Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control

Winter 2019/2020

Lecturer: Sebastian Korczak, PhD Eng.

Lecture 13

Stability criteria.
Gain margin and phase margin.
System correction.
Summing of Bode plots.
State-space representation.

STABILITY CRITERIA

General stability criterion

Hurwitz criterion

Nyquist stability criterion

General stability criterion – definition

LTI SISO system is asymptotically stable if real part of every pole of the system's transfer function is less than zero.

$$H(s) = \frac{(s - z_1)(s - z_2)\ldots(s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

$$\operatorname{Re} p_1 < 0 \ \wedge \ \operatorname{Re} p_2 < 0 \ \wedge \ldots \wedge \ \operatorname{Re} p_n < 0$$

Hurwitz criterion

Hurwitz criterion ≠ Routh criterion
(1895) (1876)

<u>mathematics</u>

a necessary and sufficient condition whether all the roots of the polynomial are in the left half of the complex plane

$Re(p_i) < 0$

<u>control theory</u>

a necessary and sufficient condition whether all the poles of transfer function of a linear system have negative real parts

Note: Liénard–Chipart criterion – modification of Hurwitz criterion

Hurwitz criterion – definition

LTI SISO system with a transfer function

$$H(s)=\boxed{\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}}$$

is stable if:

1. $a_n>0,\quad a_{n-1}>0,\quad \ldots,\quad a_1>0,\quad a_0>0$
2.

Hurwitz criterion – definition

LTI SISO system with a transfer function

$$H(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_1 s+b_0}{a_n s^n + a_{n-1}s^{n-1}+\ldots+a_1 s+a_0}$$

is stable if:

1. $a_n>0,\quad a_{n-1}>0,\quad \ldots,\quad a_1>0,\quad a_0>0$

2. $\det\Delta_2>0$
   $\det\Delta_3>0$
   $\ldots$
   $\det\Delta_{n-1}>0$

$$M_n=\begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Δ_2, Δ_3, Δ_{n-1}

Δ_i - *leading principal minor of order i*

Hurwitz criterion – definition

Hurwitz matrix

$$M_n = \begin{bmatrix} a_{n-1} & a_n & 0 & 0 & 0 & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} & a_n & 0 & 0 \\ a_{n-5} & a_{n-4} & a_{n-3} & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & 0 & a_0 \end{bmatrix}$$

Hurwitz criterion

Example 1

$n=2$

$$G(s)=\frac{5s+3}{10s^2+3s+1}$$

$a_2=10$; $a_1=3$; $a_0=1$

(1) $a_2>0$; $a_1>0$; $a_0>0$ o.k.

(2) o.k.

→ SYST. IS STABLE

Hurwitz criterion

Example 2

$n = 3$

$$G(s) = \frac{2s}{2s^3 + s + 20}$$

$a_3 = 2; \quad a_2 = 0; \quad a_1 = 1, \quad a_0 = 20$

(1) $a_3 > 0; \quad \boxed{a_2 = 0}; \quad a_1 > 0; \quad a_0 > 0$

→ SYSTEM IS USTABLE

Hurwitz criterion

$\boxed{\text{Example 3}}$

$n = 3$

$$G(s) = \frac{3s - 5}{s^{\boxed{3}} + 4s^2 + 3s + 10}$$

$a_3 = 1,\ a_2 = 4,\ a_1 = 3,\ a_0 = 10$

1. $a_3, a_2, a_1, a_0 > 0$

2. $\det \Delta_2 > 0$

$$\det \begin{bmatrix} a_2 & a_3 \\ a_0 & a_1 \end{bmatrix} > 0$$

$$M_3 = \begin{bmatrix} \boxed{\begin{matrix} a_2 & a_3 \\ a_0 & a_1 \end{matrix}}^{\,\Delta_2} & 0 \\ & a_2 \\ 0 \quad 0 & a_0 \end{bmatrix}$$

$a_2 a_1 - a_0 a_3 > 0$

$4 \cdot 3 - 10 \cdot 1 > 0$

$2 > 0 \longrightarrow$ OK.

