Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control

Winter 2019/2020

Lecturer: Sebastian Korczak, PhD Eng.

Lecture 5 cont.

Dynamics of planar mechanisms.

Dynamics of planar mechanisms

Inverse dynamics problem

Calculation of forces and torques that cause given motion of a mechanism (kinetostatics)

0. Mechanism and it's geometry, driving and operating forces/torques, displacement, velocity and acceleration functions are given.

1. Calculation of inertia forces and torques acting moving members of the mechanism.

2. Decomposition of the mechanism with reaction disclosure.

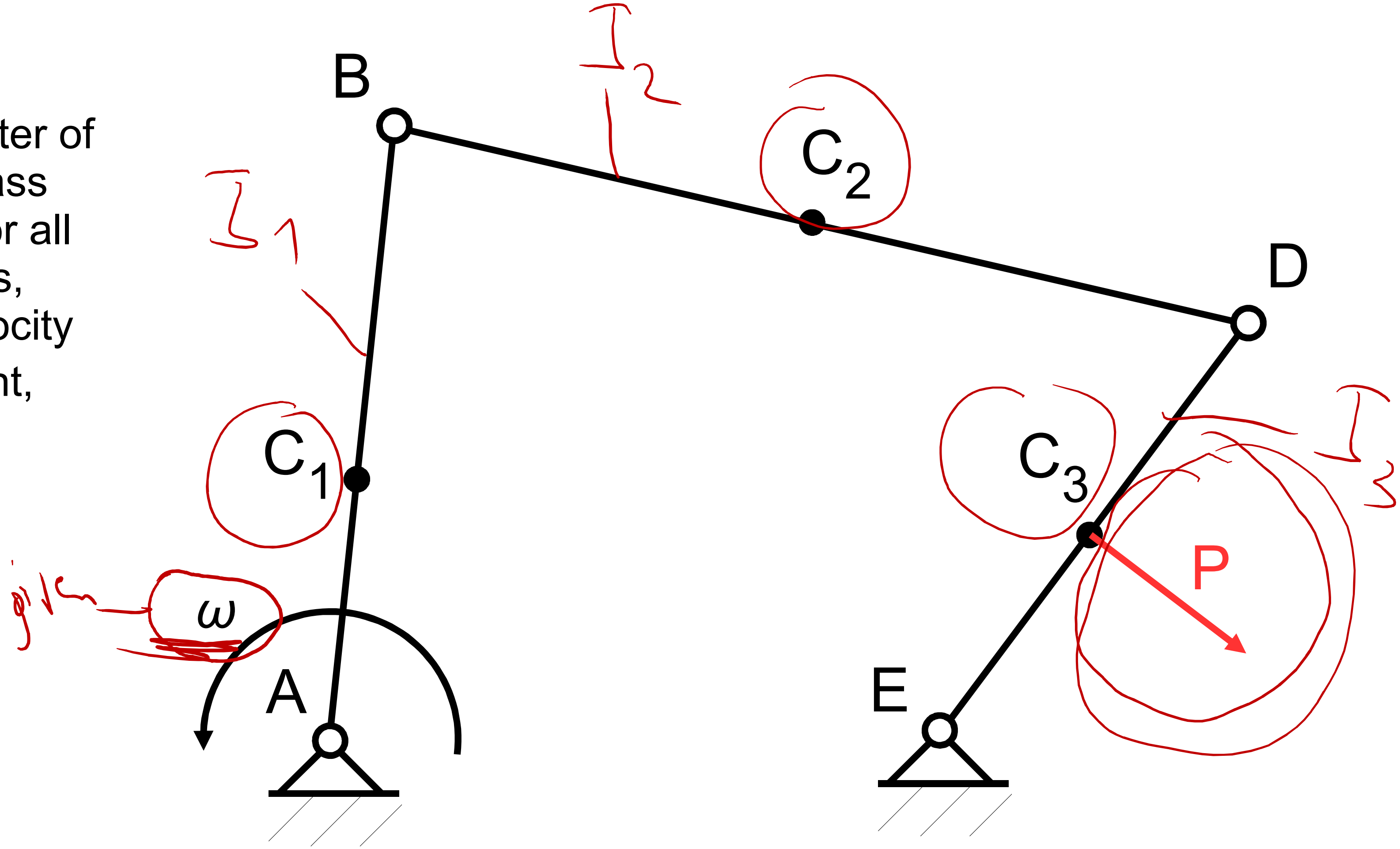
3. Write down vector sums of external forces, reactions and inertia forces (d'Alembert equations).

4. Solve the equations with graphical and/or analytical method.

Dynamics of planar mechanisms

Inverse dynamic problem – example

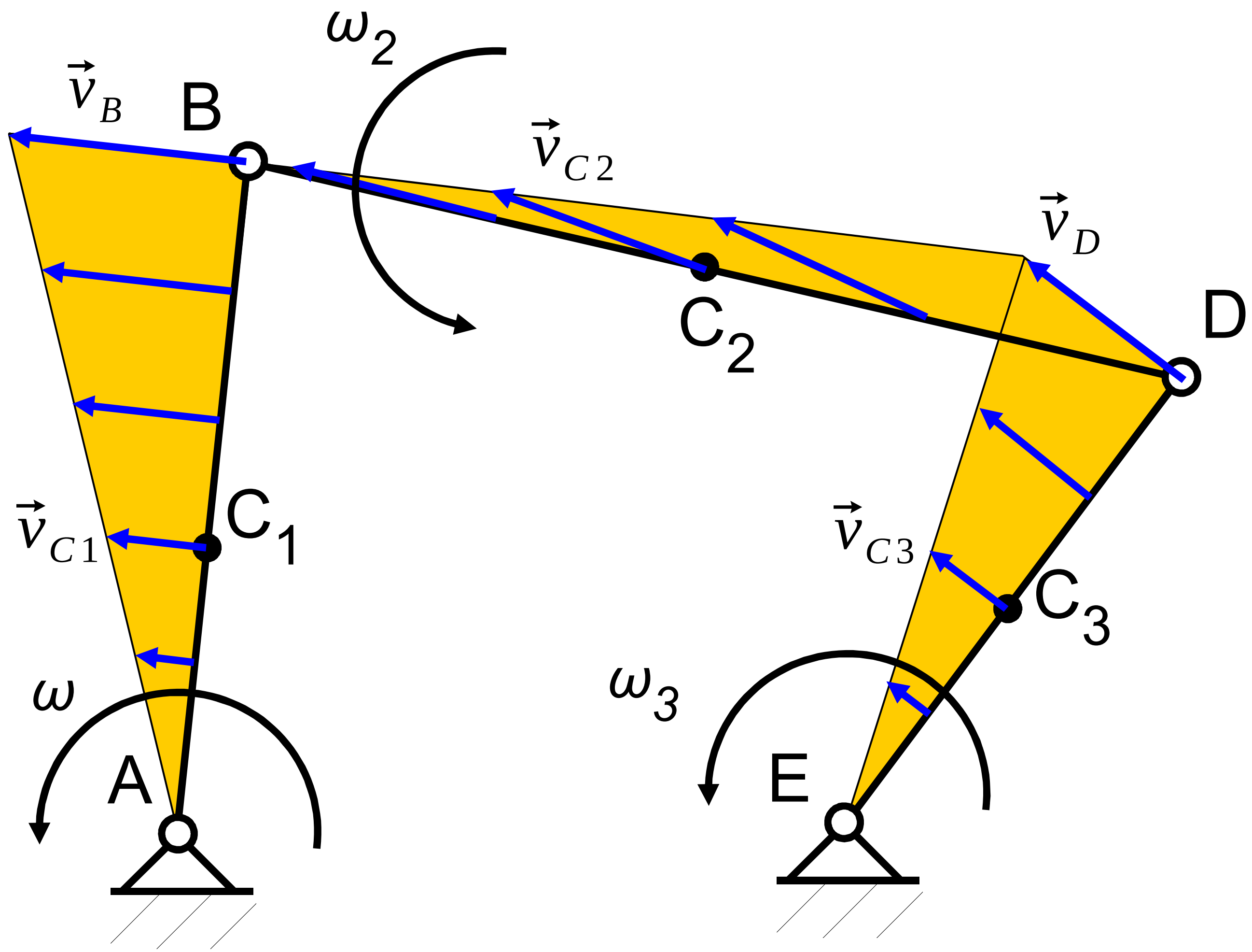
Given:
geometry, mass, center of a mass locations, mass moments of inertia for all mechanism members, constant angular velocity ω of a driven element, operating force P.



