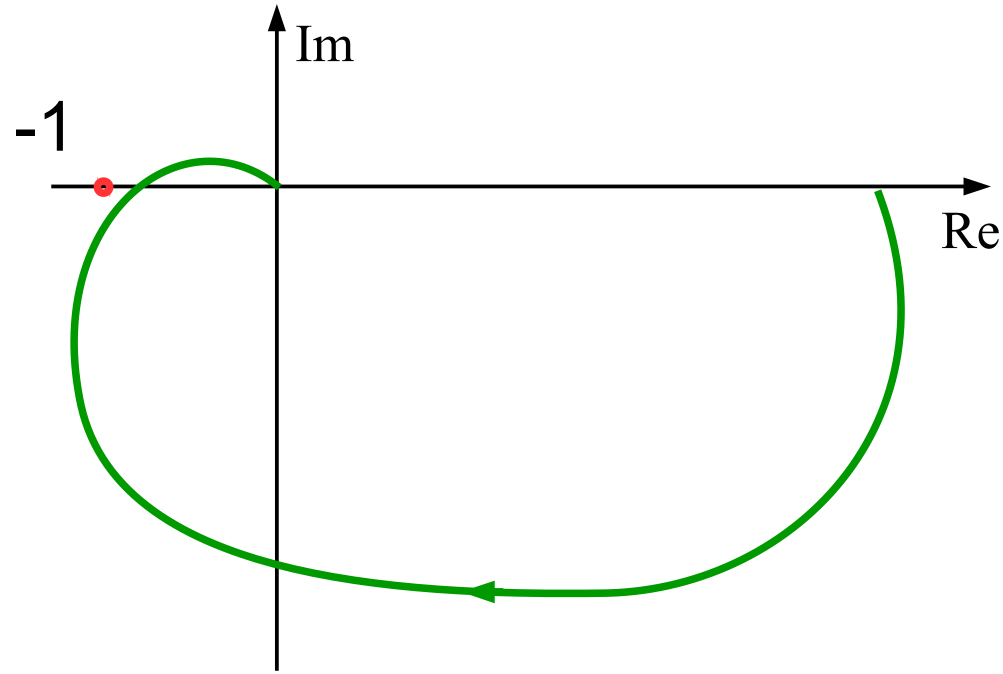


Higher gain margin,
higher phase margin,
higher steady state error

$k \cdot G(s)$

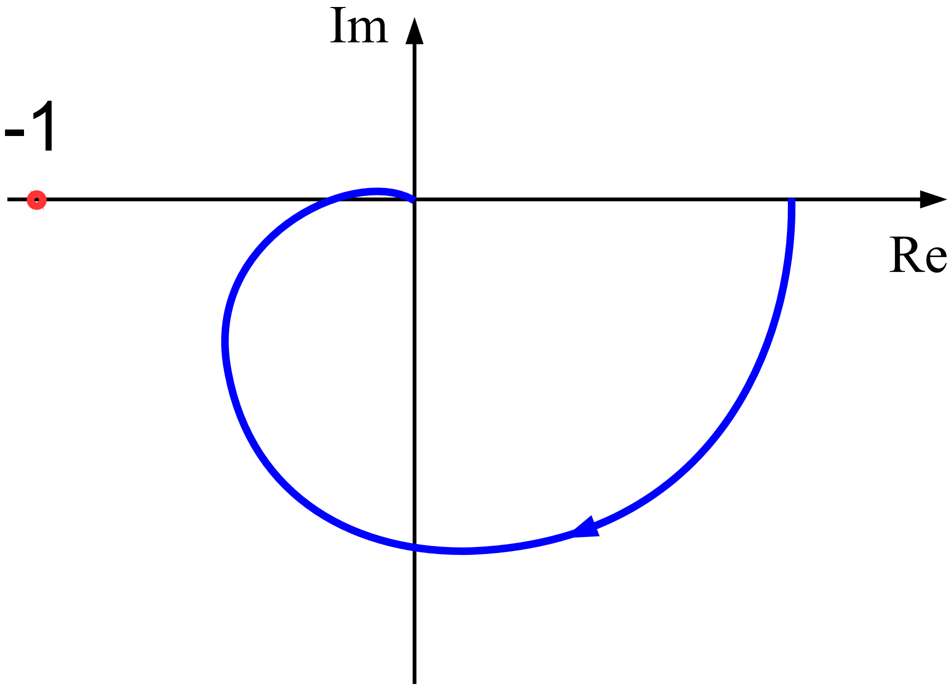


Lower gain margin,
lower phase margin,
lower steady state error

Correction of the system

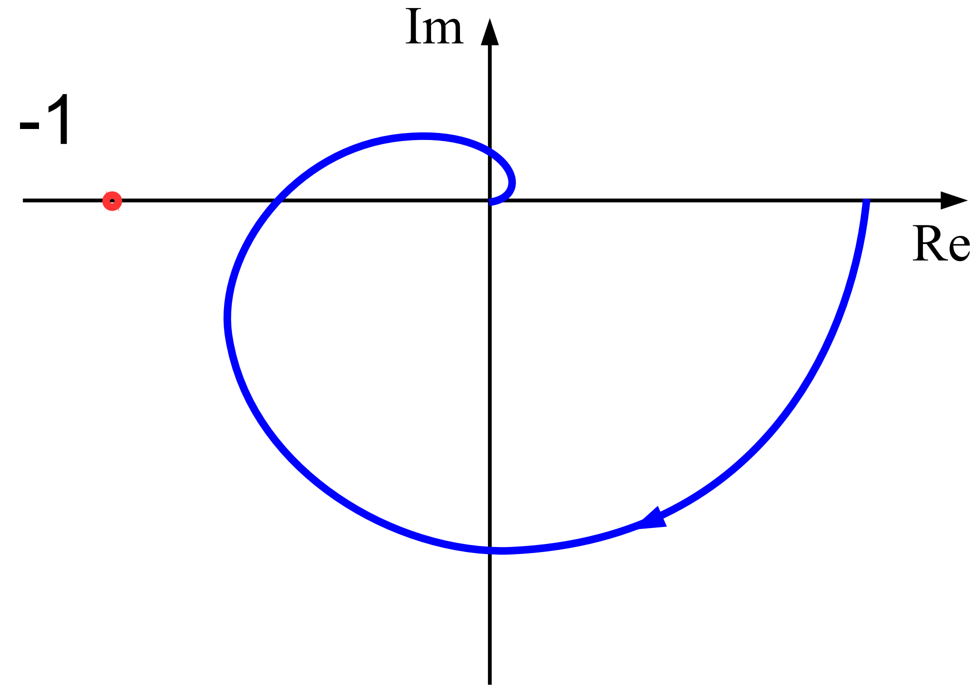
Correction by delay

$G(s)$



Higher gain and phase margins

$G(s) \cdot e^{-\tau s}$



Lower gain and phase margins

Correction of the system

Derivative

$$K(s)=\frac{1+T\,s}{1+a\,s+b\,s^2}$$

Proportional-derivative

$$K(s)=k_P\frac{T\,s+1}{\alpha\,T\,s+1},\ \alpha<1$$

Integral

$$K(s)=1+\frac{k}{1+T\,s}$$

Proportional-integral

$$K(s)=\alpha\frac{T\,s+1}{\alpha\,T\,s+1},\ \alpha>1$$

Proportional-integral-derivative

$$K(s)=k\left(T_d\,s+1\right)\left(1+\frac{1}{T_i\,s}\right)$$

Materials for exam – lectures from 1 to 13 (>1100 slides...)

Lecture 14 – material repeat, supplementary info,
informations about the exam,
WUT questionnaires,
consultations

Lecture 15 – modern control theory overview,
experiment with control system,
consultations

Exam: Thursday, 1st February
Thursday, 8th February