SYSTEM IS STABLE

Hurwitz criterion

$n=4$

Example 4

$$G(s)=\frac{1}{\underbrace{3}_{a_4}s^4+\underbrace{4}_{a_3}s^3+\underbrace{6}_{a_2}s^2+\underbrace{4}_{a_1}s+\underbrace{5}_{a_0}}$$

1) $a_4, a_3, a_2, a_1, a_0 > 0$

$$M_4=\begin{bmatrix} a_3 & a_4 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & 0 & a_0 \end{bmatrix}$$

2) $\det\Delta_2>0 \quad \det\Delta_3>0$

$$\det\Delta_2=\det\begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \end{bmatrix}=\begin{bmatrix} 4 & 3 \\ 4 & 6 \end{bmatrix}=4\cdot6-4\cdot3>0 \quad \text{ok}$$

$$\det\Delta_3=\begin{bmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{bmatrix}=\det\begin{bmatrix} 4 & 3 & 0 \\ 4 & 6 & 4 \\ 0 & 5 & 4 \end{bmatrix}=4\cdot6\cdot4-3\cdot4\cdot4-4\cdot5\cdot4=$$

$$=4\cdot4(6-3-5)<0$$

SYSTEM IS UNSTABLE

Hurwitz criterion

Example 5

Choose k parameter to satisfy Hurwitz criterion

$$\frac{k\,s}{4\,s^3+3\,s^2+k\,s+1}$$

$n=3$; $a_3=4$; $a_2=3$; $a_1=k$; $a_0=1$

① $a_3>0$; $a_2>0$; $a_1>0$; $a_0>0$

$\rightarrow$ $k>0$

② $\det \Delta_2 = \det \begin{bmatrix} a_2 & a_3 \\ a_0 & a_1 \end{bmatrix} = \begin{vmatrix} 3 & 4 \\ 1 & k \end{vmatrix} = 3k-4 > 0$