Dynamics of planar mechanisms

Inverse dynamic problem – example

velocity scheme



Dynamics of planar mechanisms

Inverse dynamic problem – example

acceleration scheme

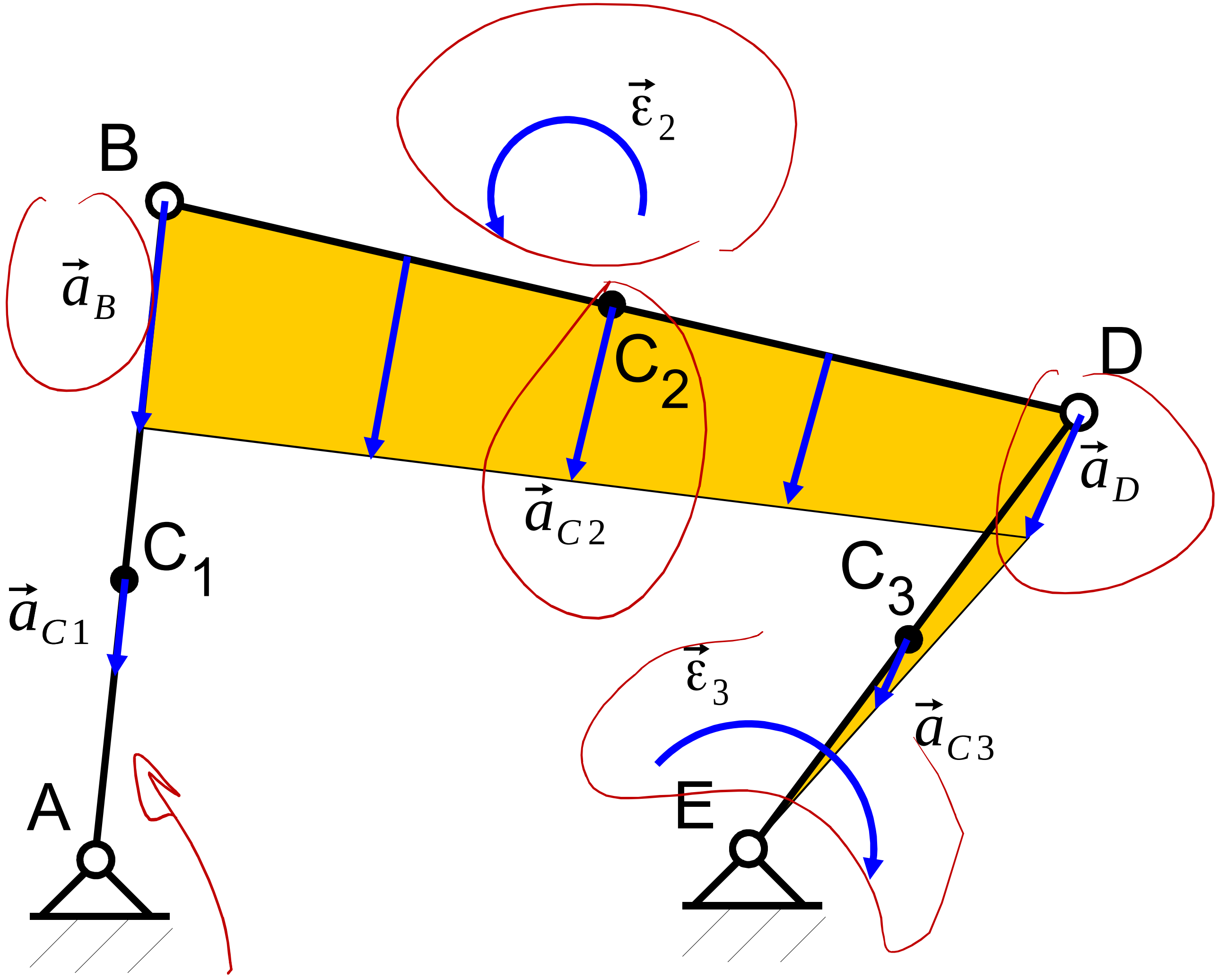
$$\bar{B}_1 = -m_1 \bar{a}_{c1}$$

$$\bar{B}_2 = -m_2 \bar{a}_{c2}$$

$$\bar{B}_3 = -m_3 \bar{a}_{c3}$$

$$\bar{M}_2 = -I_{c2} \bar{\varepsilon}_2$$

$$\bar{M}_3 = -I_{c3} \bar{\varepsilon}_3$$

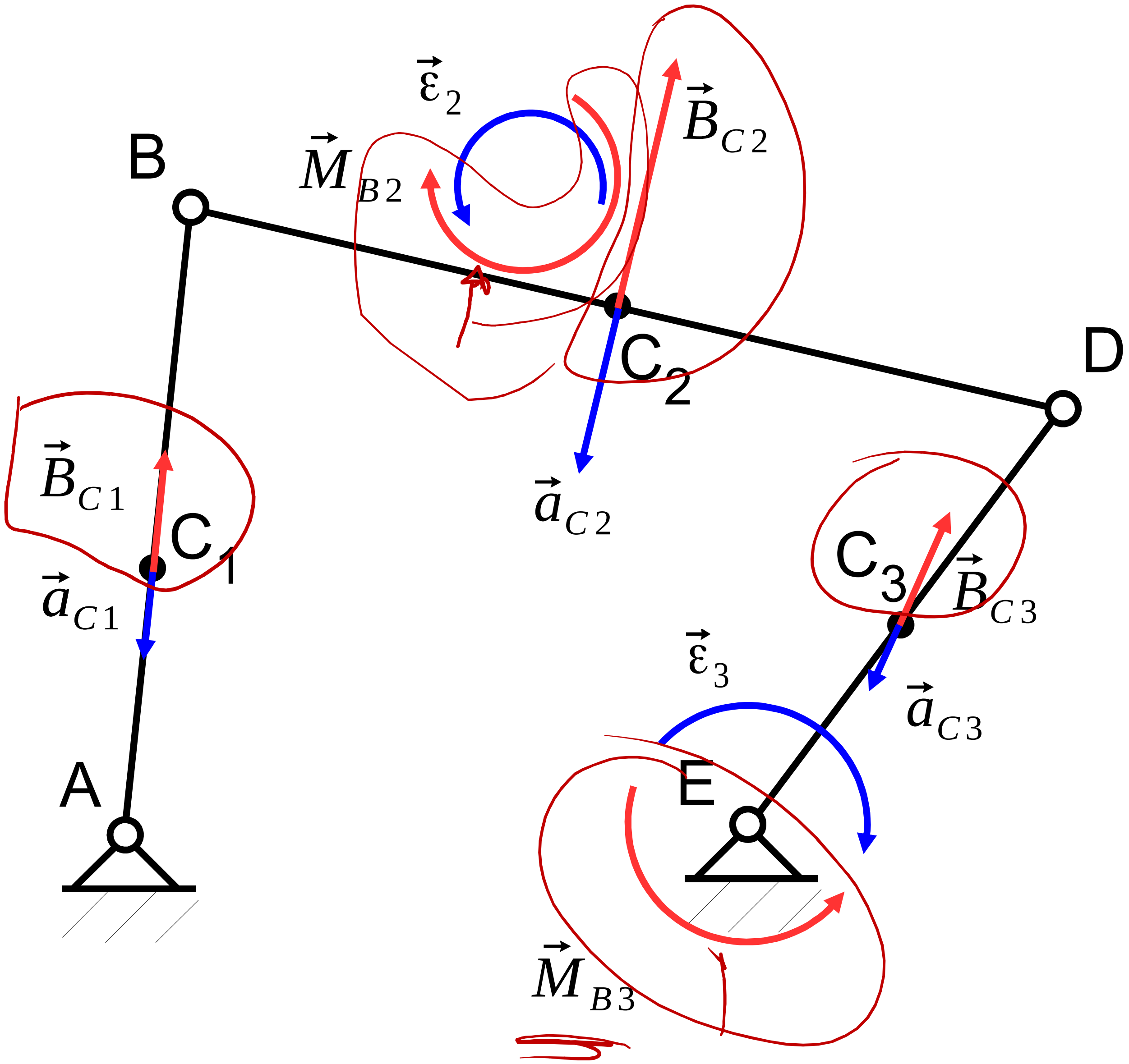


$\varepsilon_1 = 0$ $\bar{M}_1 = 0$

Dynamics of planar mechanisms

Inverse dynamic problem – example

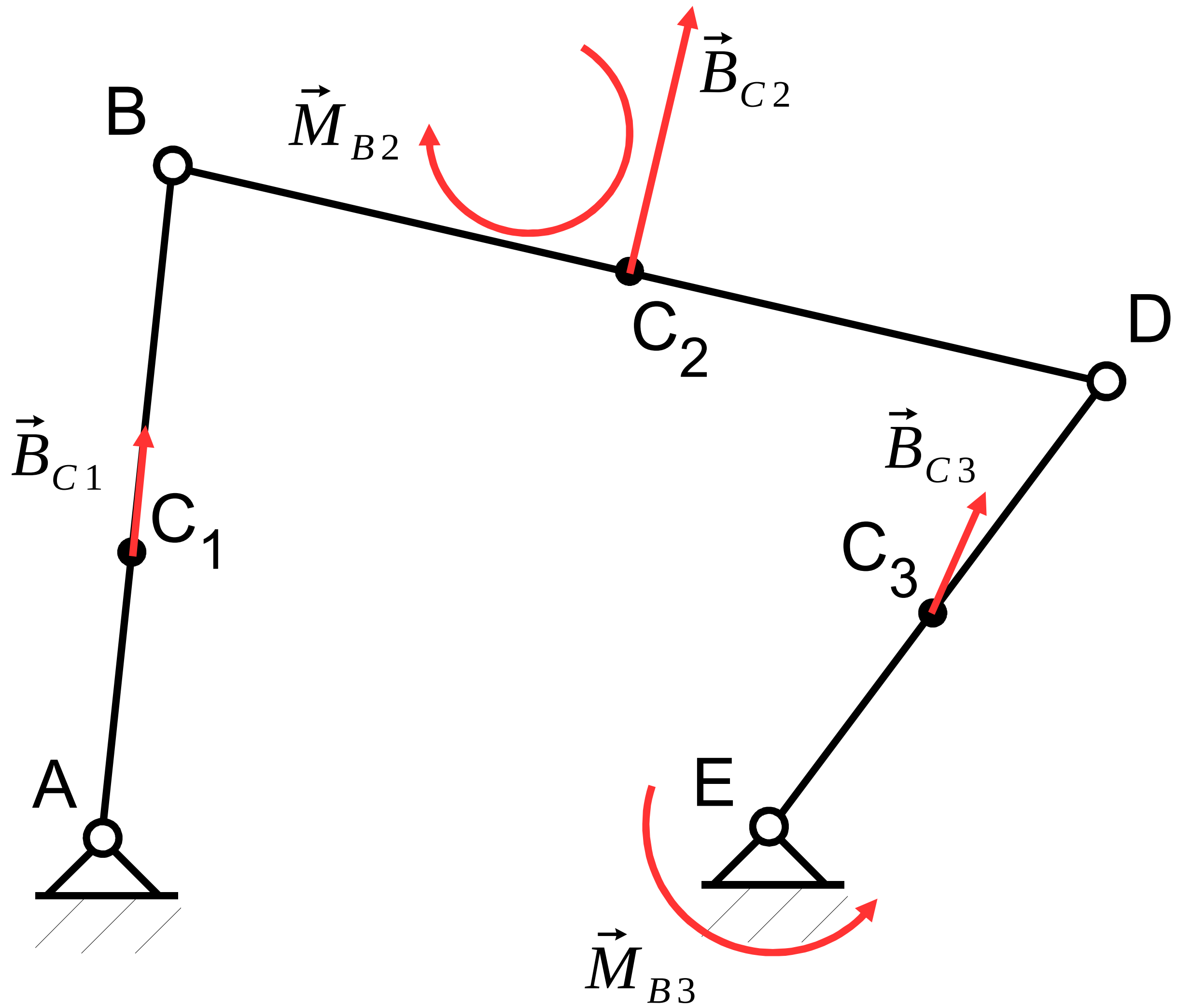
inertial forces and torques



Dynamics of planar mechanisms

Inverse dynamic problem – example

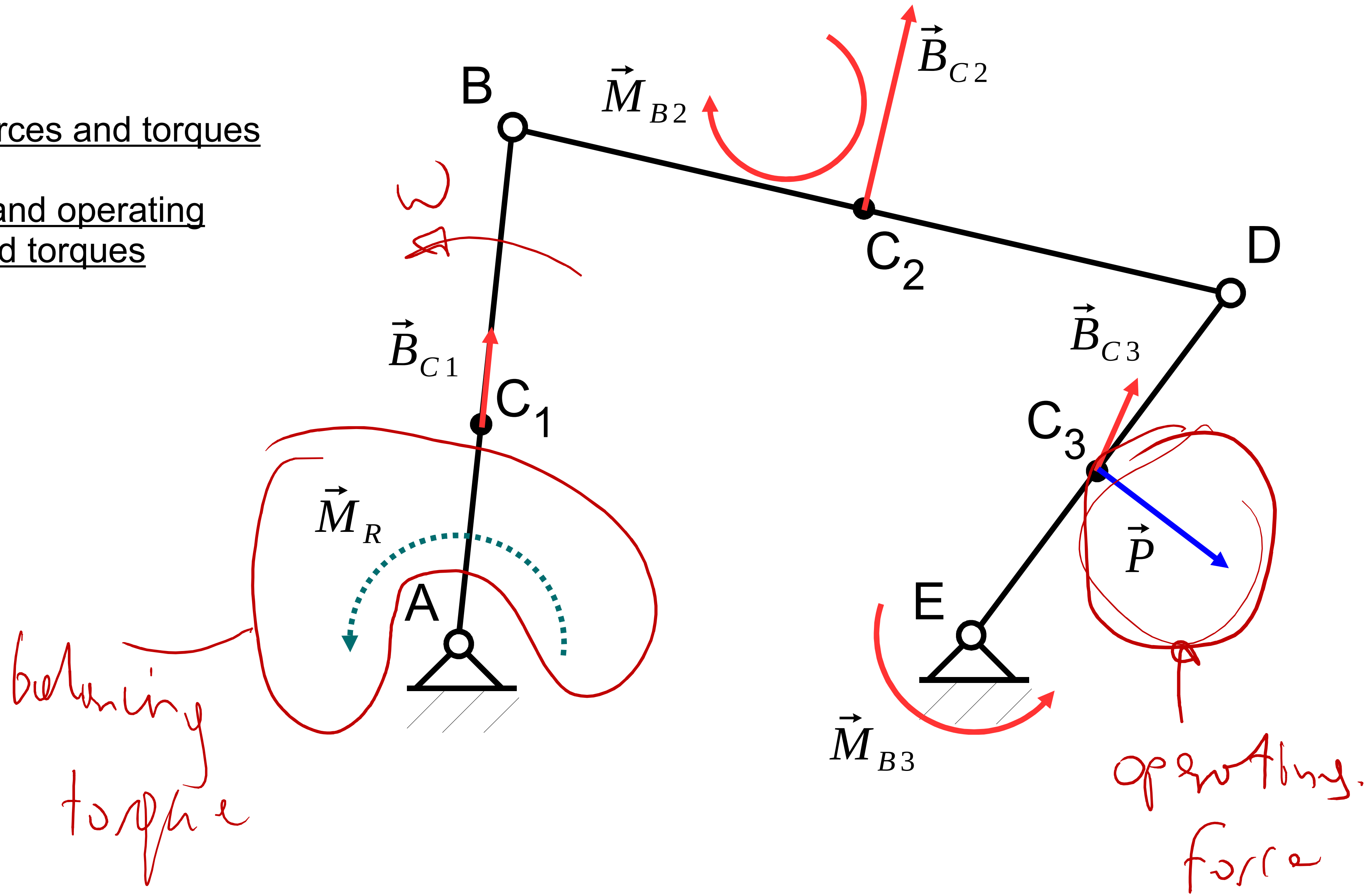
inertial forces and torques



Dynamics of planar mechanisms

Inverse dynamic problem – example

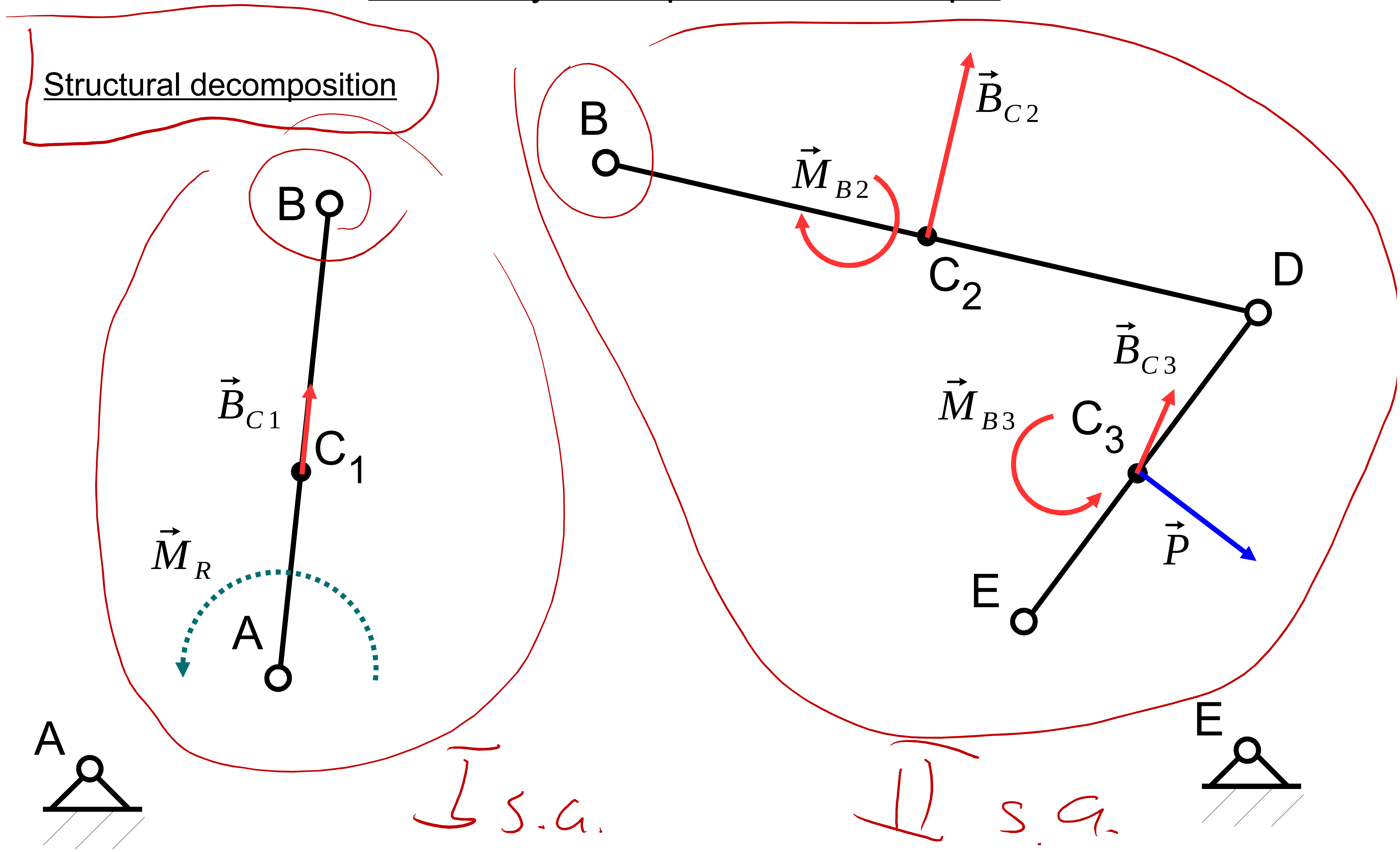
inertial forces and torques
+
external and operating
forces and torques



Dynamics of planar mechanisms

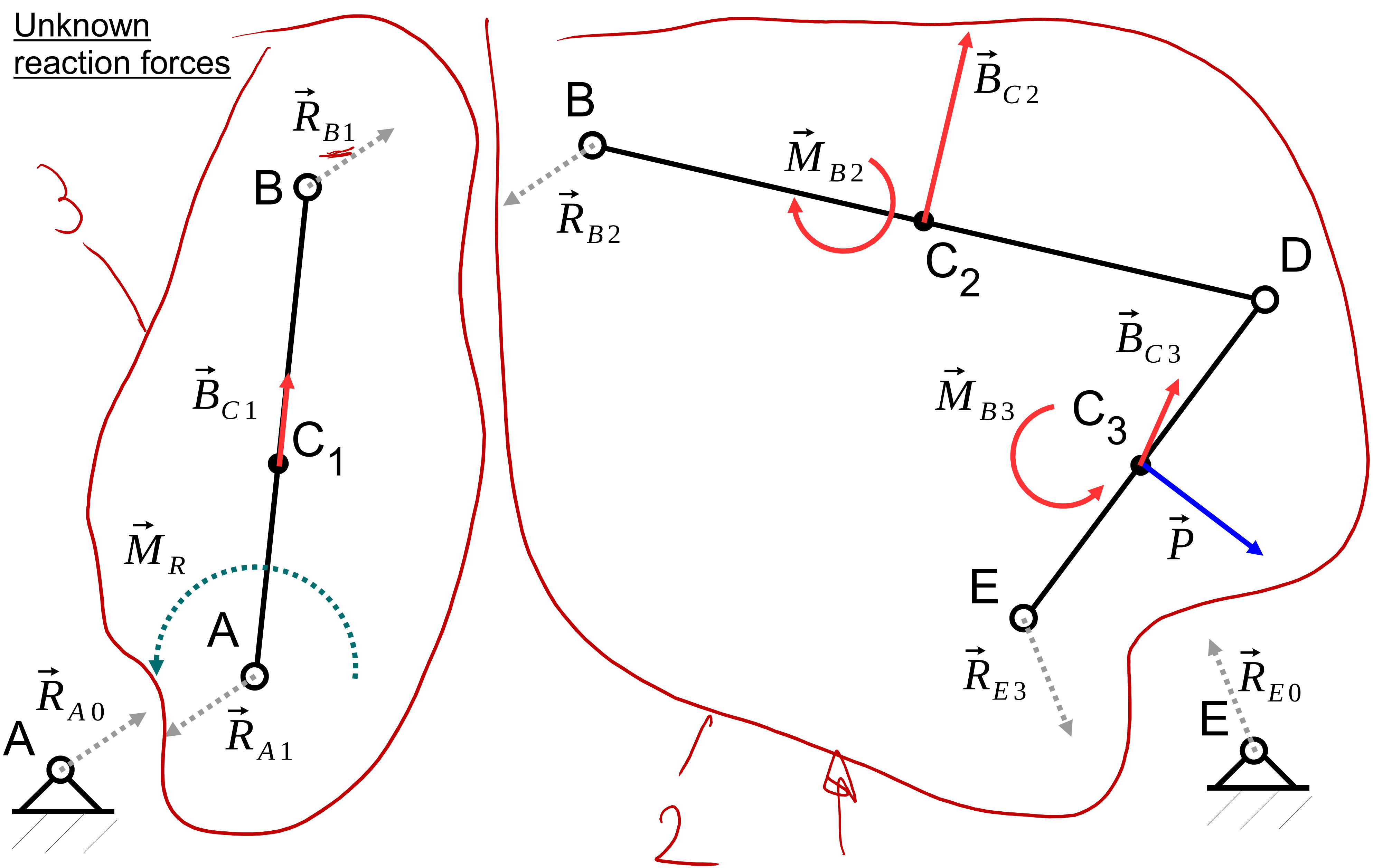
Inverse dynamic problem – example

Structural decomposition



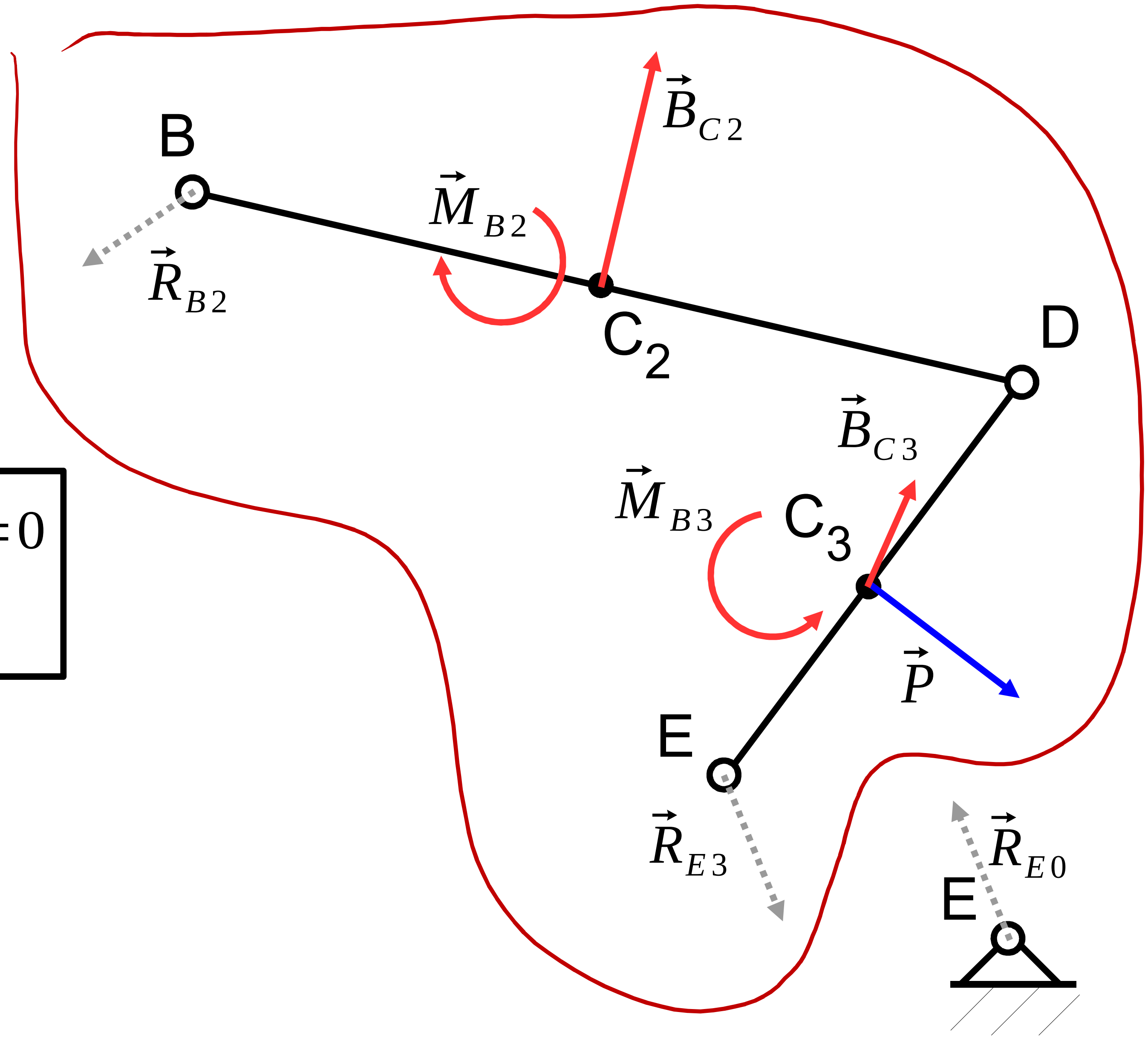
Dynamics of planar mechanisms

Inverse dynamic problem – example



Dynamics of planar mechanisms

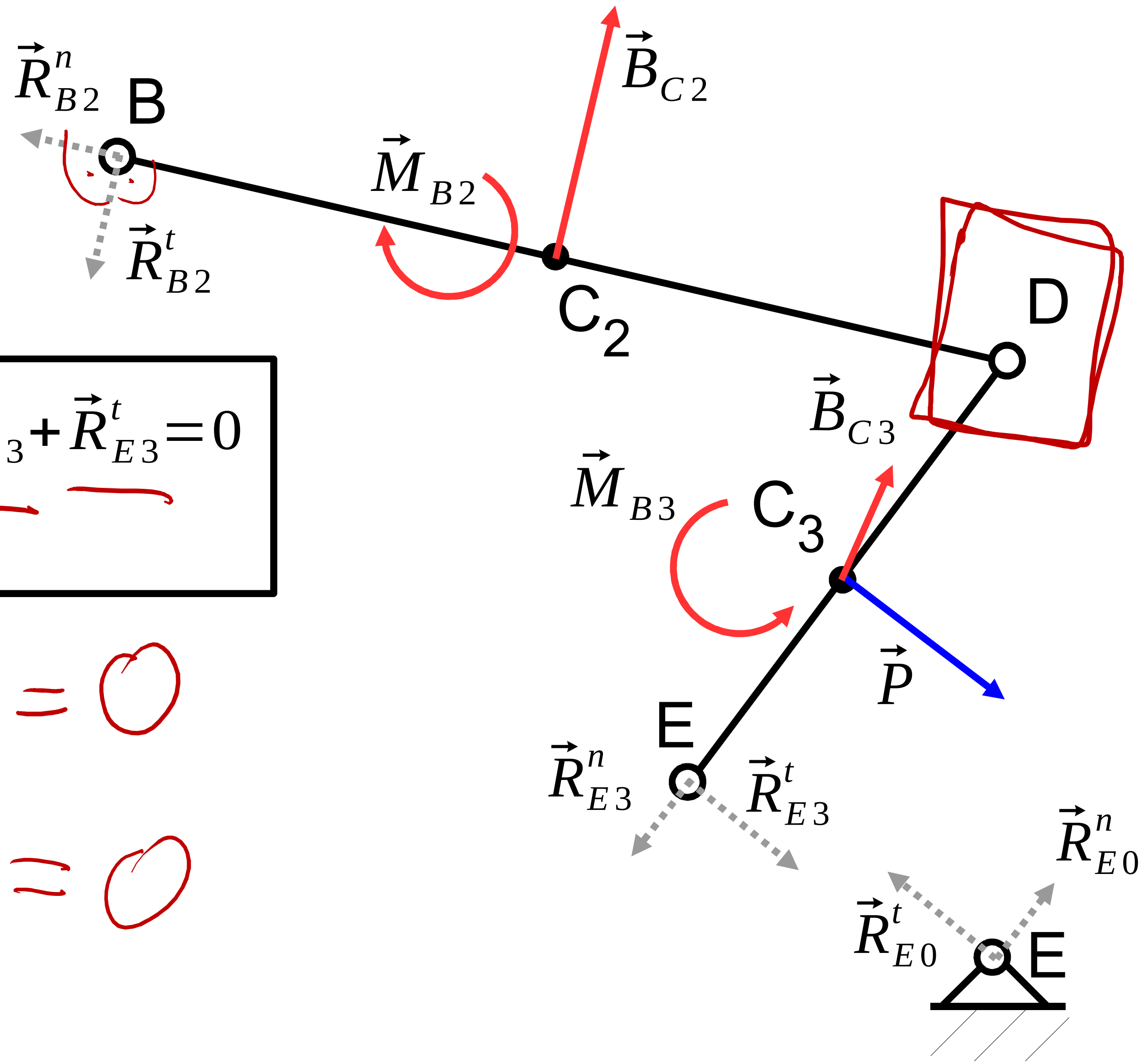
Inverse dynamic problem – example



$$\vec{R}_{B2}+\vec{B}_{C2}+\vec{B}_{C3}+\vec{P}+\vec{R}_{E3}=0$$

Dynamics of planar mechanisms

Inverse dynamic problem – example



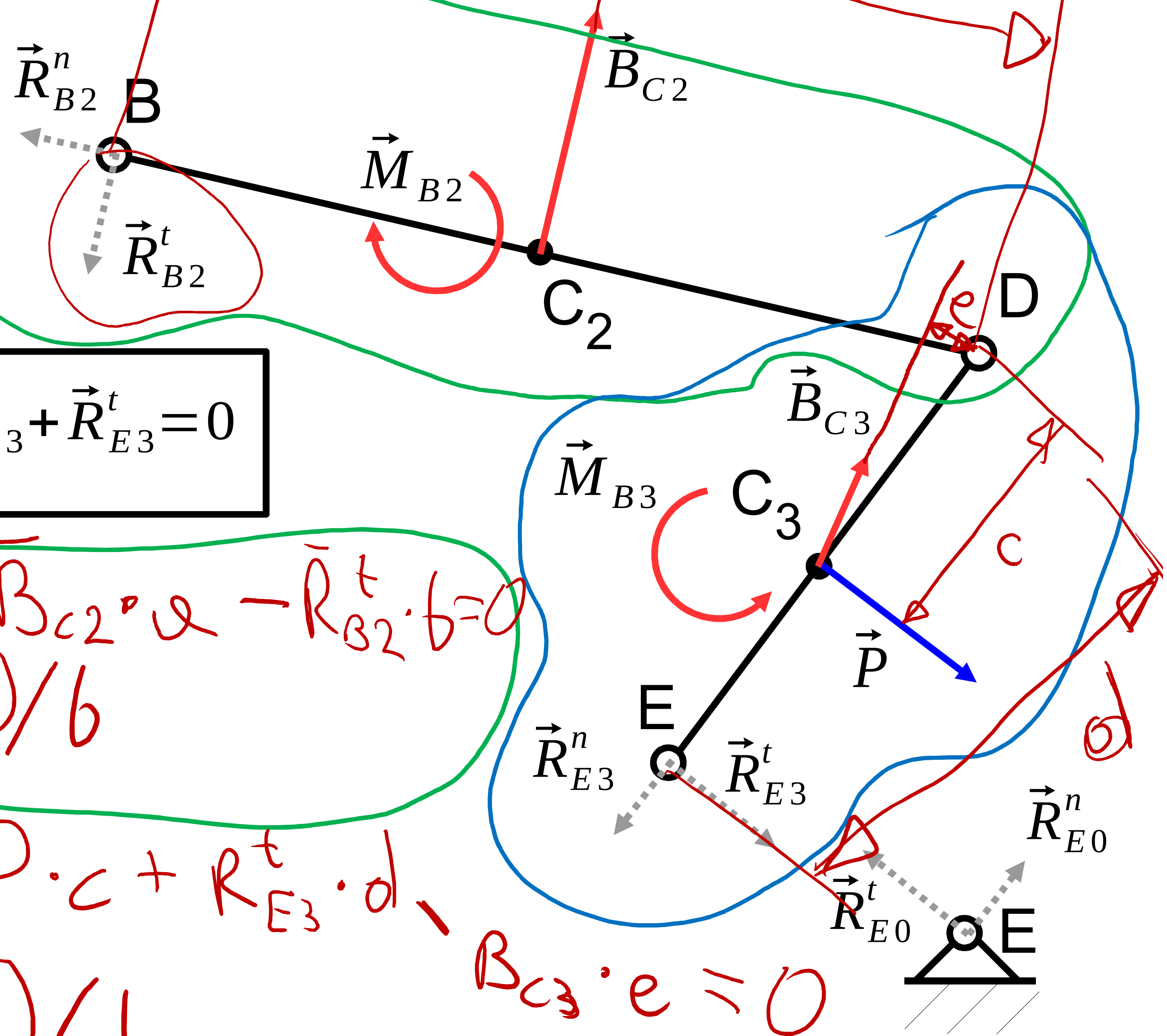
$$\vec{R}^{n}_{B2}+\vec{R}^{t}_{B2}+\vec{B}_{C2}+\vec{B}_{C3}+\vec{P}+\vec{R}^{n}_{E3}+\vec{R}^{t}_{E3}=0$$

$\Sigma M_{D2} = \ldots = 0$

$\Sigma M_{D3} = \ldots = 0$

Dynamics of planar mechanisms

Inverse dynamic problem – example



$$\vec{R}^n_{B2}+\vec{R}^t_{B2}+\vec{B}_{C2}+\vec{B}_{C3}+\vec{P}+\vec{R}^n_{E3}+\vec{R}^t_{E3}=0$$

part BD: $\sum M_D : M_{B2} + \bar{B}_{C2}\cdot a - \bar{R}^t_{B2}\cdot b = 0$

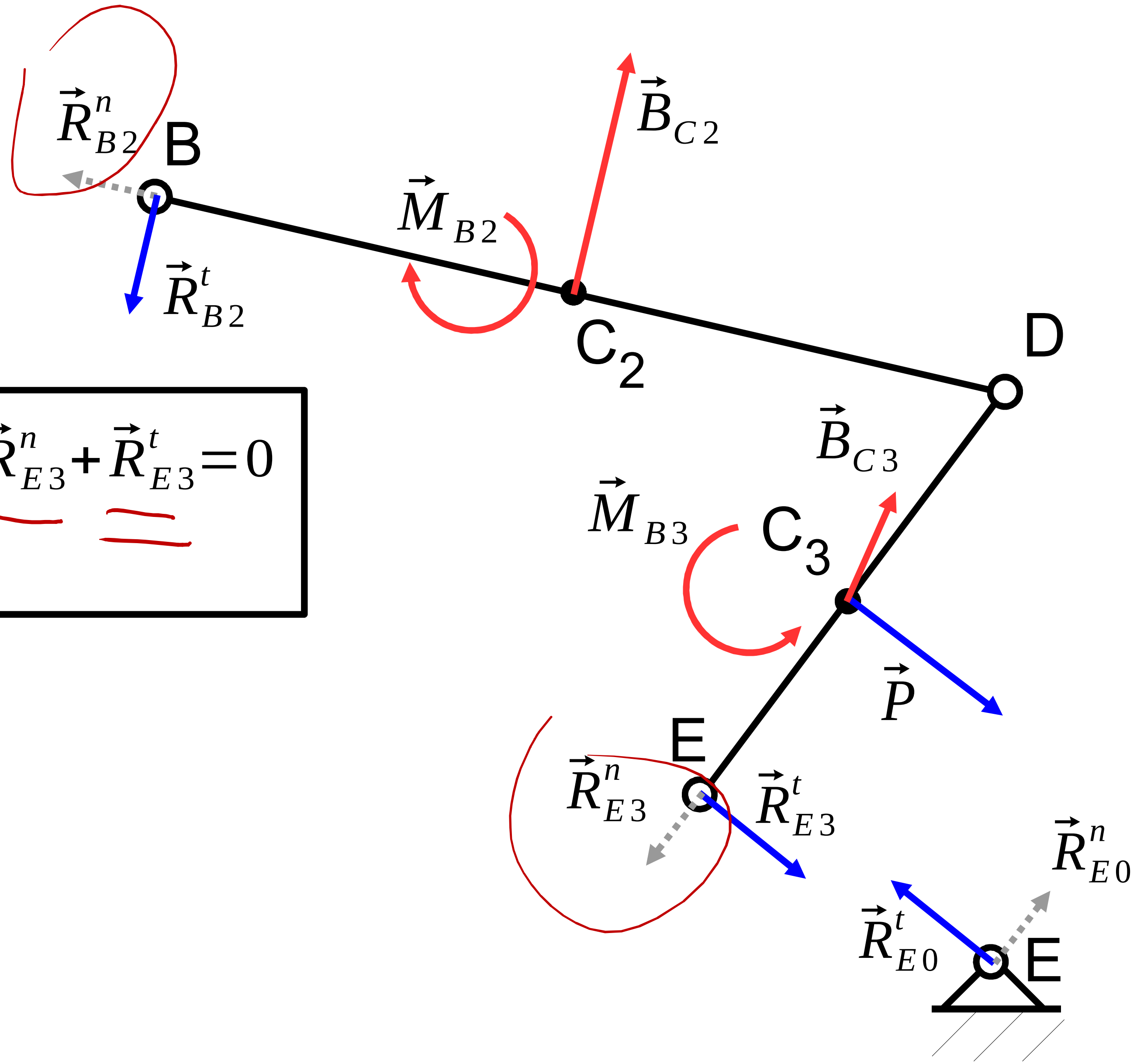
$\vec{R}^t_{B2} = (M_{B2} + \bar{B}_{C2}\cdot a)/b$

part ED: $\sum M_D : M_{B3} + P\cdot c + R^t_{E3}\cdot d - B_{C3}\cdot e = 0$

$\vec{R}^t_{E3} = \ldots$ $(B_{C3}\cdot e - M_{B3} - P\cdot c)/d$

Dynamics of planar mechanisms

Inverse dynamic problem – example

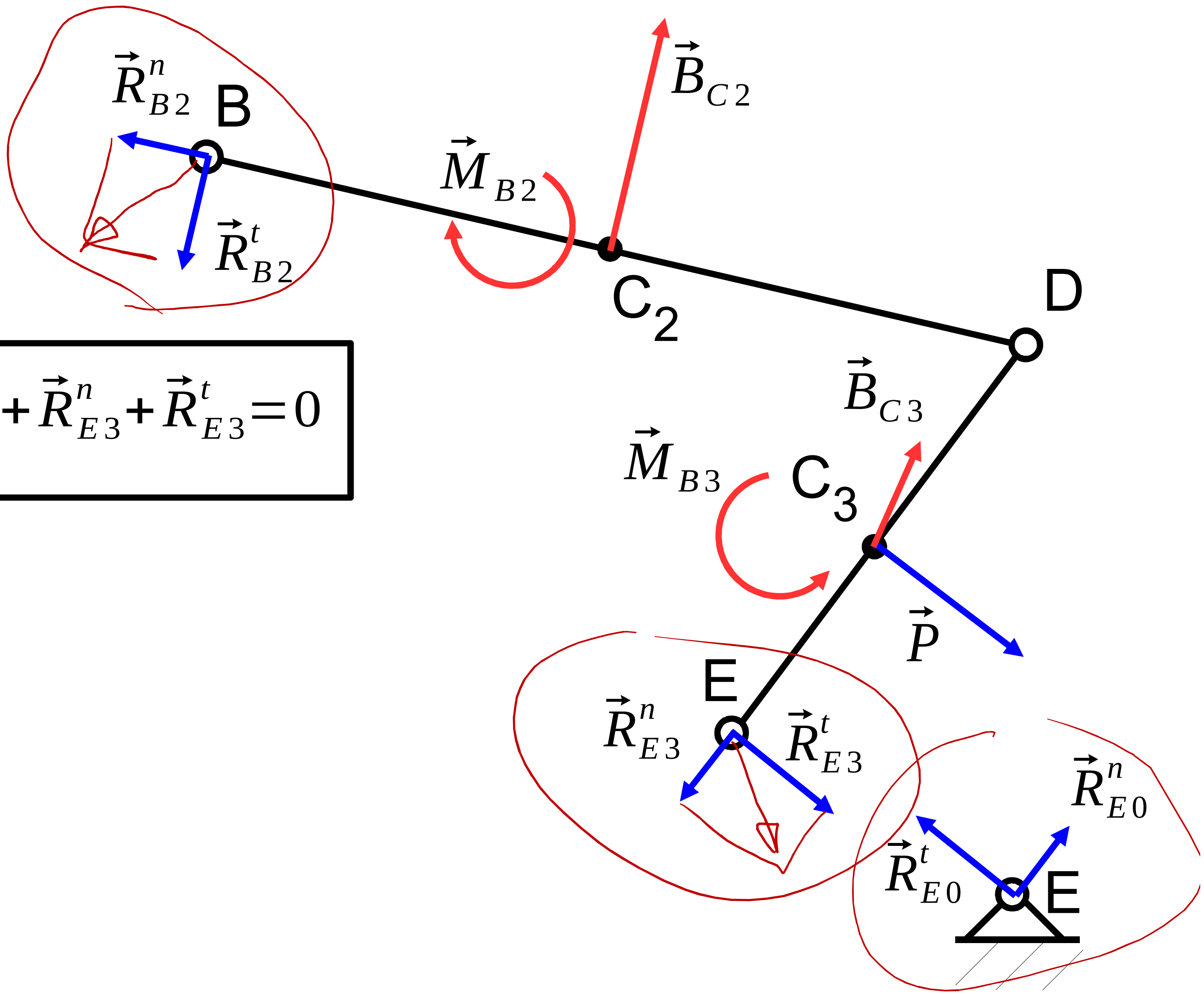


$$\vec{R}^n_{B2}+\vec{R}^t_{B2}+\vec{B}_{C2}+\vec{B}_{C3}+\vec{P}+\vec{R}^n_{E3}+\vec{R}^t_{E3}=0$$

→ graph.

Dynamics of planar mechanisms

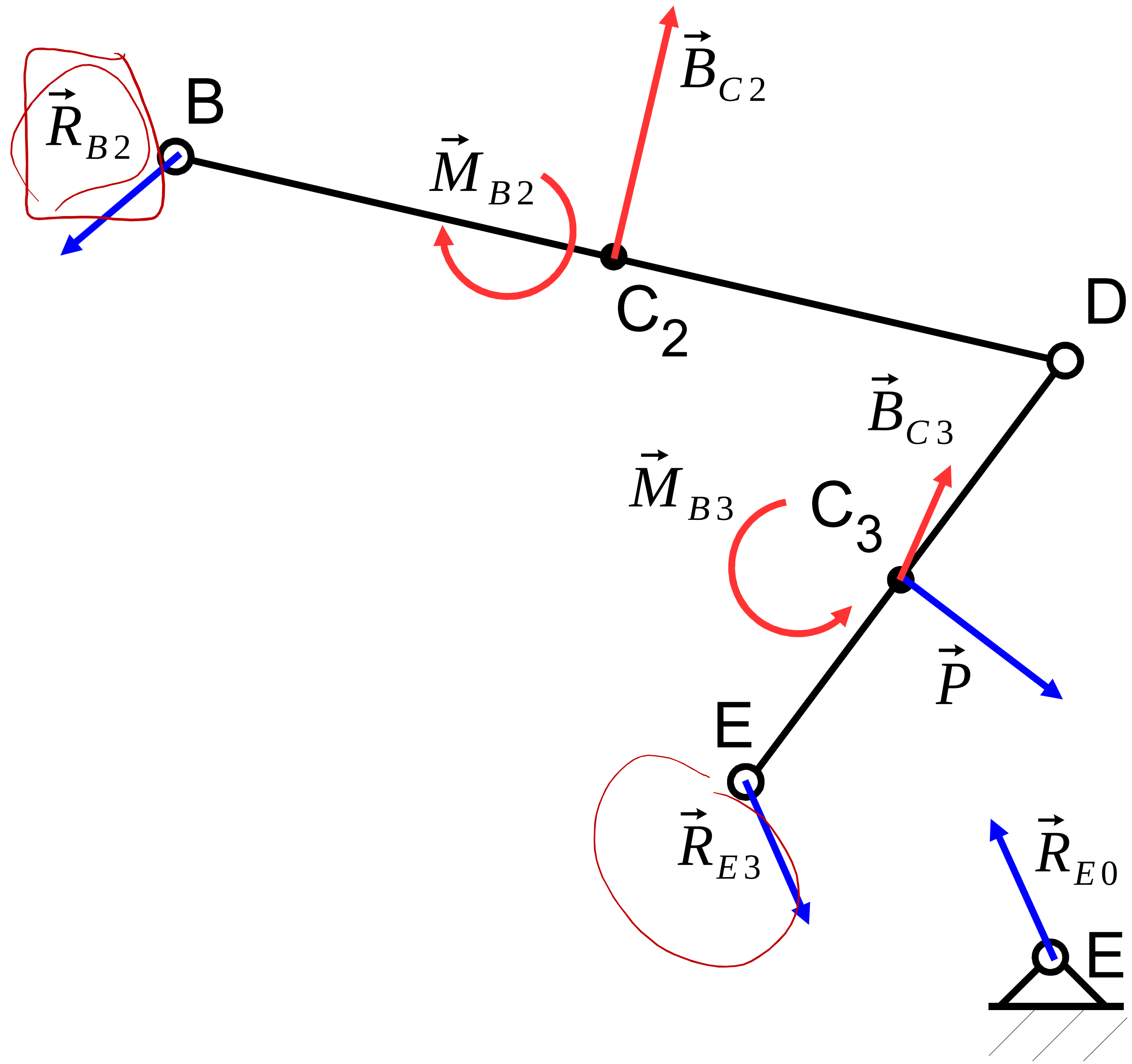
Inverse dynamic problem – example



$$\vec{R}^n_{B2}+\vec{R}^t_{B2}+\vec{B}_{C2}+\vec{B}_{C3}+\vec{P}+\vec{R}^n_{E3}+\vec{R}^t_{E3}=0$$

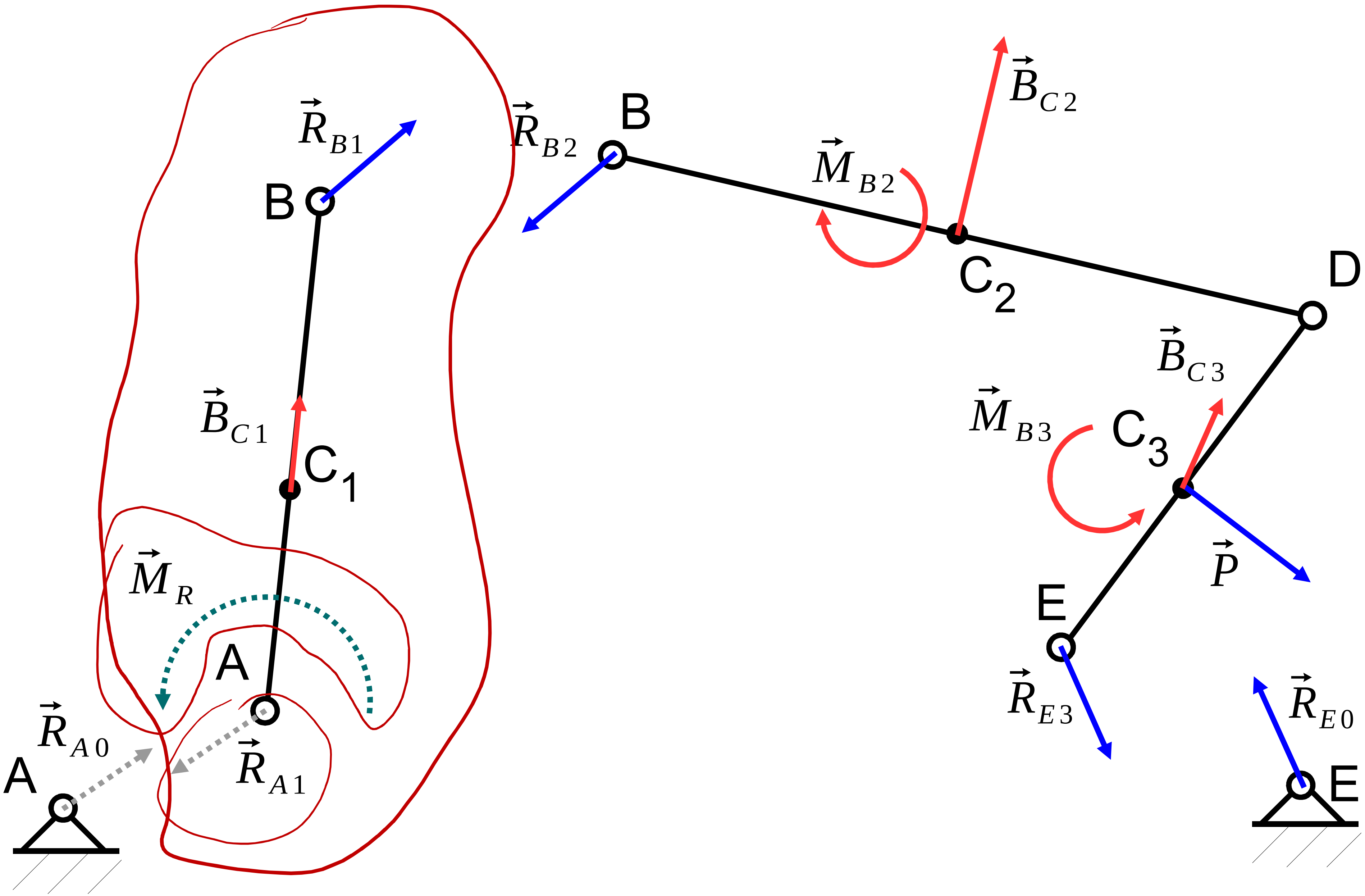
Dynamics of planar mechanisms

Inverse dynamic problem – example



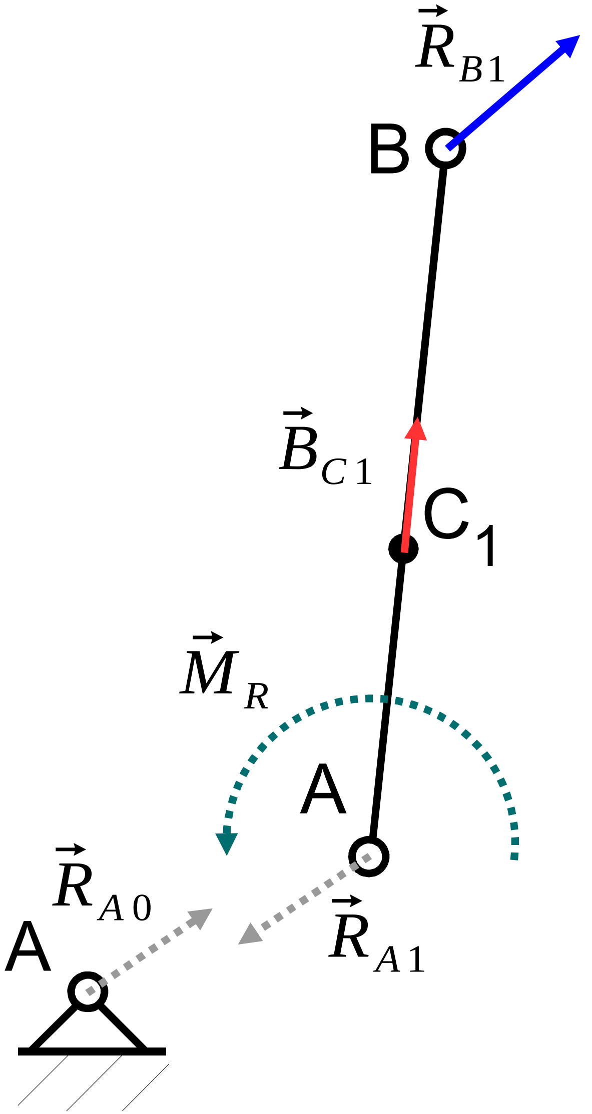
Dynamics of planar mechanisms

Inverse dynamic problem – example



Dynamics of planar mechanisms

Inverse dynamic problem – example

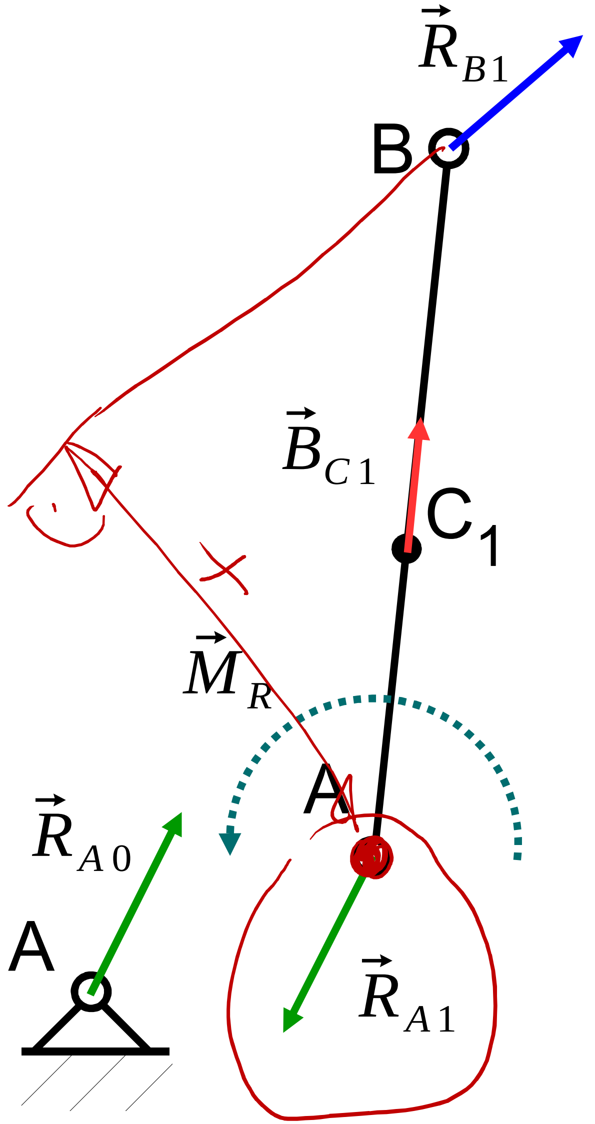


$$part\ AB:\ \vec{R}_{A1}+\vec{B}_{C1}+\vec{R}_{B1}=0$$

$$\vec{R}_{A1}=-\bar{B}_{C1}-\bar{R}_{B1}$$

Dynamics of planar mechanisms

Inverse dynamic problem – example



$\sum M_A : \ldots M_R - R_{B1} \cdot x = 0$

$M_R = \ldots R_{B1} \cdot x$

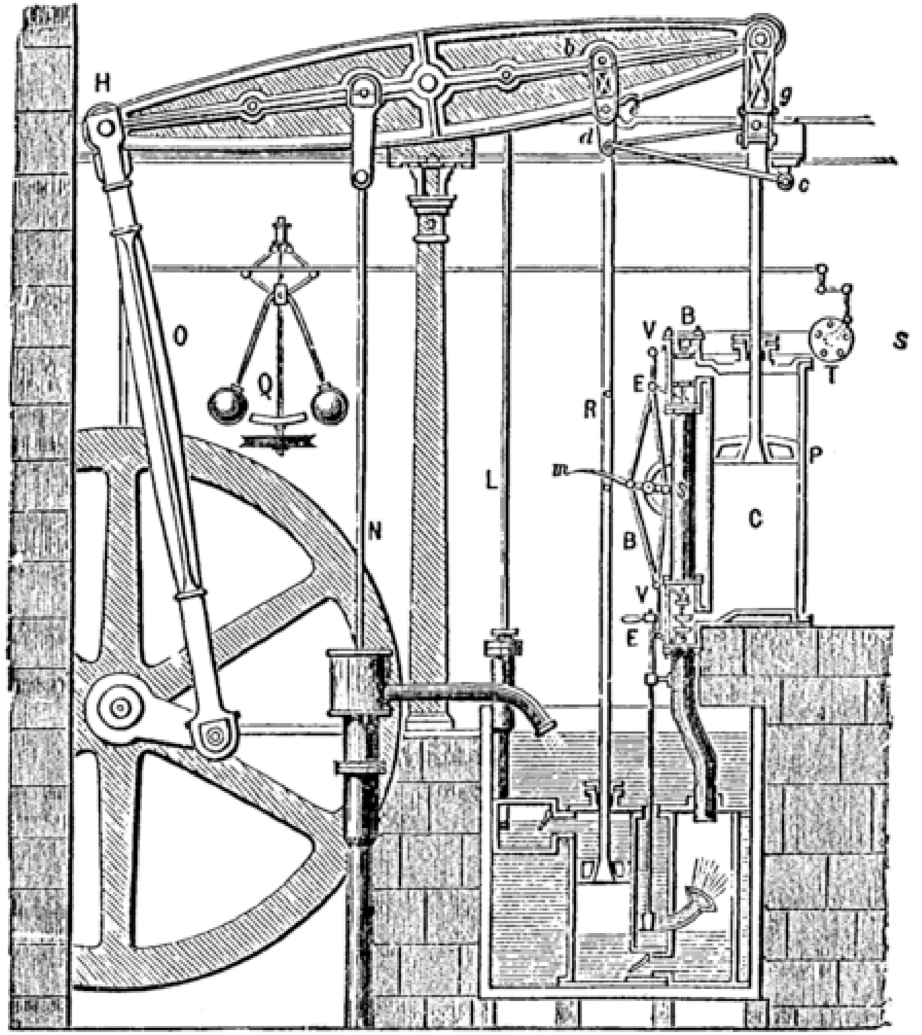
Lecture 6

Machine dynamics.
Reduction of masses and forces.
Machine equation of motion.