$k > \frac{4}{3}$

FINALY SYSTEM IS STABLE FOR $k > \frac{4}{3}$

Hurwitz criterion

Example 6

Choose k parameter to satisfy Hurwitz criterion

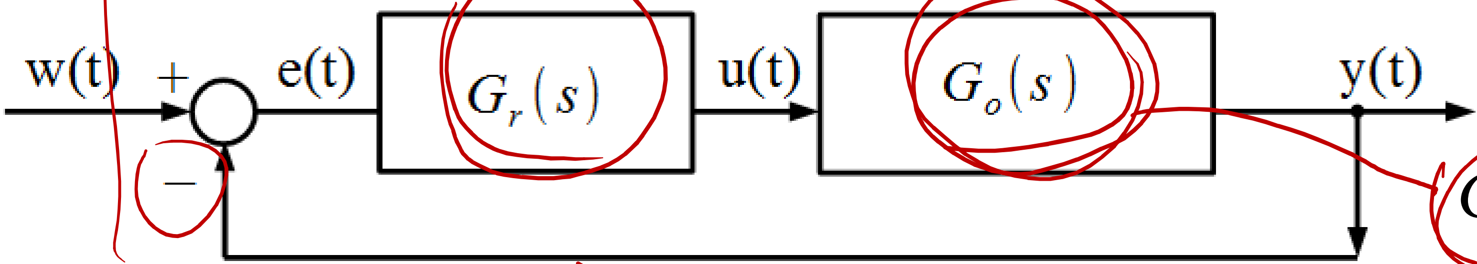
$$\frac{2}{2s^3+ks^2+(1+k)s+3}$$

Homework

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s) = \frac{4s}{Ts+1}$$

$$G_o(s) = \frac{1}{s^3+2s^2+s+1}$$

$$G_{RED}(s) = \frac{Y(s)}{W(s)} = \frac{G_r \cdot G_o}{1 + G_r \cdot G_o} =$$

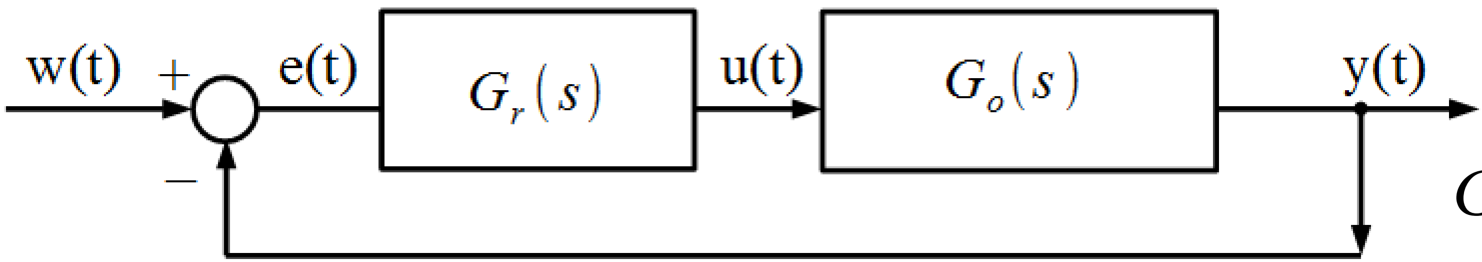
$$= \frac{\frac{4s}{Ts+1} \cdot \frac{1}{s^3+2s^2+s+1}}{1 + \frac{4s}{Ts+1} \cdot \frac{1}{s^3+2s^2+s+1}} = \frac{4s}{(Ts+1)(s^3+2s^2+s+1) + 4s}$$

$$= \frac{4s}{Ts^4+2Ts^3+Ts^2+Ts+s^3+2s^2+s+1+4s} = \frac{4s}{Ts^4+(2T+1)s^3+(T+2)s^2+(T+5)s+1}$$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

$$G_o(s)=\frac{1}{s^3+2s^2+s+1}$$

$$G_z(s)=\frac{G_r G_o}{1+G_r G_o G_p}=\frac{4s}{a_4 s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

$a_4=T,\ a_3=2T+1,$
$a_2=T+2,\ a_1=T+5,\ a_0=1$

$a_4>0 \rightarrow T>0$

$a_3>0 \rightarrow 2T+1>0 \rightarrow T>-\frac{1}{2}$

$a_2>0 \rightarrow T+2>0 \rightarrow T>-2$

$a_1>0 \rightarrow T+5>0 \rightarrow T>-5$

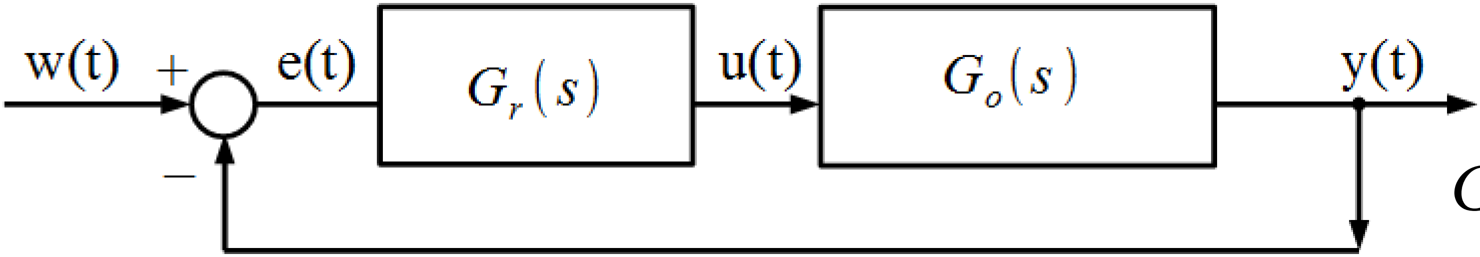
$a_0>0 \rightarrow 1>0$

$T>0$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

$$G_o(s)=\frac{1}{s^3+2s^2+s+1}$$

$$G_z(s)=\frac{G_rG_o}{1+G_rG_oG_p}=\frac{4s}{a_4s^4+a_3s^3+a_2s^2+a_1s+a_0}$$

$$a_4=T,\ a_3=2T+1,\ a_2=T+2,\ a_1=T+5,\ a_0=1$$

(1) $a_4>0,\ a_3>0,\ a_2>0,\ a_1>0,\ a_0>0 \rightarrow T>0$

(2) $\det\Delta_2>0 \rightarrow \det\begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \end{bmatrix}>0 \Rightarrow a_3a_2-a_4a_1>0$