Machine dynamics

Overview

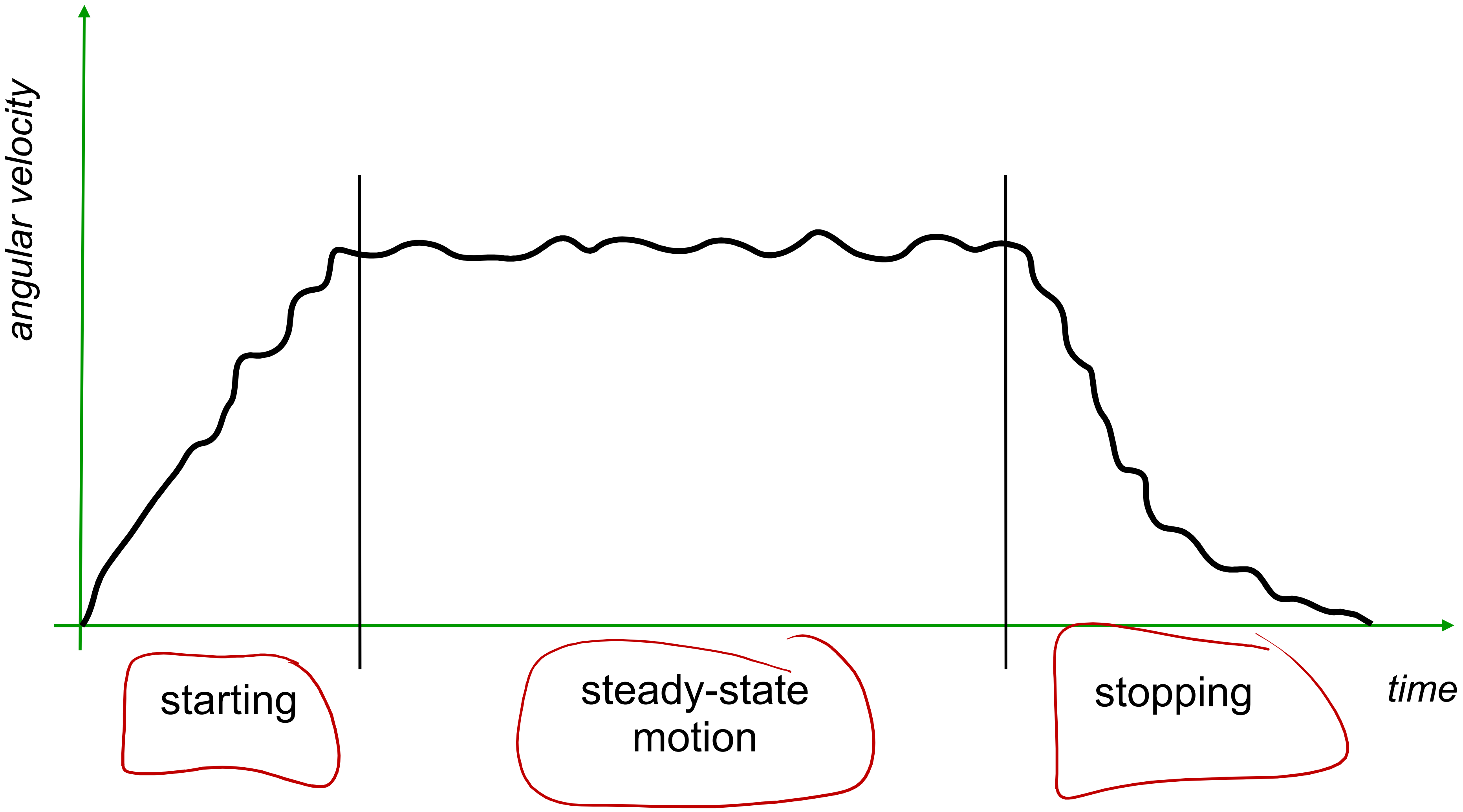
Machine – a tool containing one or more parts that uses energy to perform an intended action. Machines are assembled from components.



source: wikipedia.org, The Boulton & Watt Steam Engine, 1784

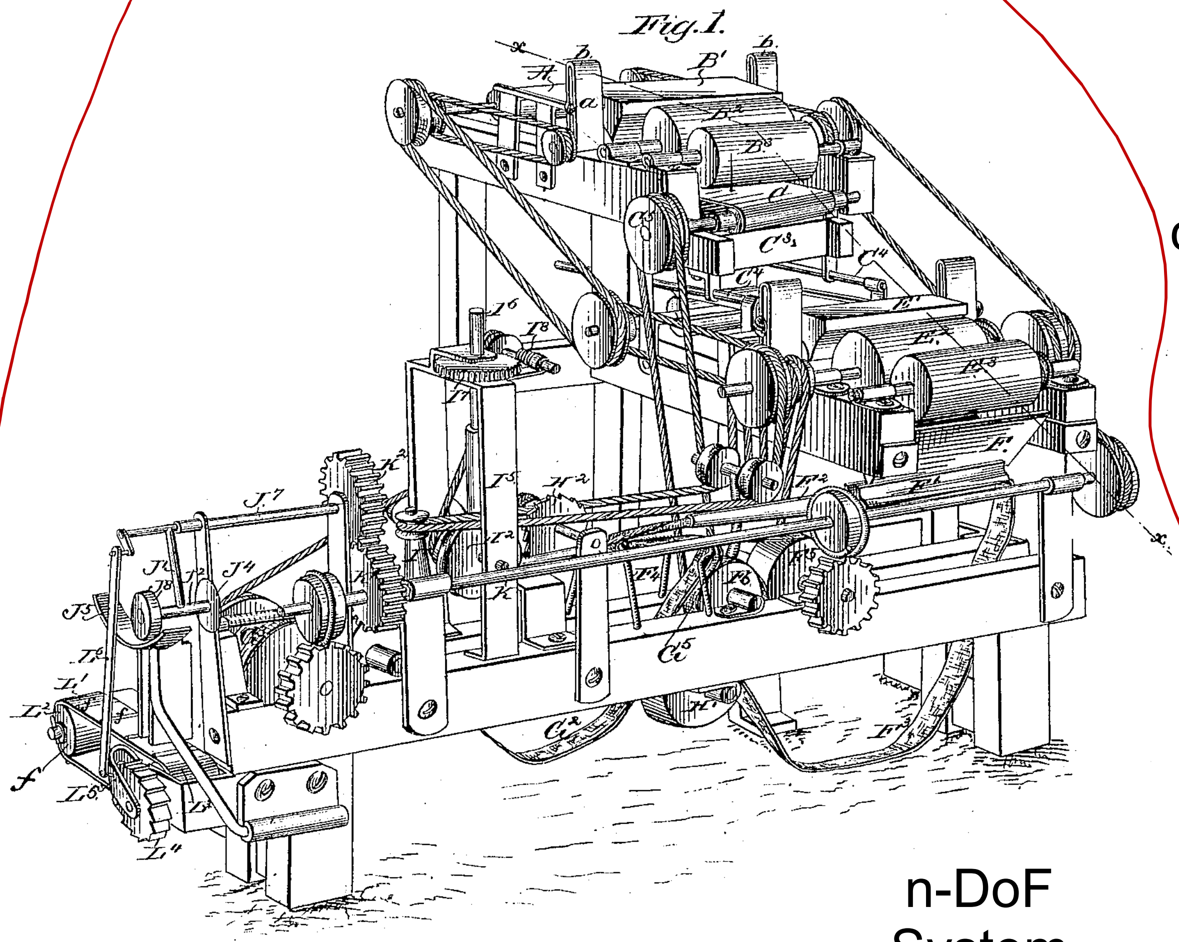
Machine dynamics

Overview



Reduction of masses and forces

Idea of reduction



Source: *James Albert Bonsack (1859 – 1924) - U.S. patent 238,640 cigarette rolling machine, invented in 1880 and patented in 1881*

n-DoF System

complicated

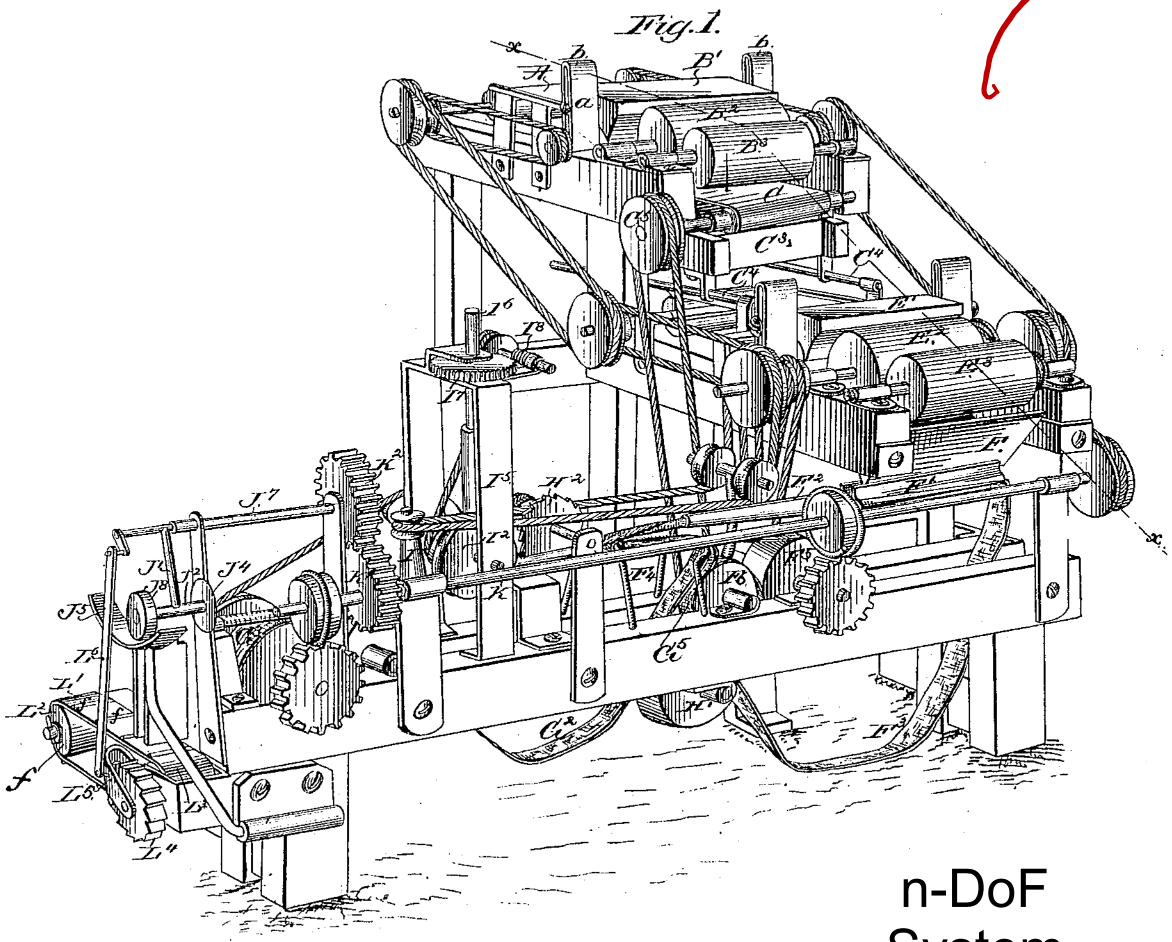
$$\ddot{x}_1(t) = F_1(x_1, x_2, \ldots, t)$$
$$\ddot{x}_2(t) = F_2(x_1, x_2, \ldots, t)$$
$$\ldots$$
$$\ddot{x}_n(t) = F_n(x_1, x_2, \ldots, t)$$

+constraints

+ limitations

Reduction of masses and forces

Idea of reduction

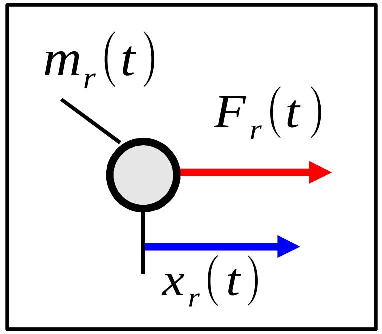


Source: *James Albert Bonsack (1859 – 1924) - U.S. patent 238,640 cigarette rolling machine, invented in 1880 and patented in 1881*

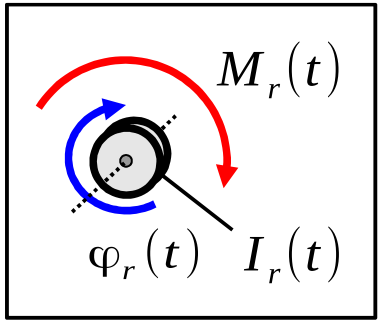
n-DoF System

simpler, but not always possible

1-DoF System

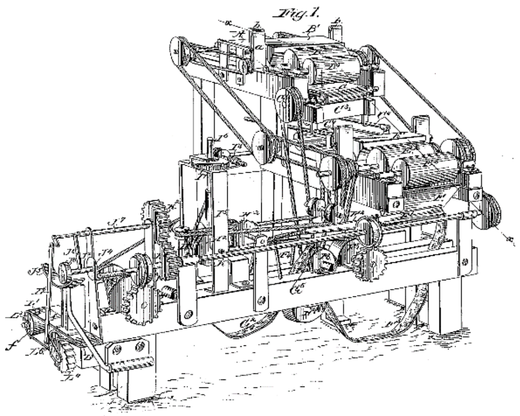


or



Reduction of masses

Kinetic energy

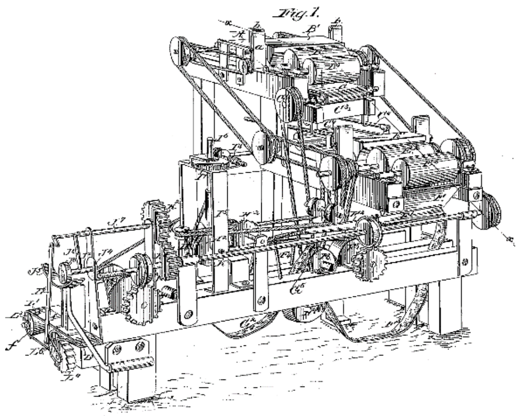


Total kinetic energy

$$T = \sum_{i=1}^{n} \left(\frac{1}{2} m_i v_i^2 + \frac{1}{2} I_i \omega_i^2 \right)$$

Reduction of masses

Kinetic energy

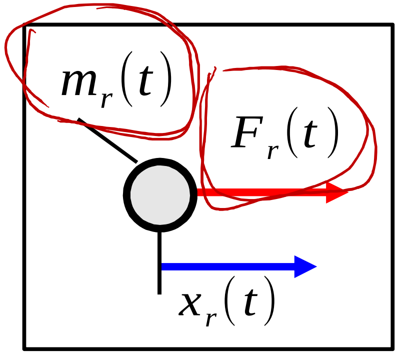


Total kinetic energy

$$T=\sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2+\frac{1}{2}I_i\omega_i^2\right)$$

$$T=\frac{1}{2}m_r v_r^2$$

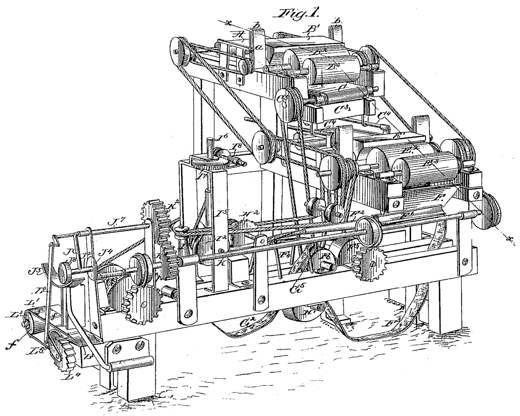
reduced mass



$$v_r=\frac{dx_r(t)}{dt}$$

Reduction of masses

Kinetic energy



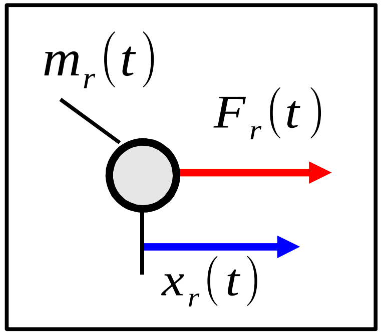
Total kinetic energy

$$T=\sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2+\frac{1}{2}I_i\omega_i^2\right)$$

or

$$T=\frac{1}{2}m_r\,v_r^2$$

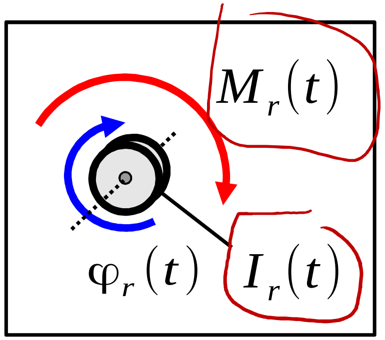
reduced mass



$$v_r=\frac{dx_r(t)}{dt}$$

$$T=\frac{1}{2}I_r\,\omega_r^2$$

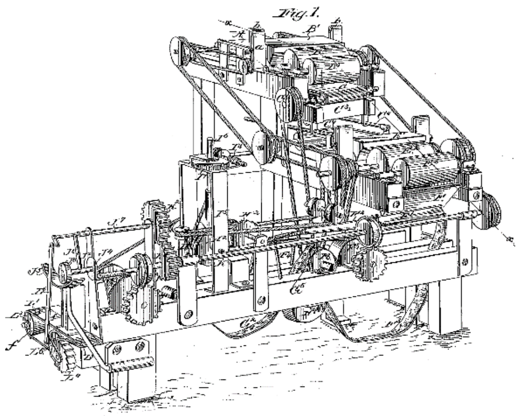
reduced moment of inertia



$$\omega_r=\frac{d\varphi_r(t)}{dt}$$

Reduction of forces

System power

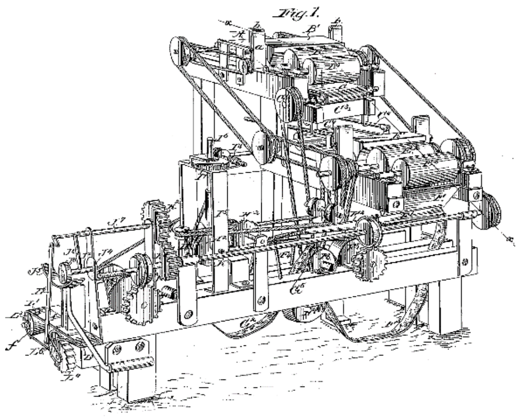


Total system's power

$$P(F_i, M_i, \omega_i, v_i, \ldots)$$

Reduction of forces

System power

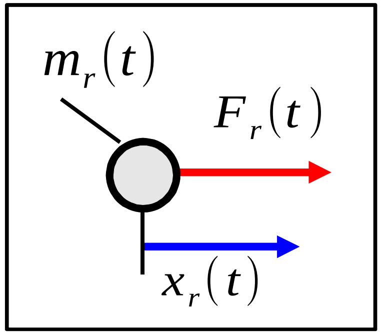


Total system's power

$$P\left(F_i, M_i, \omega_i, v_i, \ldots\right)$$

$$P = F_r v_r$$

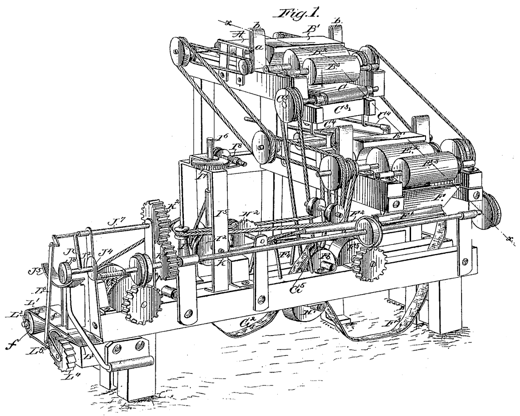
reduced force



$$v_r = \frac{dx_r(t)}{dt}$$

Reduction of forces

System power



Total system's power

$$P\left(F_i, M_i, \omega_i, v_i, \ldots\right)$$

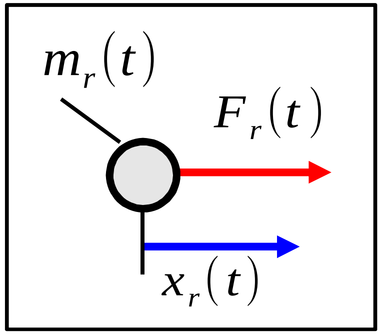
or

$$P = F_r v_r$$

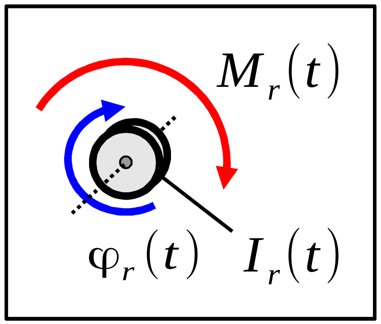
reduced force

$$P = M_r \omega_r$$

reduced torque



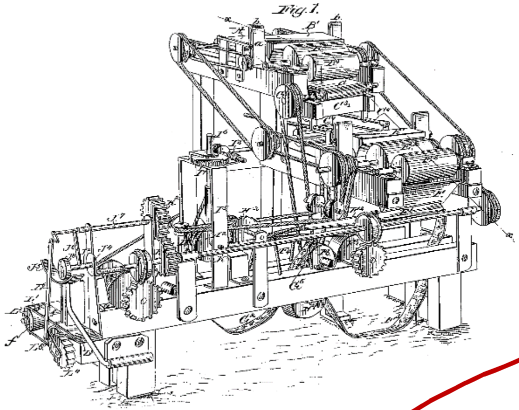
$$v_r = \frac{dx_r(t)}{dt}$$



$$\omega_r = \frac{d\varphi_r(t)}{dt}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2}m_r V_r^2 = \sum \frac{1}{2} m_i V_i^2 + \sum \frac{1}{2} I_j \omega_j^2 \quad / \cdot 2 \quad : V_r^2$$

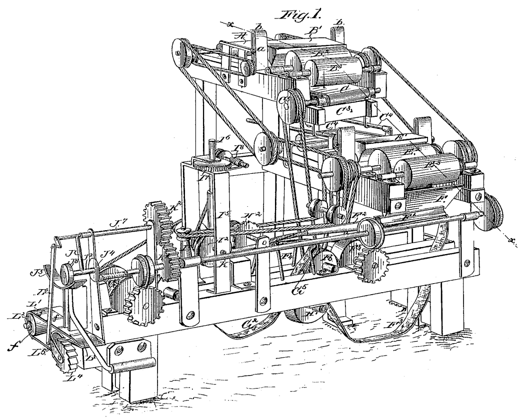
$$m_r = \sum m_i \frac{V_i^2}{V_r^2} + \sum I_j \frac{\omega_j^2}{V_r^2}$$

$$\frac{1}{2} I_r \omega_r^2 = \sum \frac{1}{2} m_i v_i^2 + \sum \frac{1}{2} I_j \omega_j^2$$

$$I_r = \sum m_i \frac{V_i^2}{\omega_r^2} + \sum I_j \frac{\omega_j^2}{\omega_r^2}$$

Reduction of masses – details

Kinetic energy



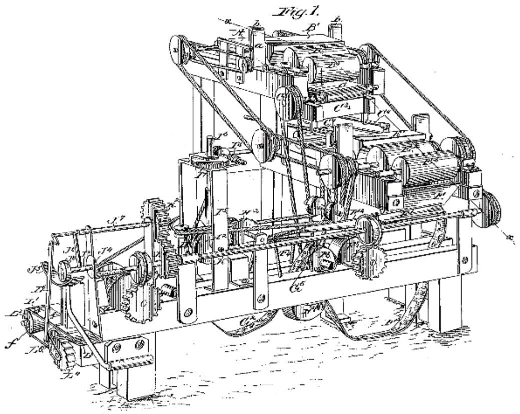
$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 + \sum_{j=1}^{k} \frac{1}{2} I_j \omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2} m_r v_r^2 = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 + \sum_{j=1}^{k} \frac{1}{2} I_j \omega_j^2$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

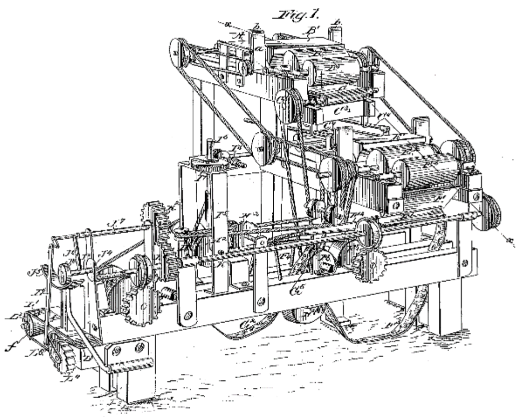
n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

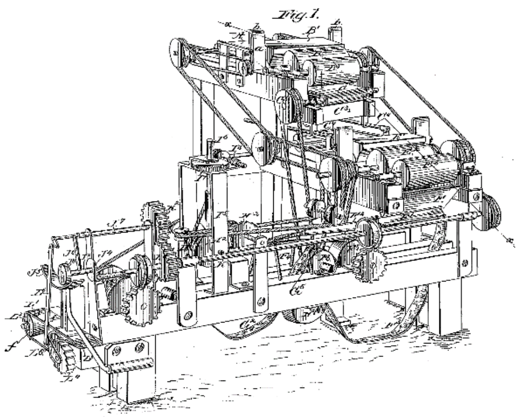
$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$\frac{1}{2}I_r\omega_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

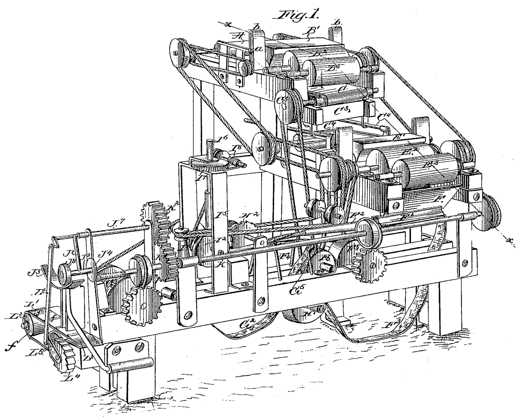
$$\frac{1}{2}I_r\omega_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

$$I_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{\omega_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{\omega_r^2}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$\frac{1}{2}I_r \omega_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

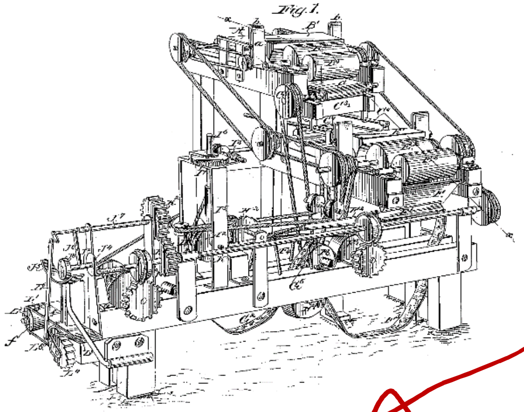
$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

$$I_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{\omega_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{\omega_r^2}$$

$v_r,\ \omega_r$ – arbitrary chosen velocities

Reduction of forces – details

Work



$$dW = \sum_{i=1}^{n} P_i \, ds_i \cos \alpha_i + \sum_{j=1}^{k} M_j \, d\varphi_j$$

$$\alpha_i = \sphericalangle (P_i, ds_i)$$

n – translating elements

k – rotating elements

$$M_r \, d\varphi_r = \ldots$$

$$P_r \cdot ds_r = \sum P_i \, ds_i \cos \alpha_i + \sum M_j \, d\varphi_j$$

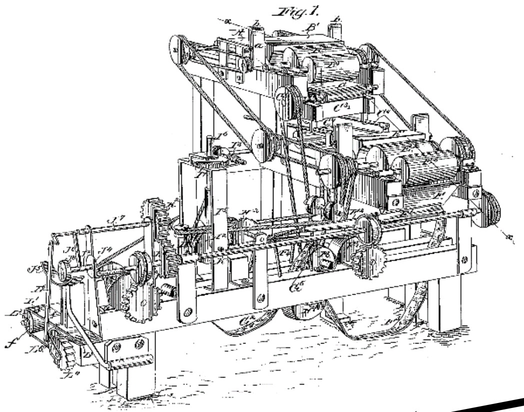
$$P_r = \sum \frac{P_i \, ds_i \cos \alpha_i}{ds_r} + \sum M_j \frac{d\varphi_j}{ds_r}$$

$$ds_i = v_i \cdot dt \; ; \; ds_r = v_r \, dt \; ; \; d\varphi_j = \omega_j \, dt$$

$$P_r = \sum P_i \frac{v_i \, dt}{v_r \, dt} \cos \alpha_i + \sum M_j \frac{\omega_j \, dt}{v_r \, dt}$$

Reduction of forces – details

<u>Work</u>



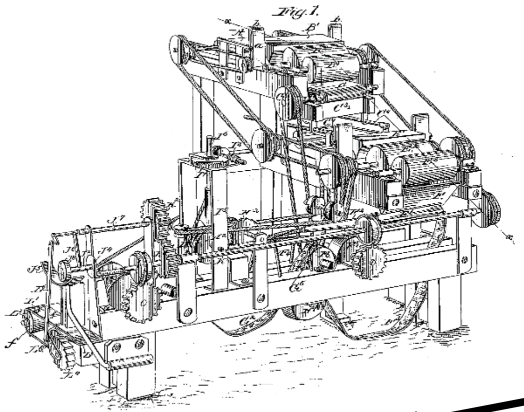
$$dW = \sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r ds_r = \sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

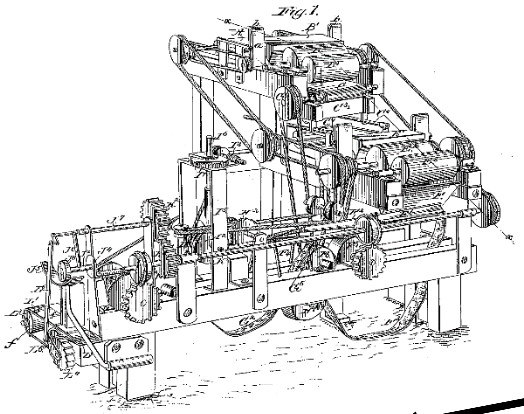
n – translating elements
k – rotating elements

$$P_r ds_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

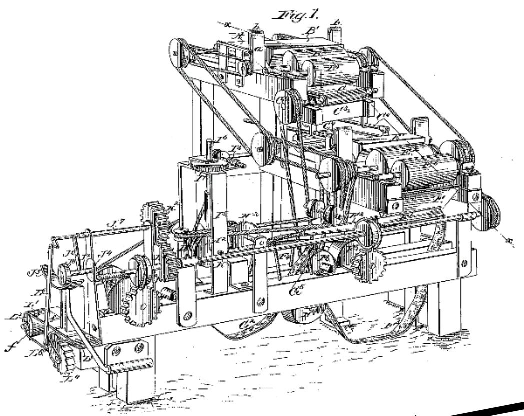
$$P_r ds_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i dt}{v_r dt} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{\omega_j dt}{v_r dt}$$

Reduction of forces – details

Work



$$dW = \sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r ds_r = \sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

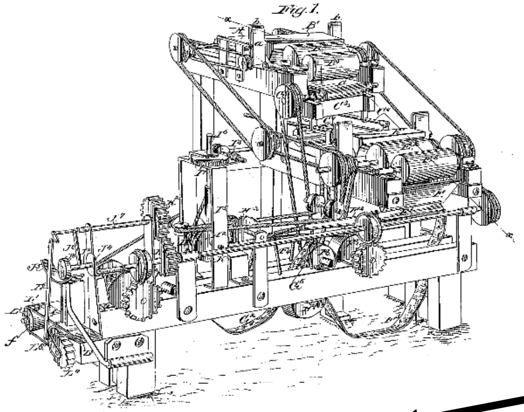
$$P_r = \sum_{i=1}^{n} P_i \frac{ds_i}{ds_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r = \sum_{i=1}^{n} P_i \frac{v_i dt}{v_r dt} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{\omega_j dt}{v_r dt}$$

$$P_r = \sum_{i=1}^{n} P_i \frac{v_i}{v_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{\omega_j}{v_r}$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r ds_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i dt}{v_r dt}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j dt}{v_r dt}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i}{v_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{v_r}$$

$$M_r d\varphi_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

$$M_r=\sum_{i=1}^{n} P_i \frac{ds_i}{d\varphi_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{d\varphi_r}$$

$$M_r=\sum_{i=1}^{n} P_i \frac{v_i dt}{\omega_r dt}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j dt}{\omega_r dt}$$

$$M_r=\sum_{i=1}^{n} P_i \frac{v_i}{\omega_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{\omega_r}$$

Reduction of masses/moments of inertia

$$m_r=\sum_{i=1}^{n} m_i \frac{v_i^2}{v_r^2}+\sum_{j=1}^{k} I_j \frac{\omega_j^2}{v_r^2}$$

$$I_r=\sum_{i=1}^{n} m_i \frac{v_i^2}{\omega_r^2}+\sum_{j=1}^{k} I_j \frac{\omega_j^2}{\omega_r^2}$$

Reduction of forces/torques

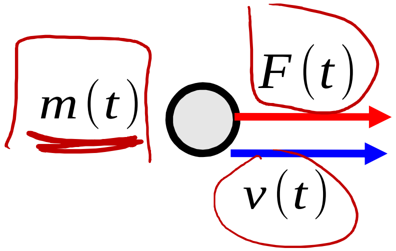
$$P_r=\sum_{i=1}^{n} P_i \frac{v_i}{v_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{v_r}$$

$$M_r=\sum_{i=1}^{n} P_i \frac{v_i}{\omega_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{\omega_r}$$

Machine equation of motion

$$d(f \cdot g) = df \cdot g + f \cdot dg$$

Linear motion



$$dT = dW$$

$$d\left(\frac{1}{2} m v^2\right) = F \cdot dx$$

$$\frac{1}{2} dm\, v^2 + \frac{1}{2} m\, 2v \cdot dv = F \cdot dx$$

$$\frac{1}{2} dm \frac{dx}{dt} \cdot v + m \frac{dx}{dt} dv = F dx$$

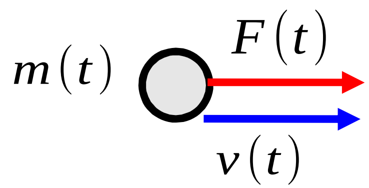
$$\frac{dm}{dt} \cdot \frac{v}{2} + m \frac{dv}{dt} = F$$

$$\frac{dm_r(t)}{dt} \frac{v(t)}{2} + m_r(t) \frac{dv(t)}{dt} = F_r(t)$$

$$\text{if } m = \text{const.} \rightarrow m \frac{dv}{dt} = F \quad (m\ddot{x} = F)$$

Machine equation of motion

Linear motion



elementary change of kinetic energy → $dT = dW$ ← elementary work

complete differential of kinetic energy →

$$d\left(\frac{1}{2}m(t)v(t)^2\right) = F(t)\,dx$$

$$\frac{1}{2}dm(t)v(t)^2 + m(t)v(t)dv(t) = F(t)\,dx$$

$$\frac{1}{2}dm(t)v(t)^2 + m(t)\frac{dx(t)}{dt}dv(t) = F(t)\,dx$$

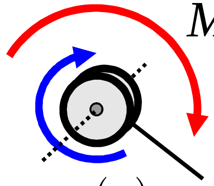
$$\frac{dm(t)}{dx}\frac{v(t)^2}{2} + m\frac{dv(t)}{dt} = F(t)$$

$$\boxed{\frac{dm(t)}{dt}\frac{v(t)}{2} + m\frac{dv(t)}{dt} = F(t)}$$

$$if\;\; m = const. \;\Rightarrow\; m\frac{dv(t)}{dt} = P(t) \quad or \quad m\ddot{x}(t) = F(t)$$

Machine equation of motion

Angular motion



$$dT = dW$$

$$d\left(I\frac{\omega(t)^2}{2}\right) = M(t)\,d\varphi$$

$$\dots$$

$$\dots$$

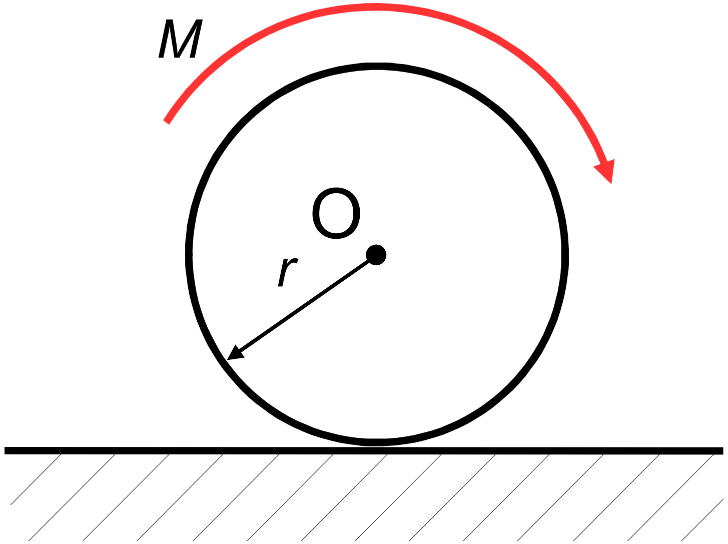
$$\frac{dI(t)}{d\varphi}\frac{\omega(t)^2}{2} + I(t)\frac{d\omega(t)}{dt} = M(t)$$

$$\boxed{\frac{dI(t)}{dt}\frac{\omega(t)}{2} + I(t)\frac{d\omega(t)}{dt} = M(t)}$$

$$if\ \ I = const. \Rightarrow\ I\frac{d\omega(t)}{dt} = M(t)\quad or\quad I\ddot{\varphi}(t) = M(t)$$

Reduction of masses and forces

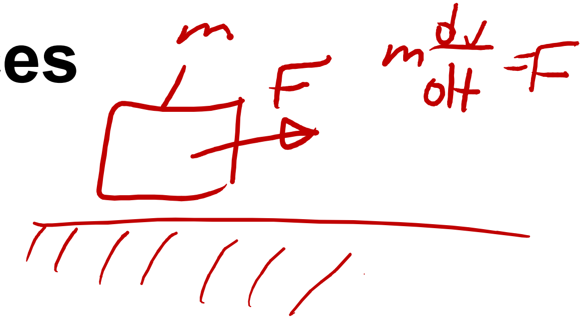
Rolling wheel (without a slip)



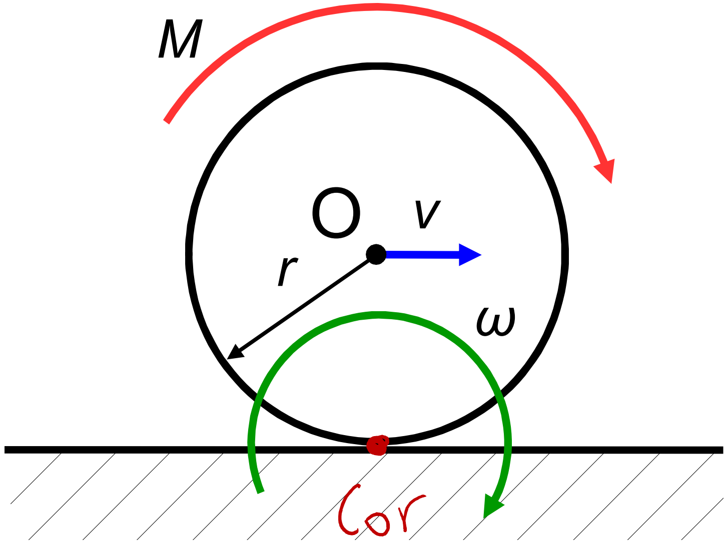
Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

1 DoF

Reduction of masses and forces



Rolling wheel (without a slip)



Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque. $M(t)$

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$V = \omega \cdot r$$

$$T(v,\omega) = \frac{1}{2} m v^2 + \frac{1}{2} I_0 \omega^2$$

$$T(v) = \frac{1}{2} m v^2 + \frac{1}{2} I_0 \frac{v^2}{r^2}$$

$$T(v) = \frac{1}{2}\left(m + \frac{I_0}{r^2}\right) v^2$$

$$m_r = \text{const.}$$

$$N(\omega) = M \cdot \omega$$

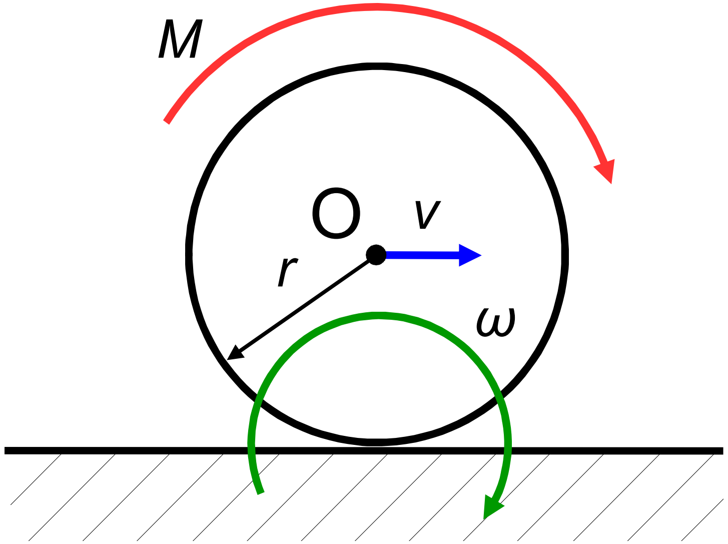
$$N(v) = M \frac{v}{r} = \frac{M}{r} \cdot v \qquad F_r \neq \text{const.}$$

$$\frac{d m_r}{dt} \frac{v}{2} + m_r \frac{dv}{dt} = F_r$$

$$\left(m + \frac{I_0}{r^2}\right) \frac{dv(t)}{dt} = \frac{M(t)}{r}$$

Reduction of masses and forces

Rolling wheel (without a slip)

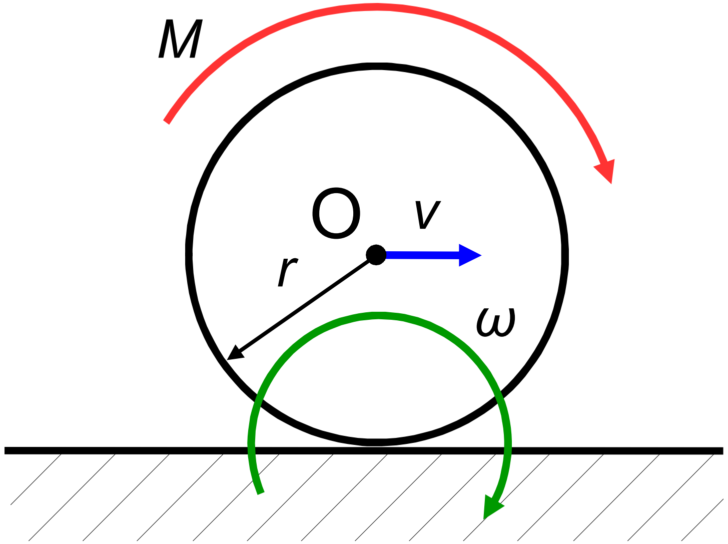


Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

Reduction of masses and forces

Rolling wheel (without a slip)

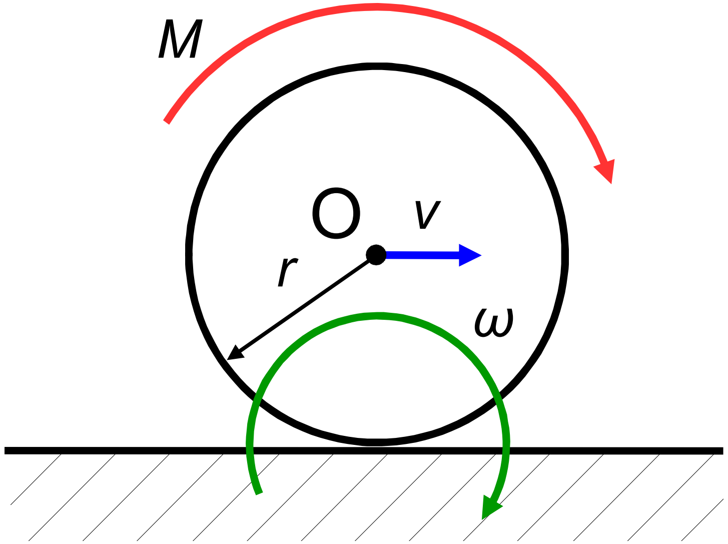


Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

Reduction of masses and forces

Rolling wheel (without a slip)



Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

$$P=M\omega$$

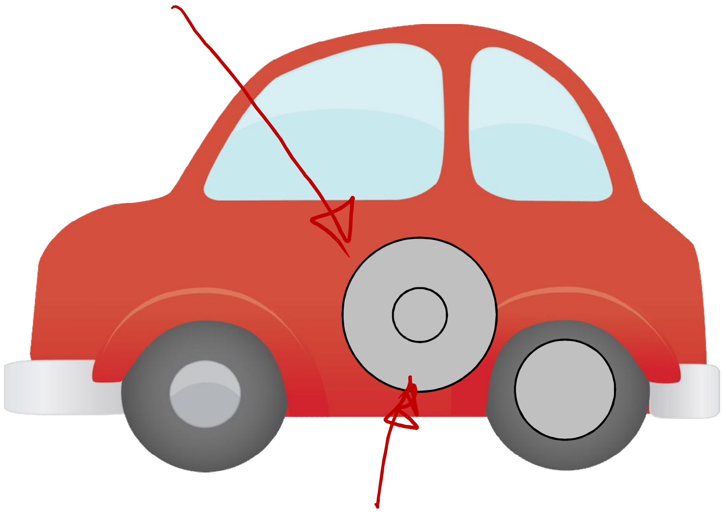
$$P=M\frac{v}{r}=\frac{M}{r}v=F_r v$$

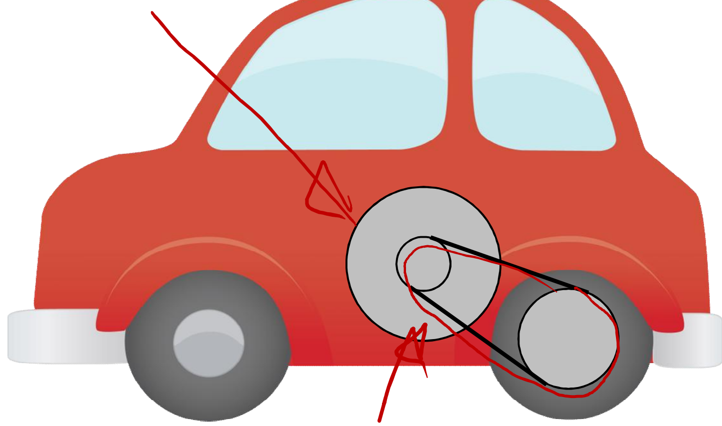
$$F_r=\frac{M}{r}$$

$$m_r\frac{dv}{dt}=F_r$$

$$\boxed{\left(m+\frac{I_0}{r^2}\right)\frac{dv}{dt}=\frac{M}{r}}$$

Reduction of masses and forces – quiz





m_1 – total mass
m_{r1} – reduced mass

m_2 – total mass
m_{r2} – reduced mass

$$m_1 = m_2$$

$$m_{r1} \; ? \; m_{r2}$$

Reduction of masses and forces

Example 1

In this example a drum winch is analyzed. It consist of:

- electric motor EM, which generates torque as a function of angular velocity ω: $M=A-B\omega$ where A and B are given constants; rotor moment of inertia is equal to I_m;
- two stage gearbox GB (reducer) with given gears moments of inertia I_1, I_2, I_3, I_4; shafts moments of inertia are equal to I_s; gears ratio are given as

$i_1=\omega_2/\omega_1$ and $i_2=\omega_4/\omega_3$

- winch's drum has diameter D and moment of inertia I_d; drum is set on two ball bearing that generates resistance M_f assumed to be constant;
- inclined plane with angle α related to the horizon line;
- box of weight G pulled up with winch. Friction between object and inclined plane is represented as dry friction with μ coefficient.