$$(2T+1)(T+2)-T(T+5)>0$$

$$2T^2+4T+T+2-T^2-5T>0$$

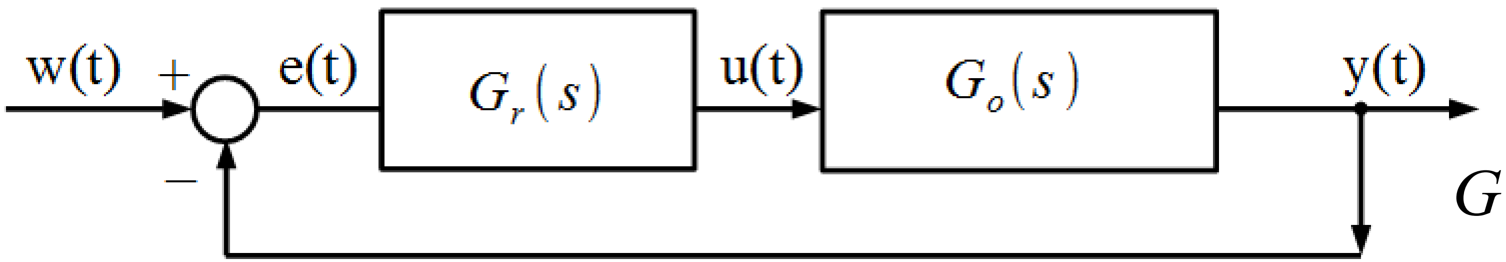
$$T^2+2>0 \Rightarrow T\in\mathbb{R}$$

$\boxed{\det\Delta_3>0}$

Hurwitz criterion

Example 7

Choose T parameter to satisfy Hurwitz criterion



$$G_r(s)=\frac{4s}{Ts+1}$$

$$G_o(s)=\frac{1}{s^3+2s^2+s+1}$$

$$G_z(s)=\frac{G_r G_o}{1+G_r G_o G_p}=\frac{4s}{a_4 s^4+a_3 s^3+a_2 s^2+a_1 s+a_0}$$

$$a_4=T,\ a_3=2T+1,\ a_2=T+2,\ a_1=T+5,\ a_0=1$$

$$a_4>0,\ a_3>0,\ a_2>0,\ a_1>0,\ a_0>0 \rightarrow T>0$$

$$\Delta_2=\begin{bmatrix} a_3 & a_4 \\ a_1 & a_2 \end{bmatrix}=T^2+2>0\ \ T\in\mathbb{R}$$

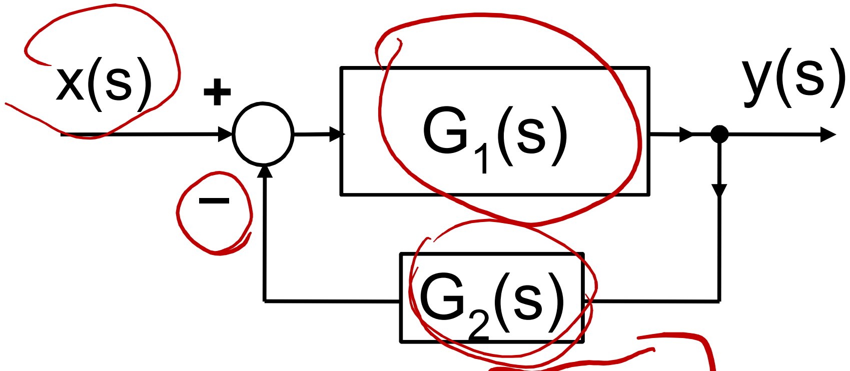
$$\Delta_3=\begin{bmatrix} a_3 & a_4 & 0 \\ a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 \end{bmatrix}$$

$$\Delta_3=\begin{bmatrix} a_{n-1} & a_n & 0 \\ a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-5} & a_{n-4} & a_{n-3} \end{bmatrix}=T^3+T^2-2T+9>0 \rightarrow T>2.83$$