Reduction of masses and forces

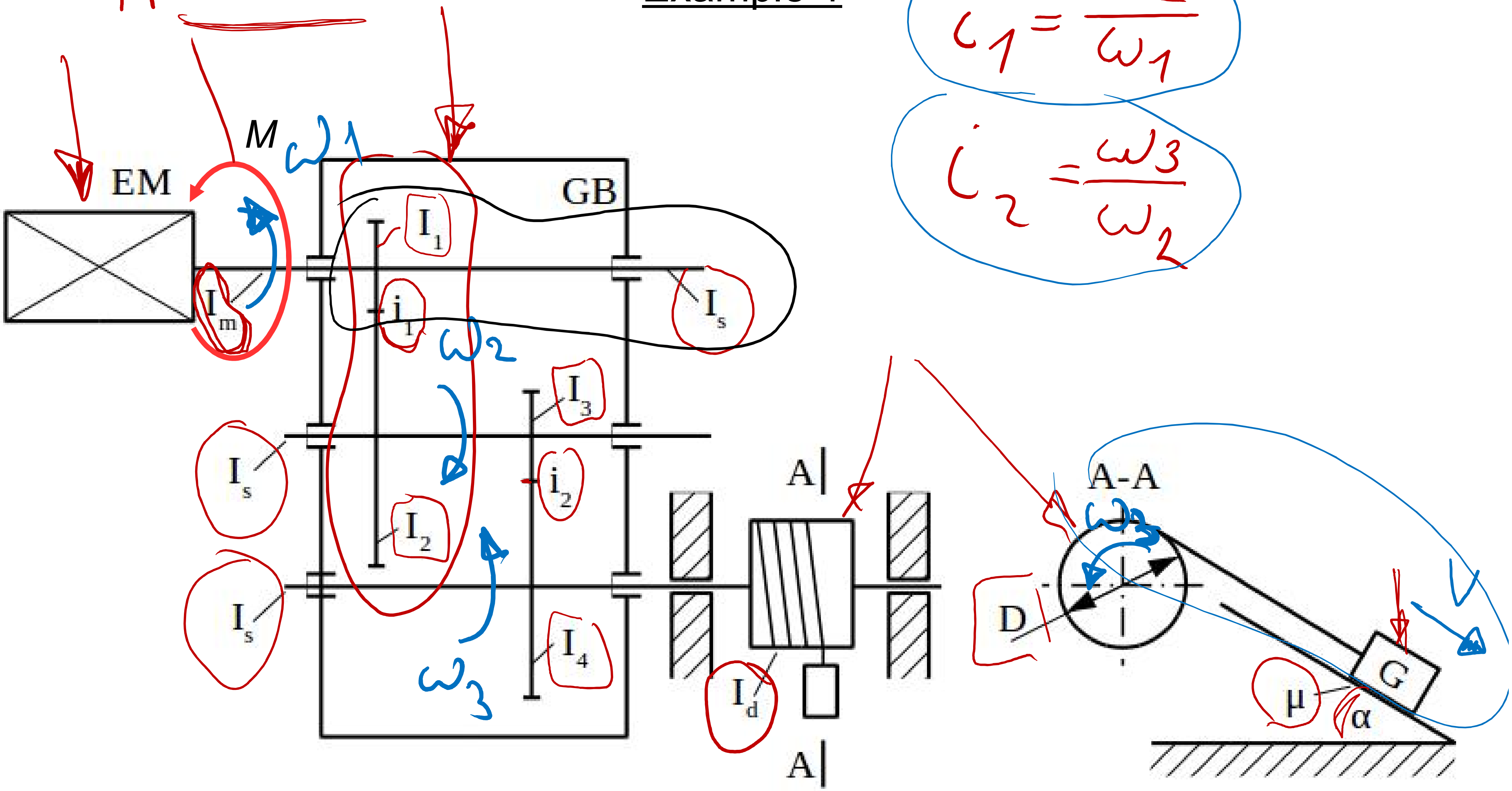
Example 1

$M = A - B\omega$

$v = \omega_3 \cdot \frac{D}{2}$

$i_1 = \frac{\omega_2}{\omega_1}$

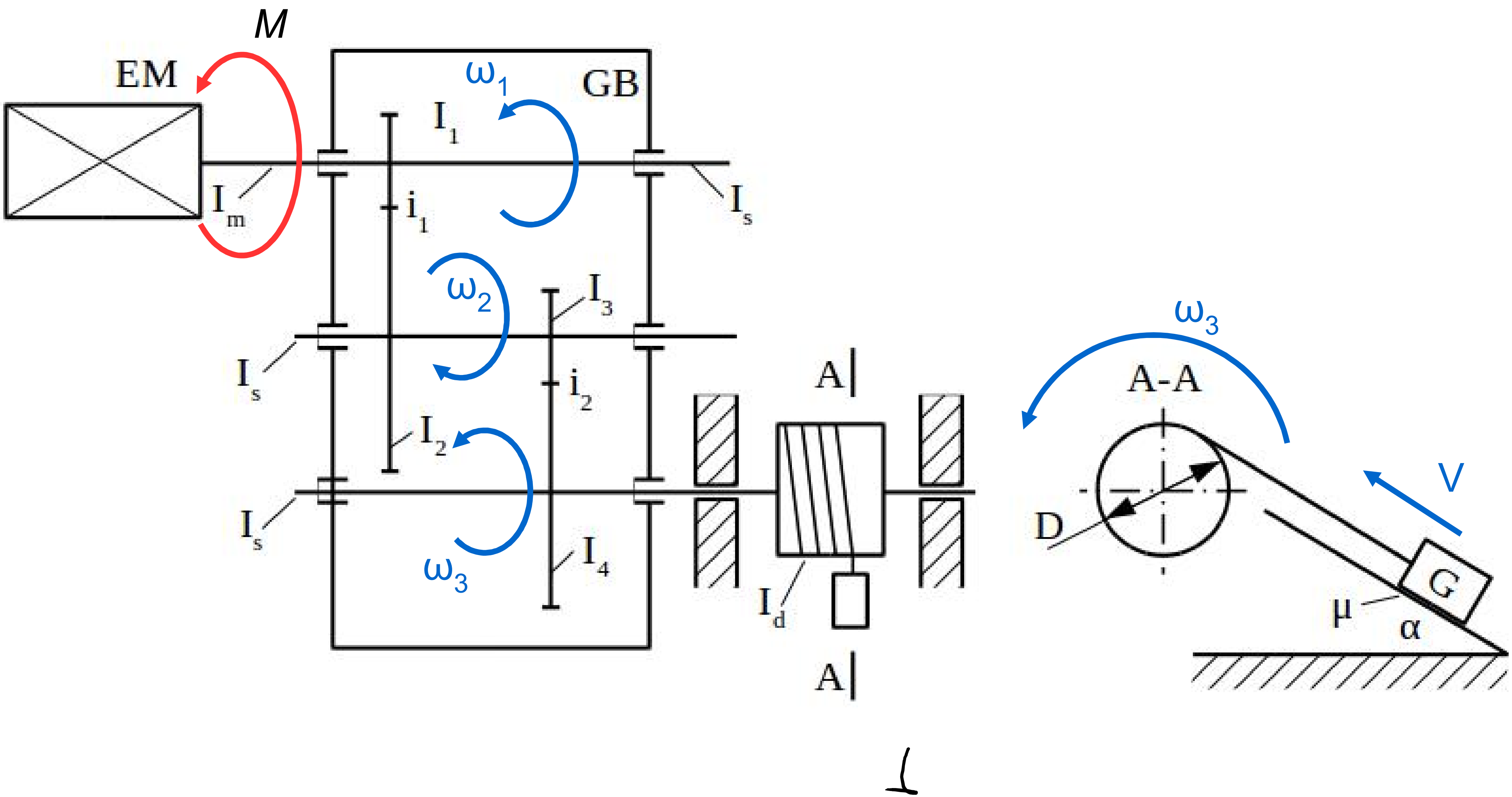
$i_2 = \frac{\omega_3}{\omega_2}$



$$T' = \frac{1}{2} I_m \omega_1^2 + \frac{1}{2}(I_s + I_1)\omega_1^2 + \frac{1}{2}(I_s + I_2 + I_3)\omega_2^2 + \frac{1}{2}(I_s + I_4 + I_d)\omega_3^2 + \frac{1}{2}\frac{G}{g}V^2$$

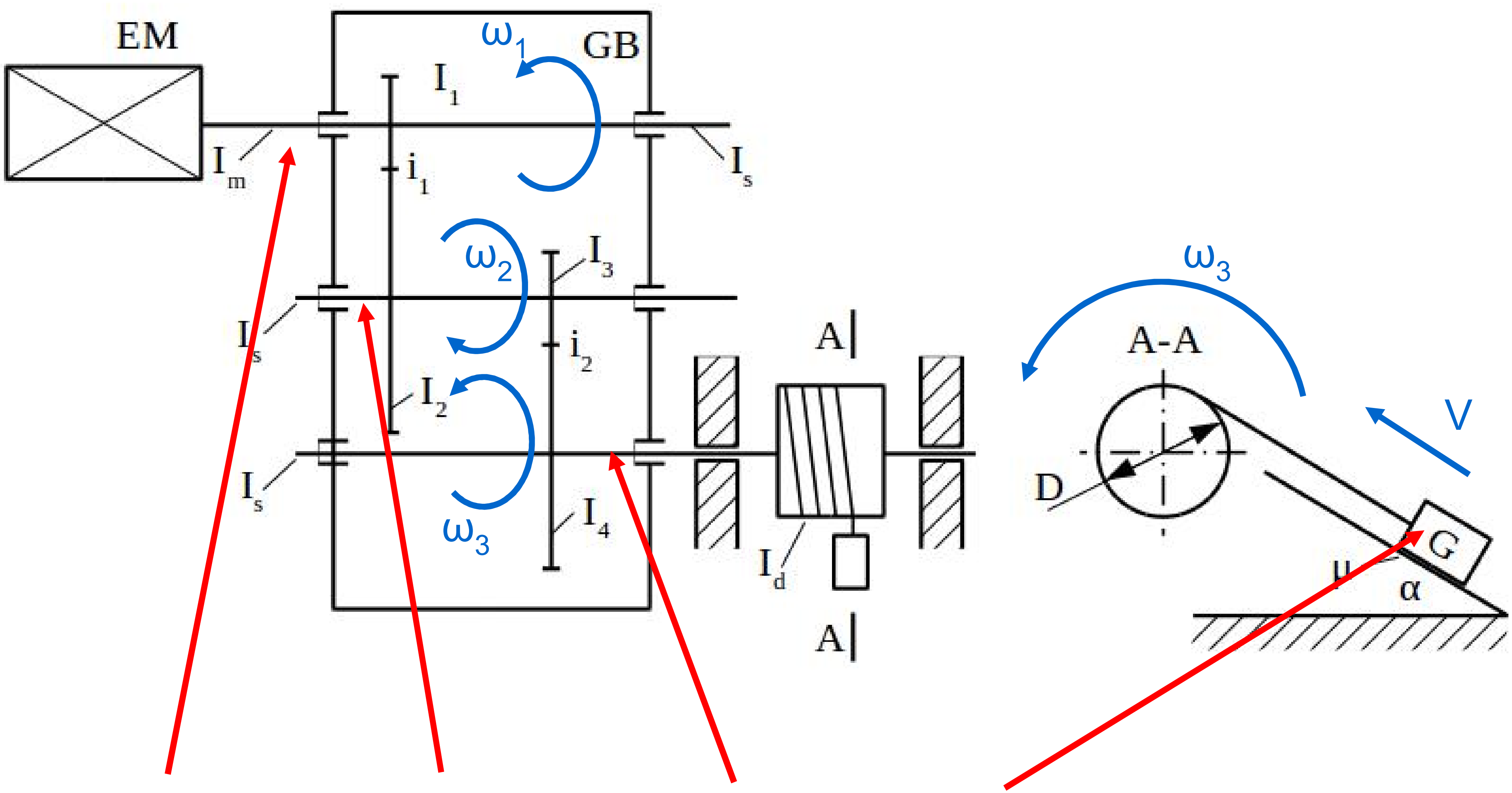
Reduction of masses and forces

Example 1



Reduction of masses and forces

Example 1



$$T=\frac{1}{2}(I_m+I_1+I_s)\omega_1^2+\frac{1}{2}(I_2+I_3+I_s)\omega_2^2+\frac{1}{2}(I_4+I_d+I_s)\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

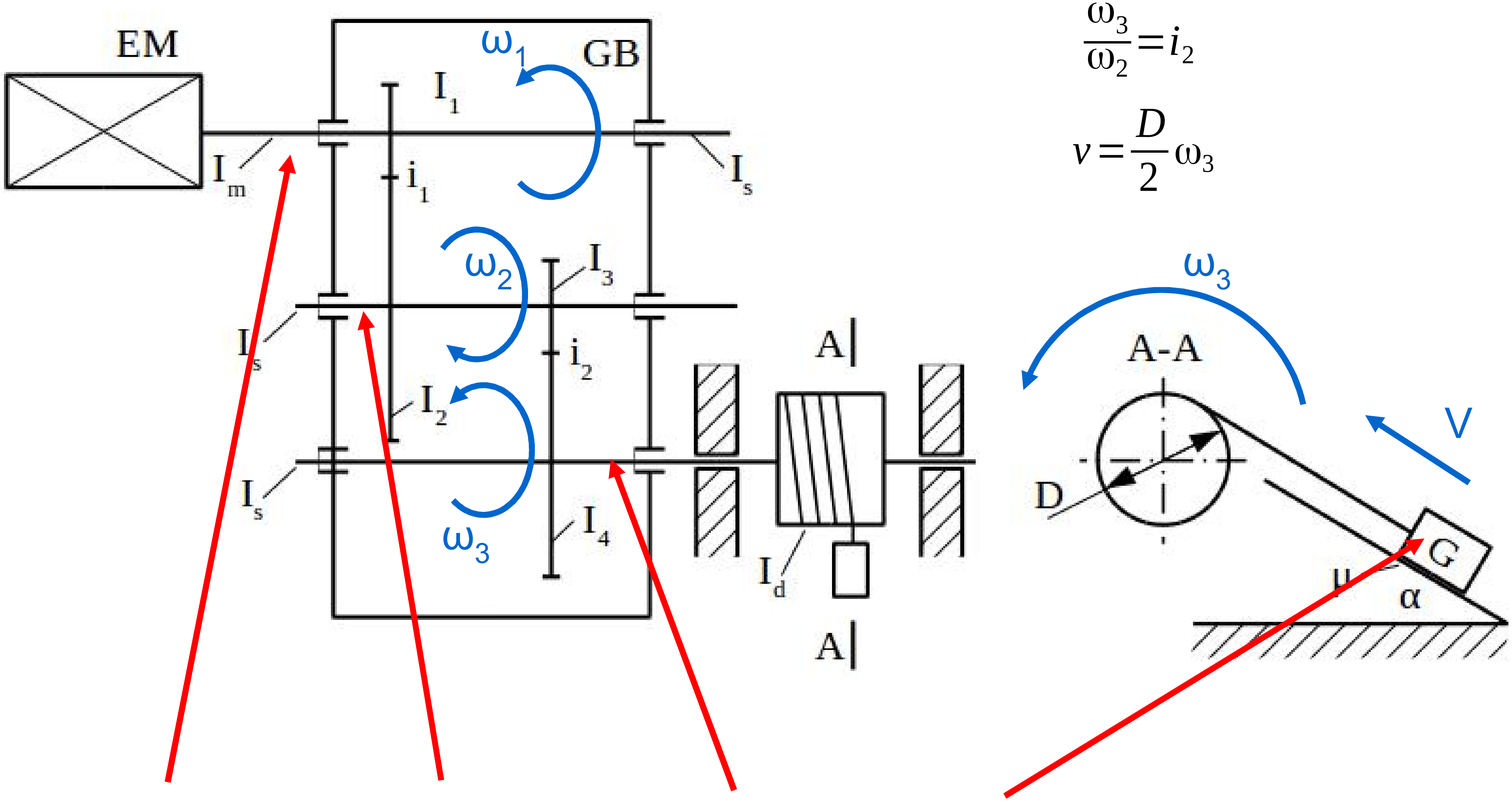
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1}=i_1$$

$$\frac{\omega_3}{\omega_2}=i_2$$

$$v=\frac{D}{2}\omega_3$$



$$T=\frac{1}{2}(I_m+I_1+I_s)\,\omega_1^2+\frac{1}{2}(I_2+I_3+I_s)\,\omega_2^2+\frac{1}{2}(I_4+I_d+I_s)\,\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

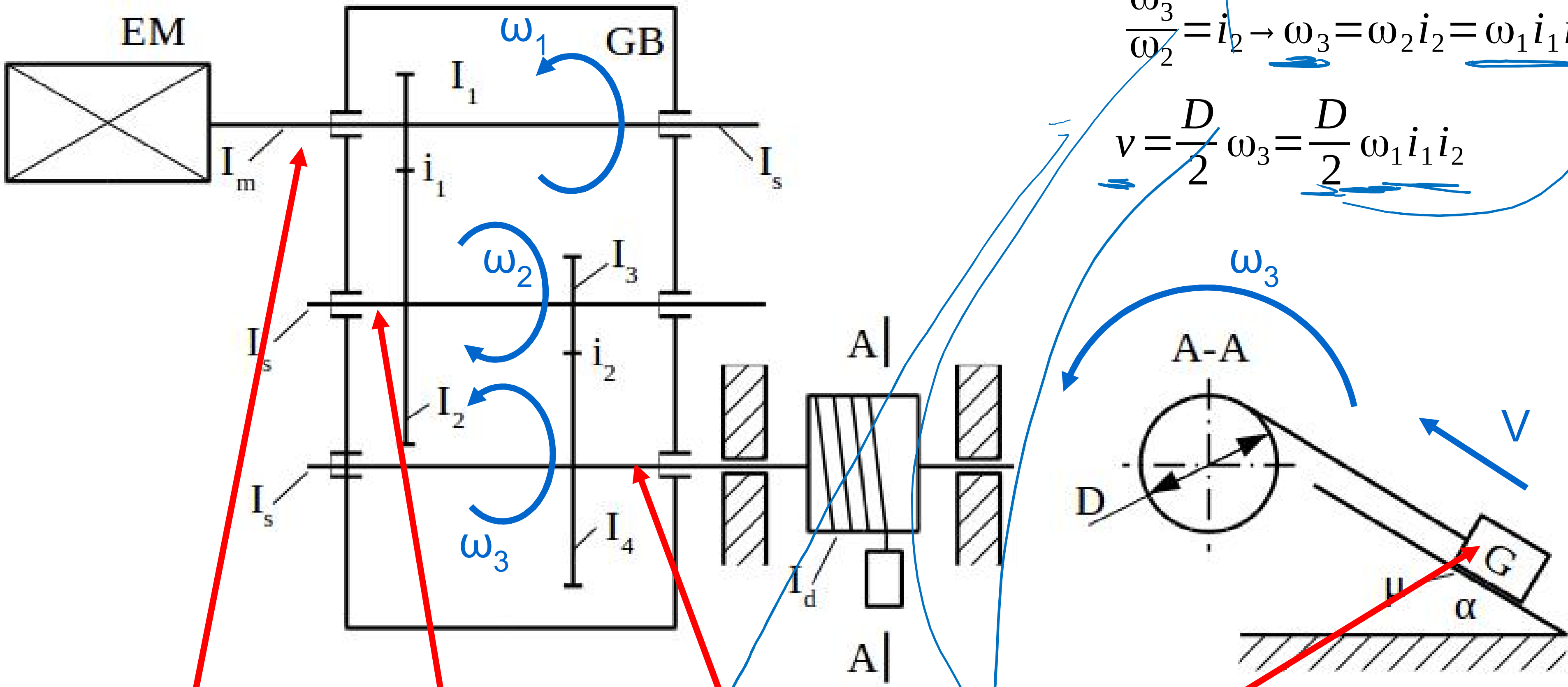
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1}=i_1 \rightarrow \omega_2=\omega_1 i_1$$

$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$



$$T=\frac{1}{2}(I_m+I_1+I_s)\omega_1^2+\frac{1}{2}(I_2+I_3+I_s)\omega_2^2+\frac{1}{2}(I_4+I_d+I_s)\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

$\rightarrow T(\omega_1)$

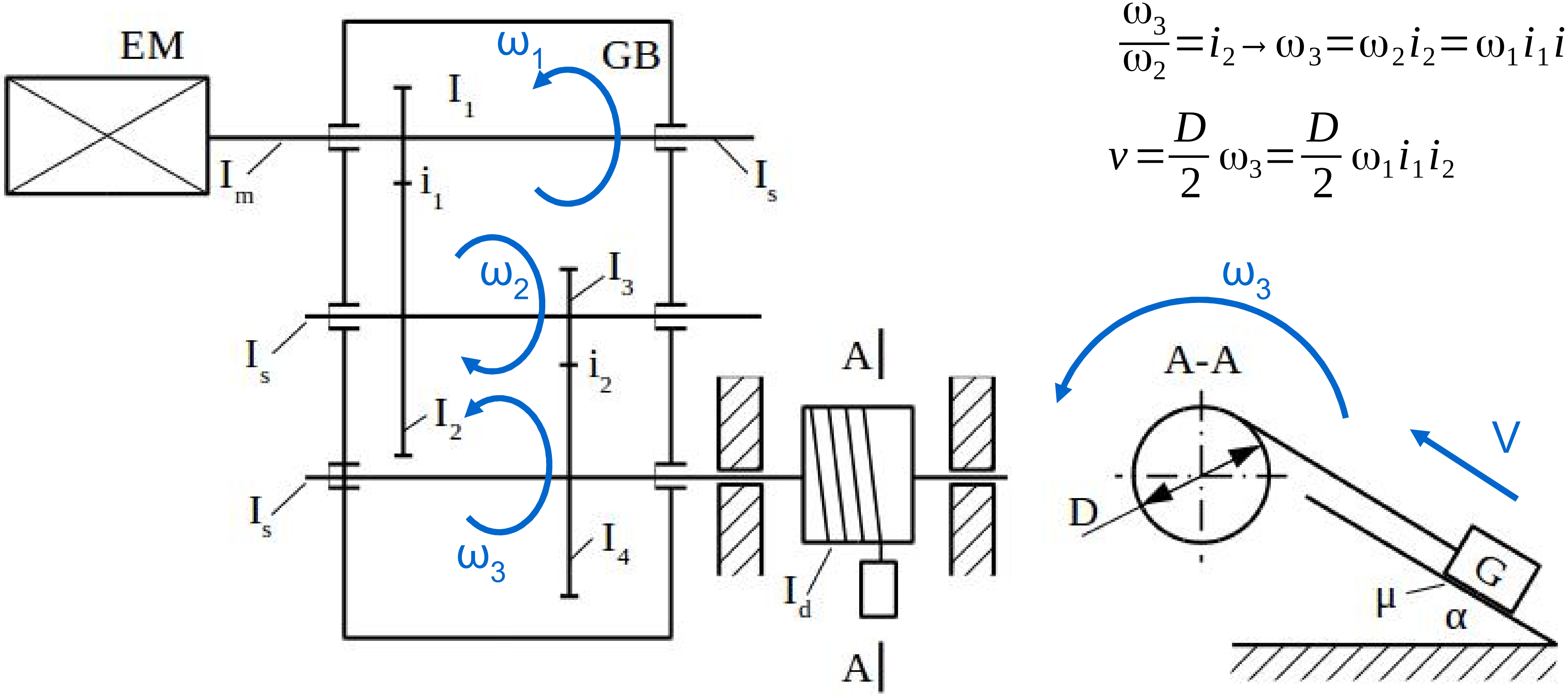
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1} = i_1 \rightarrow \omega_2 = \omega_1 i_1$$

$$\frac{\omega_3}{\omega_2} = i_2 \rightarrow \omega_3 = \omega_2 i_2 = \omega_1 i_1 i_2$$

$$v = \frac{D}{2}\omega_3 = \frac{D}{2}\omega_1 i_1 i_2$$



$$T = \frac{1}{2}(I_m + I_1 + I_s)\omega_1^2 + \frac{1}{2}(I_2 + I_3 + I_s)\omega_1^2 i_1^2 + \frac{1}{2}(I_4 + I_d + I_s)\omega_1^2 i_1^2 i_2^2 + \frac{1}{2}\frac{G}{g}\frac{D^2}{4}\omega_1^2 i_1^2 i_2^2$$

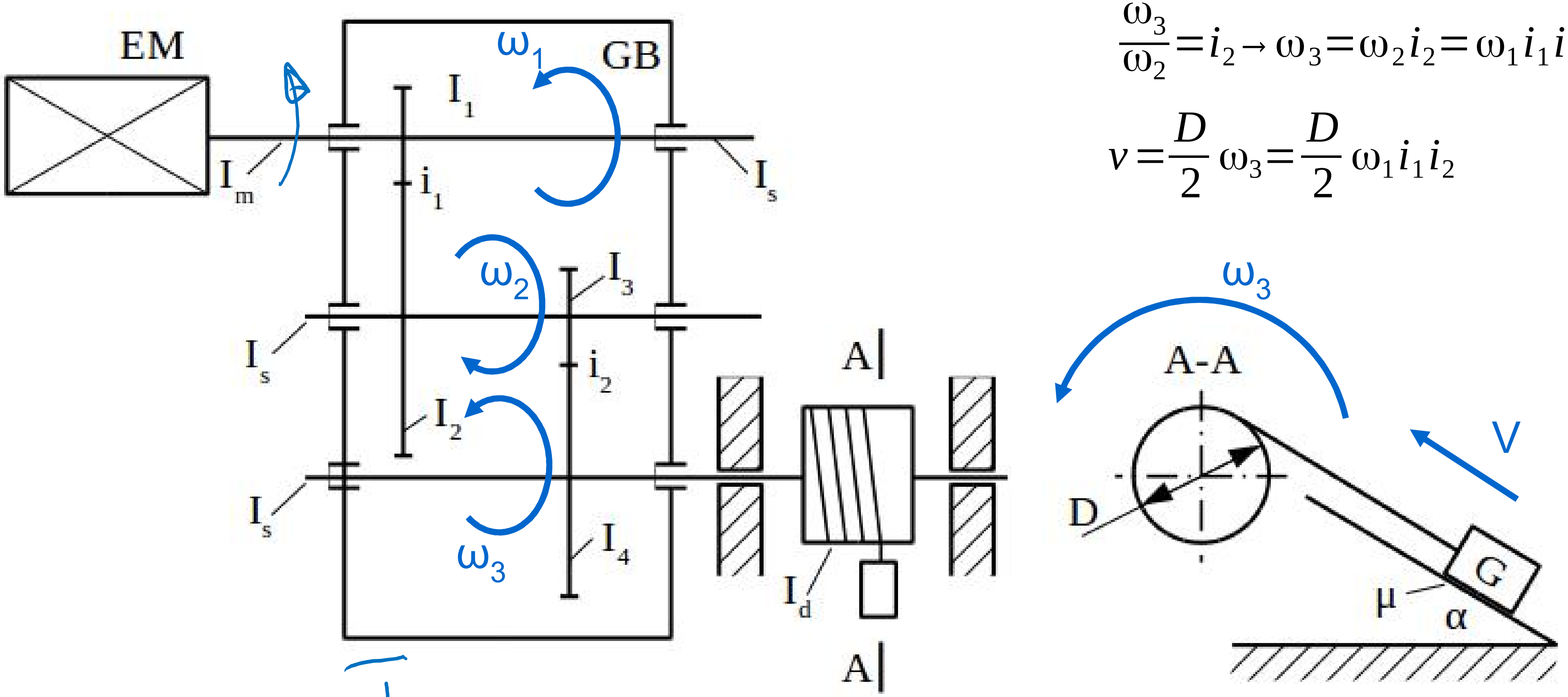
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1} = i_1 \rightarrow \omega_2 = \omega_1 i_1$$

$$\frac{\omega_3}{\omega_2} = i_2 \rightarrow \omega_3 = \omega_2 i_2 = \omega_1 i_1 i_2$$

$$v = \frac{D}{2}\omega_3 = \frac{D}{2}\omega_1 i_1 i_2$$



$$T = \frac{1}{2}\left[\left(I_m + I_1 + I_s\right) + \left(I_2 + I_3 + I_s\right) i_1^2 + \left(I_4 + I_d + I_s\right) i_1^2 i_2^2 + \frac{G}{g}\frac{D^2}{4} i_1^2 i_2^2\right]\omega_1^2$$

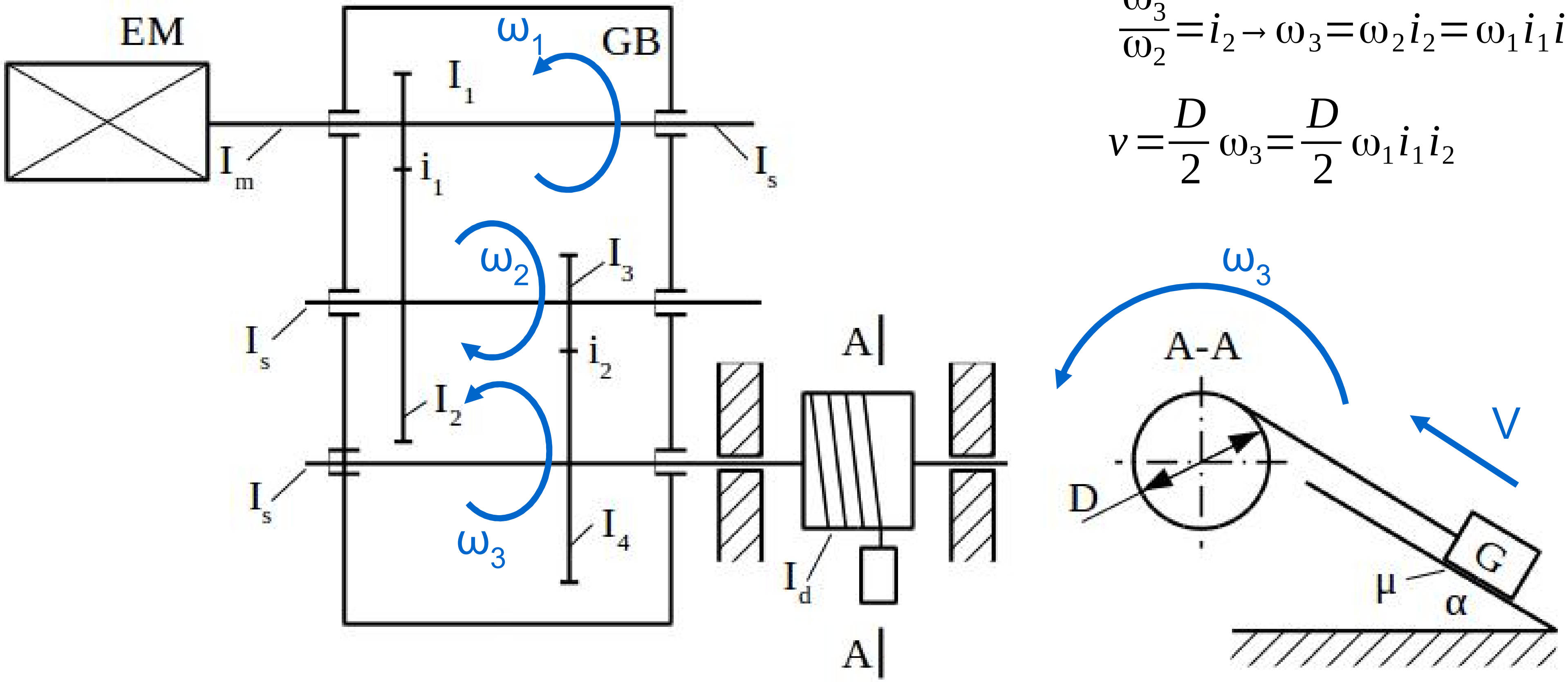
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1}=i_1 \rightarrow \omega_2=\omega_1 i_1$$

$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$

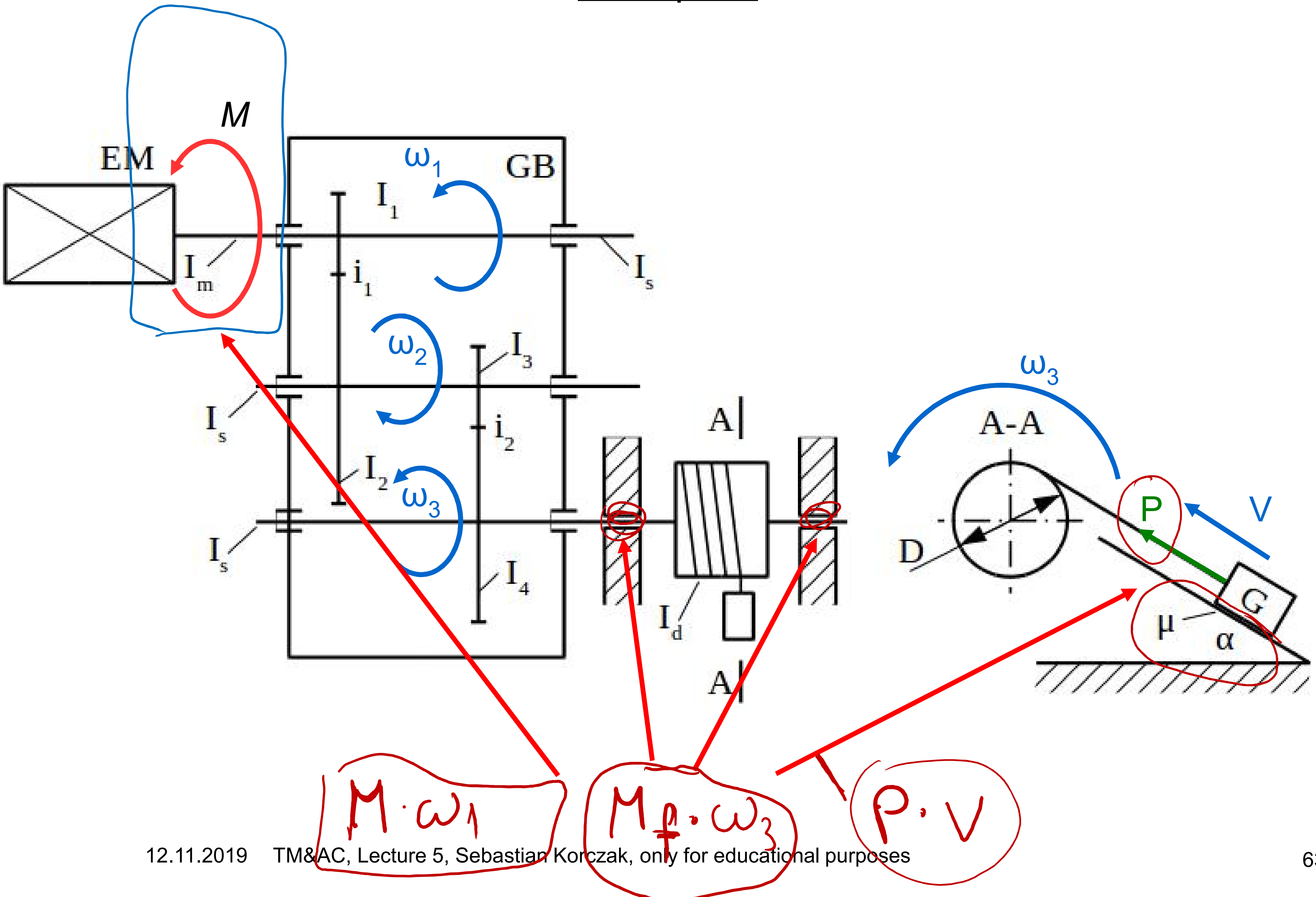


reduced moment of inertia

$$I_R=I_m+I_1+I_s+(I_2+I_3+I_s)i_1^2+(I_4+I_d+I_s)i_1^2 i_2^2+\frac{G}{g}\frac{D^2}{4}i_1^2 i_2^2$$

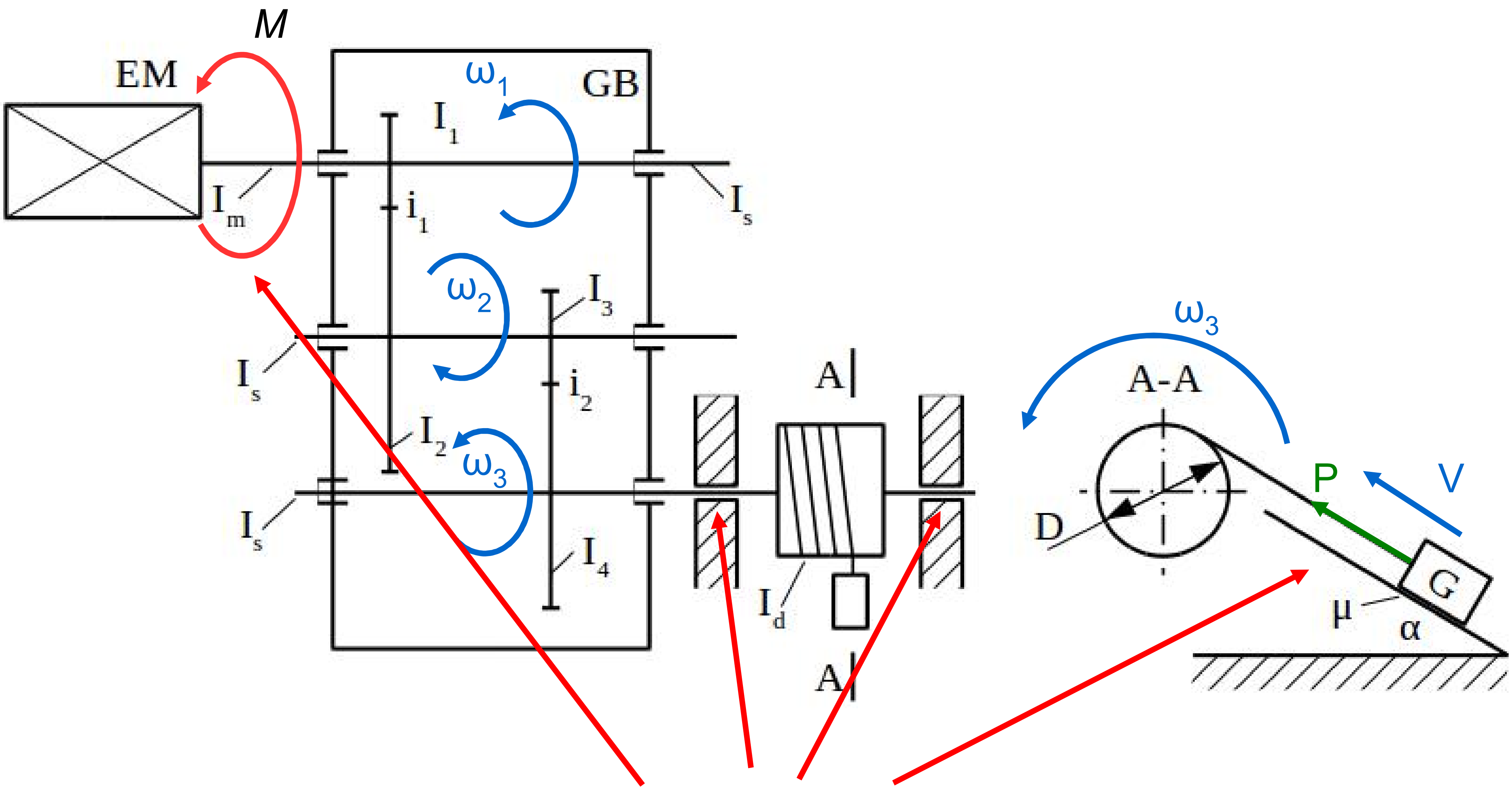
Reduction of masses and forces

Example 1



Reduction of masses and forces

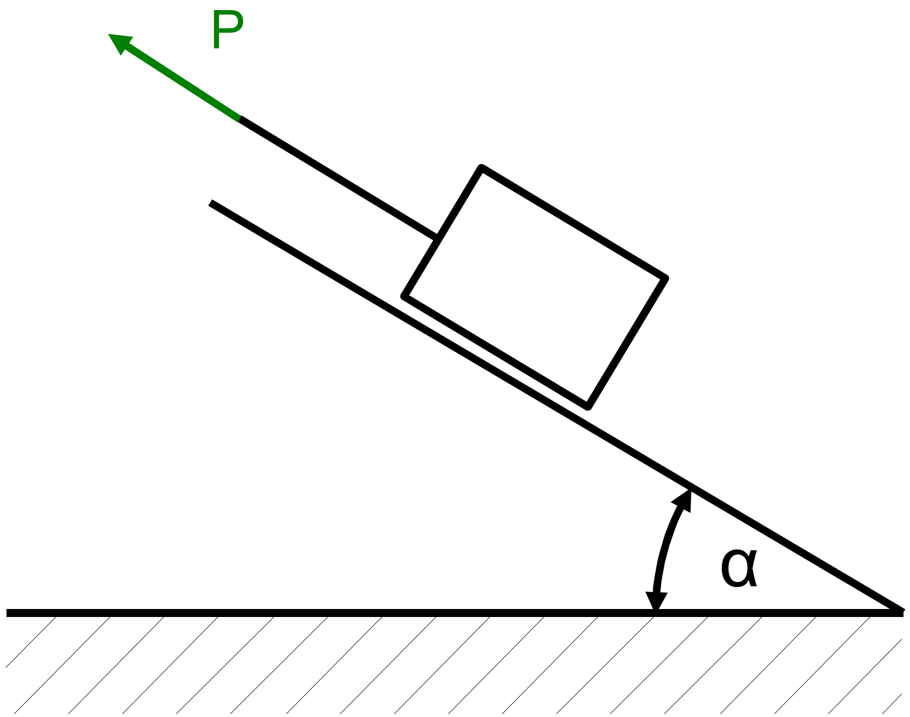
Example 1



$$N = M_s \omega_1 - M_f \omega_3 - P v$$

Reduction of masses and forces

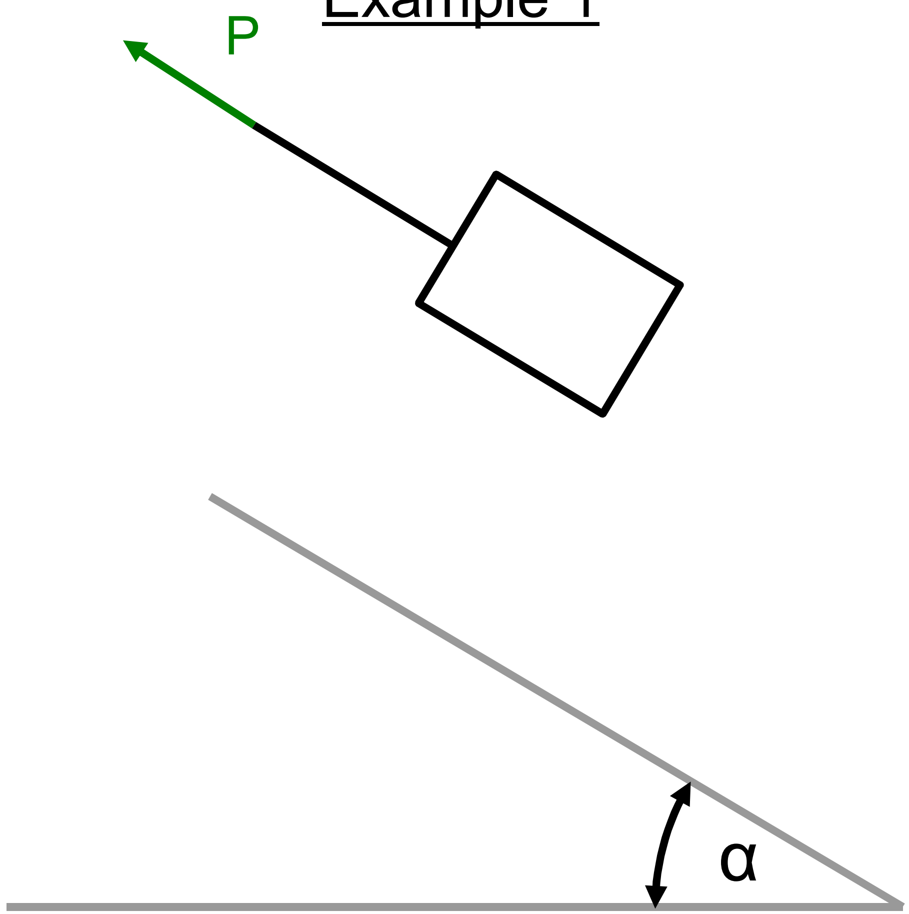
Example 1



P=…

Reduction of masses and forces

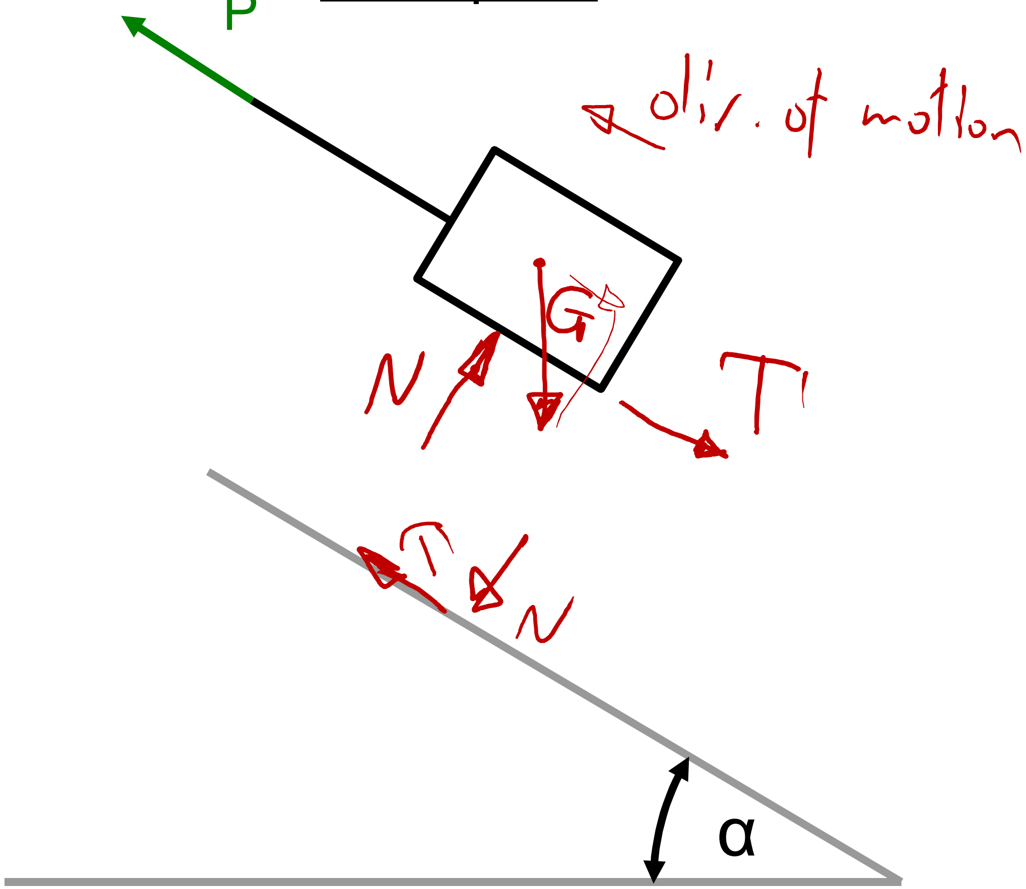
Example 1



P=…

Reduction of masses and forces

Example 1



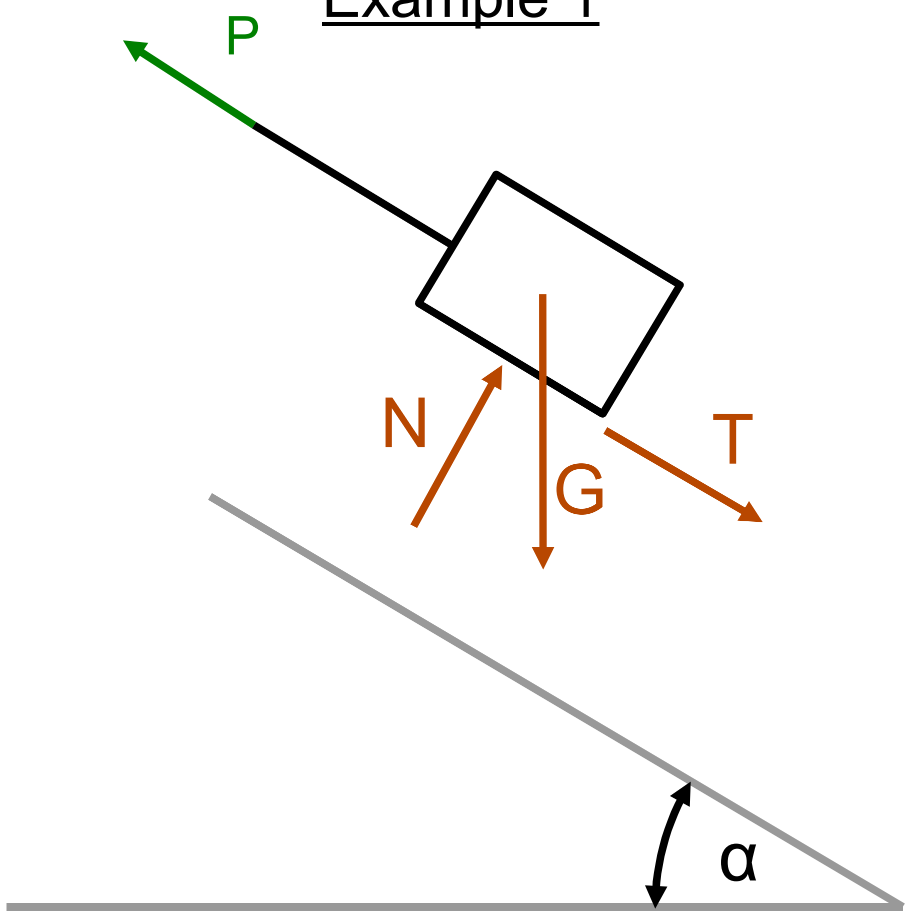
P=...

$$P = G\cdot\sin\alpha + T = G\sin\alpha + \mu N = G(\sin\alpha + \mu\cos\alpha)$$

$N = G\cos\alpha$

Reduction of masses and forces

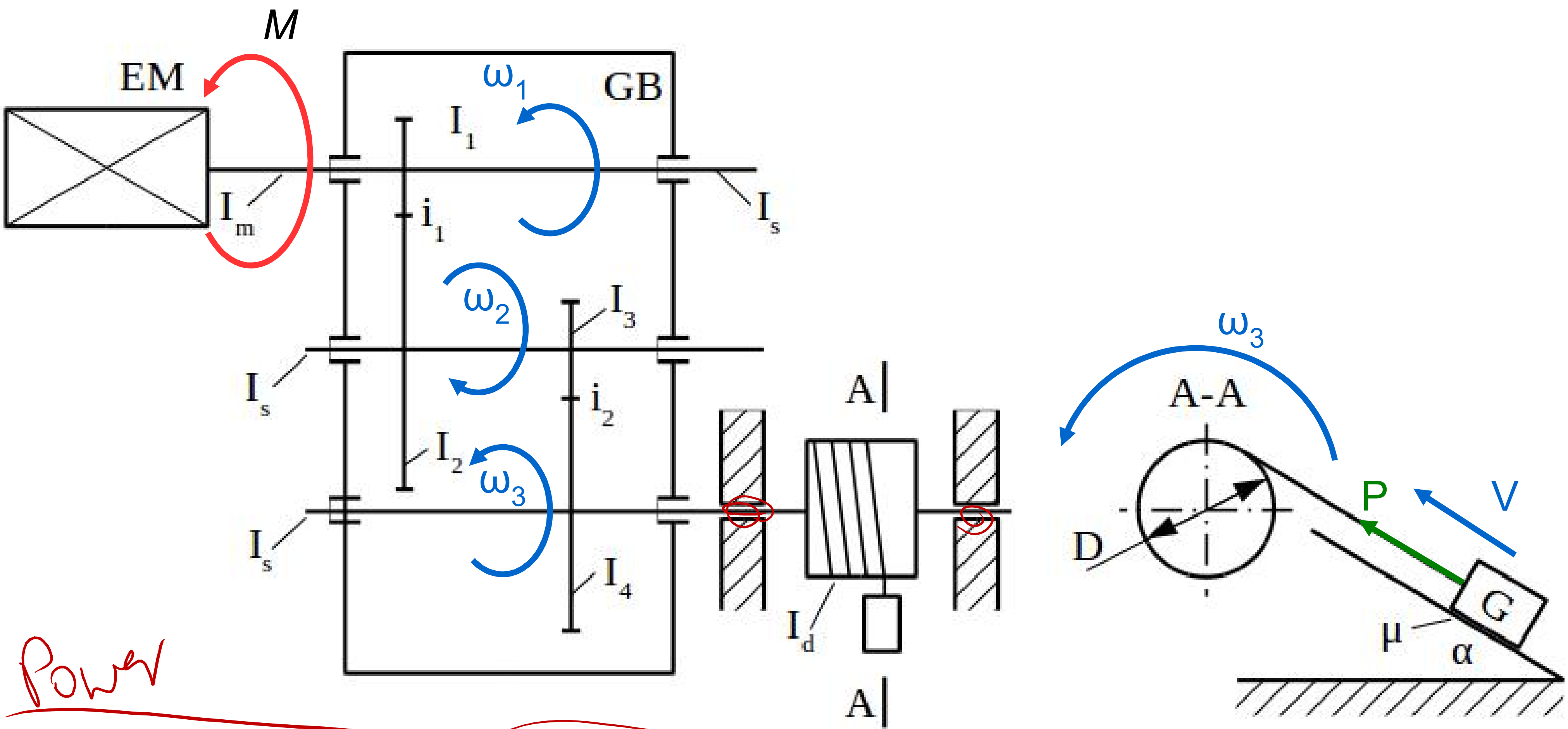
Example 1



$$P = T + G\sin\alpha = \mu N + G\sin\alpha = \mu G\cos\alpha + G\sin\alpha$$

Reduction of masses and forces

Example 1



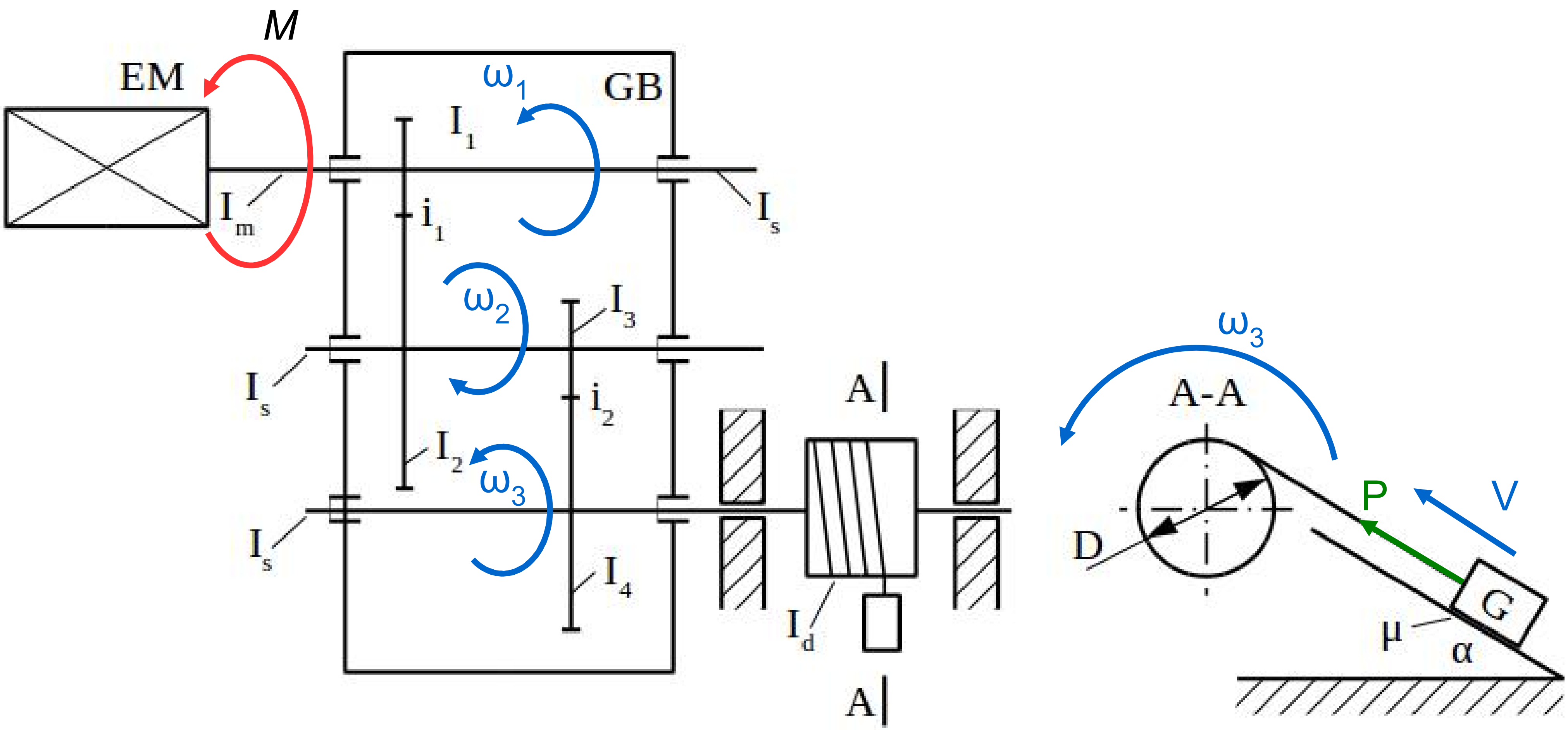
Power

$$N = M_s \omega_1 - M_f \omega_3 - (\mu G \cos\alpha + G \sin\alpha) v$$

→ $N(\omega_1)$

Reduction of masses and forces

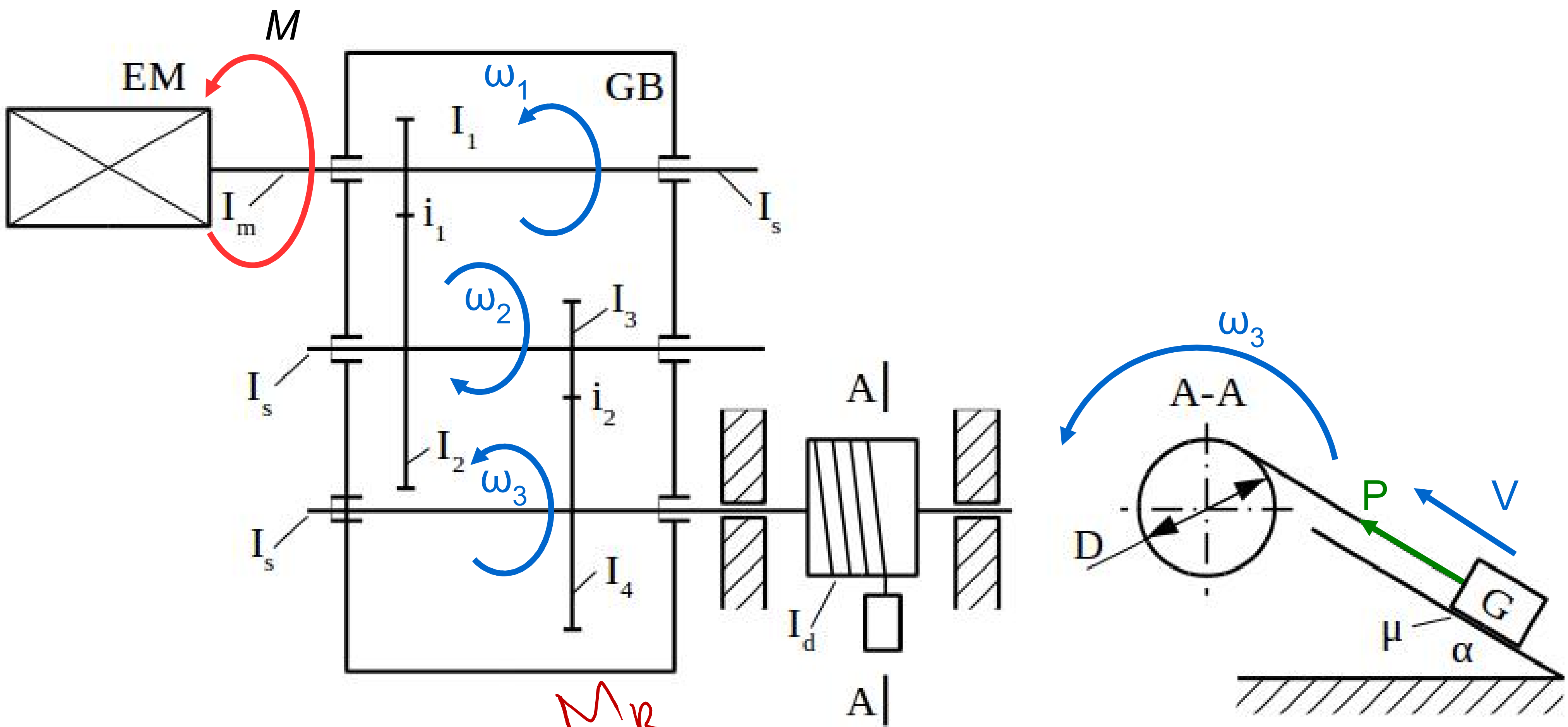
Example 1



$$N = M_s \omega_1 - M_f \omega_1 i_1 i_2 - (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} \omega_1 i_1 i_2$$

Reduction of masses and forces

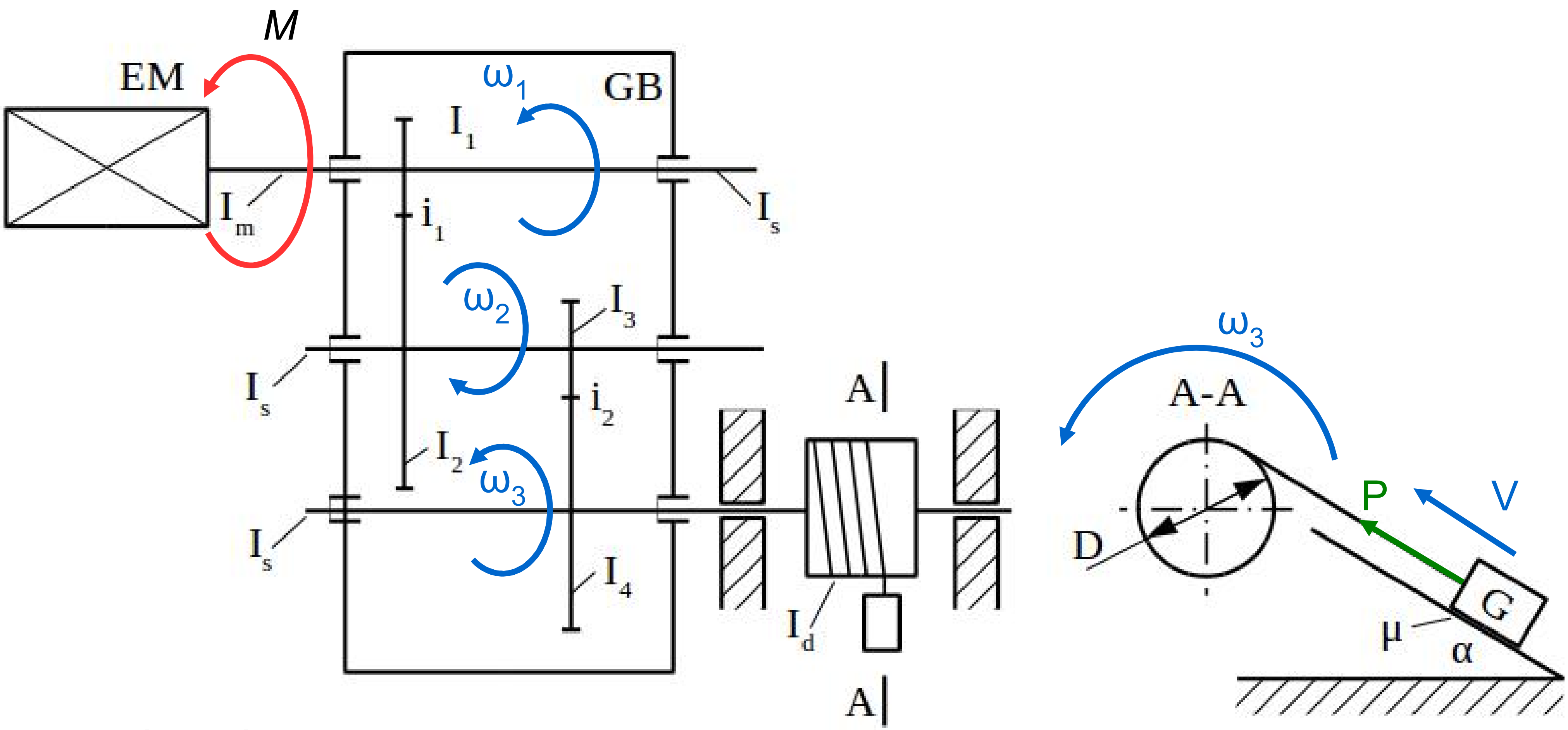
Example 1



$$N=\left[M_s - M_f i_1 i_2 - (\mu G\cos\alpha + G\sin\alpha)\frac{D}{2} i_1 i_2\right]\omega_1$$

Reduction of masses and forces

Example 1



reduced torque

$$M_R = M_s - M_f i_1 i_2 - (\mu G \cos\alpha + G \sin\alpha)\frac{D}{2} i_1 i_2$$

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = M_s - \left(M_f i_1 i_2 + (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2 \right)$$

motor $= (A - B\omega_1)$ — active

Passive torque

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = M_s - \left(M_f i_1 i_2 + (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2 \right)$$

active

driving torque M_D
(electric motor torque)

passive torque M_P

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = M_s - \left(M_f i_1 i_2 + (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2 \right)$$

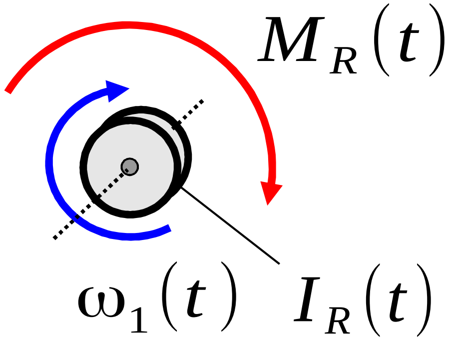
M_s: driving torque M_D (electric motor torque)

$\left(M_f i_1 i_2 + (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2 \right)$: passive torque M_P

$$M_R = M_D - M_P = (A - B\omega_1) - M_P$$

Reduction of masses and forces

Example 1



Start-up process

$$I_R \frac{d\omega_1}{dt} = M_R$$

$$M_R = A - B\omega_1 - M_P$$

const.

$$I_R \frac{d\omega_1}{dt} = A - B\omega_1 - M_P$$

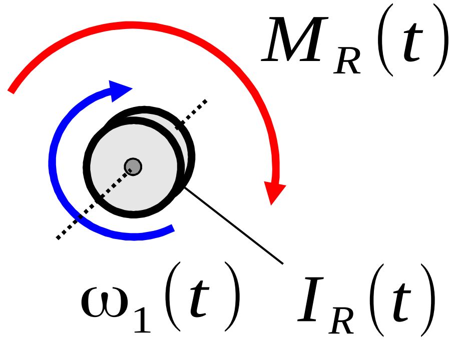
$$I_R \frac{d\omega_1}{dt} + B\omega_1 = A - M_P$$

$$\frac{d\omega_1(t)}{dt} + \frac{B}{I_R}\omega_1(t) = \frac{A - M_P}{I_R}$$

Reduction of masses and forces

Example 1

Start-up process

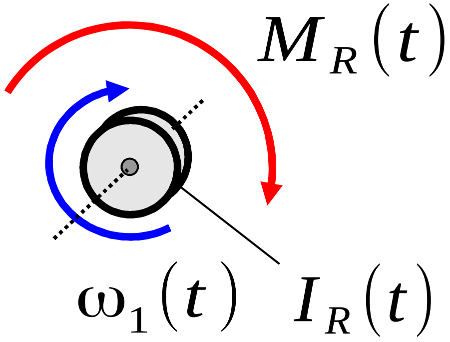


$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

Reduction of masses and forces

Example 1

Start-up process



$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

Reduction of masses and forces

Example 1

Start-up process

$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

A, M_P, B, I_R - constants

non-homogeneous
1st order ODE
with const. coef.

general solution

(2) $\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = 0$

$\rightarrow \omega_g(t) = E \cdot e^{-\frac{B}{I_R}t}$

particular solution

(1) $\omega_p(t) = F$ - const.

substitute

$F = \ldots$

initial condition

(3) $\omega_1(t) = \omega_g + \omega_p$

substitute IC

$\omega_1(t=0) = 0$

$$\omega_1(t) = \text{———}\left(1 - e^{-\frac{B}{I_R}t}\right)$$

Reduction of masses and forces

Example 1

Start-up process

$$\frac{d\omega_1}{dt}+\frac{B}{I_R}\omega_1=\frac{A-M_P}{I_R}$$

A, M_P, B, I_R - constants

non-homogeneous
1st order ODE
with const. coef.

general solution

particular solution

initial condition

Reduction of masses and forces

Example 1

Start-up process

$$\frac{d\omega_1}{dt}+\frac{B}{I_R}\omega_1=\frac{A-M_P}{I_R}$$

A, M_P, B, I_R - constants

non-homogeneous 1st order ODE with const. coef.

general solution

$$\omega_{1g}(t)=E\,e^{-\frac{B}{I_R}t}$$

particular solution

$$\omega_{1p}(t)=F$$

initial condition

$$\omega_1(t=0)=0$$

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$

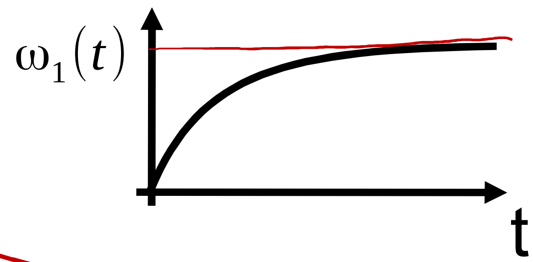
Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$

ω_{max}

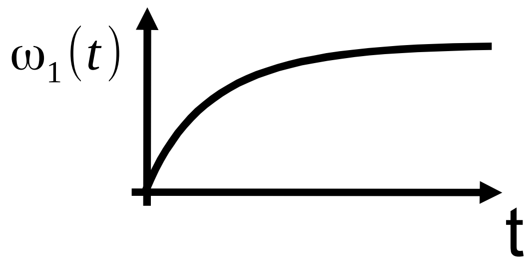


Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$



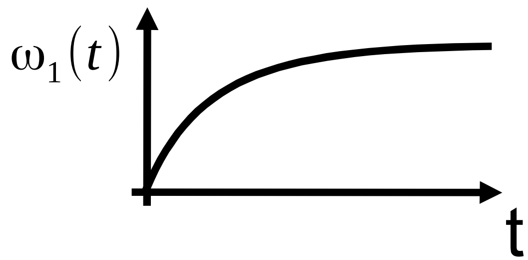
steady state value (maximum) $\omega_{max} = \frac{A - M_P}{B}$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$



steady state value (maximum) $\omega_{max} = \frac{A - M_P}{B}$

START-UP TIME
(95% of maximum value)

$$0{,}95\,\frac{A - M_P}{B} = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t_{95}}\right)$$

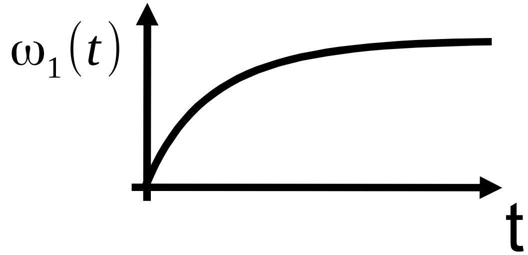
Let

Reduction of masses and forces

<u>Example 1</u>

Start-up process

$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$



steady state value (maximum) $\omega_{max} = \frac{A - M_P}{B}$

START-UP TIME
(95% of maximum value)

$$0,95\frac{A - M_P}{B} = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t_{95}}\right)$$

$$t_{95} \approx 3\frac{I_R}{B}$$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$

electric motor angular velocity

$$v(t)=\frac{D}{2}\omega_1(t)\,i_1\,i_2$$

box linear velocity