FINALLY : SYSTEM IS STABLE IF $T > 2.83$

Nyquist stability criterion – idea

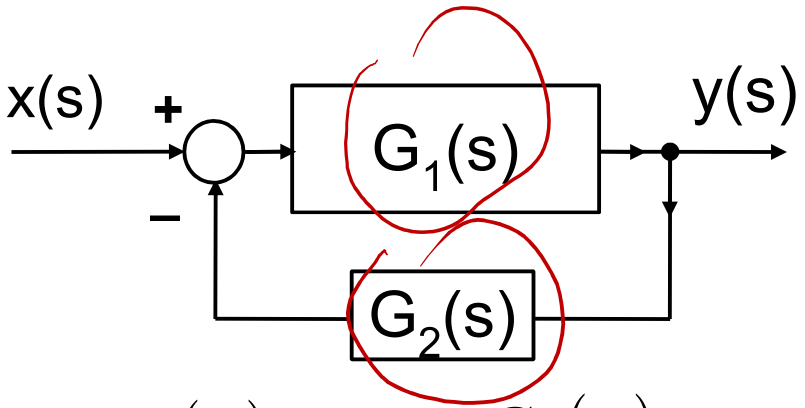
ONLY FOR SYSTEMS WITH FEEDBACK!



$$G_z(s)=\frac{y(s)}{x(s)}=\frac{\boxed{G_1(s)}}{\boxed{1+G_1(s)G_2(s)}}$$

if $G_1 \cdot G_2 = -1 \rightarrow G_z(s) \nearrow \infty$

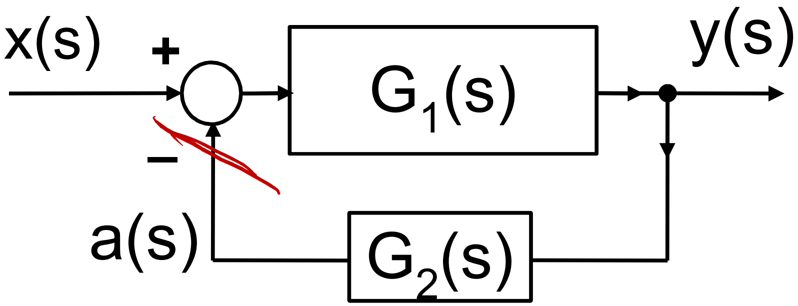
Nyquist stability criterion – idea



$$G_z(s) = \frac{y(s)}{x(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

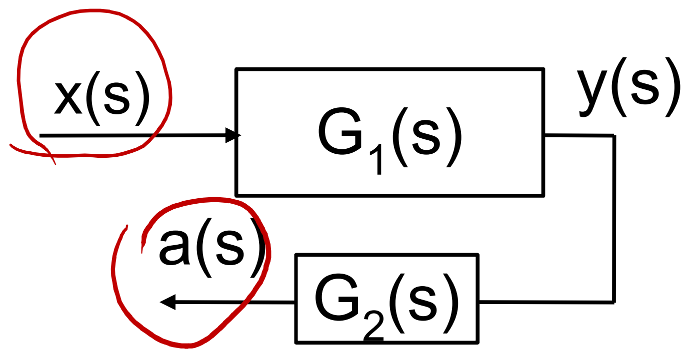
Unstable if: $\boxed{G_1 G_2} = -1$

Nyquist stability criterion – idea



$$G_z(s)=\frac{y(s)}{x(s)}=\frac{G_1(s)}{1+G_1(s)G_2(s)}$$

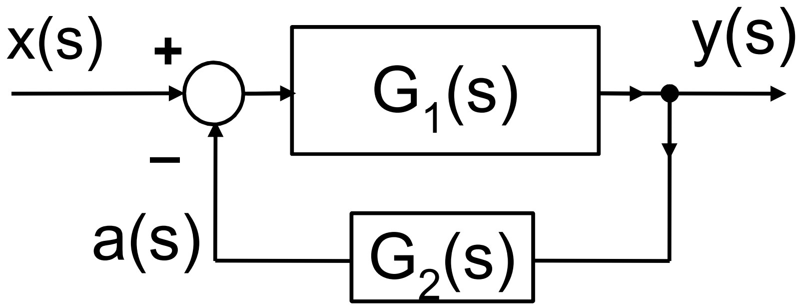
OPENED SYSTEM

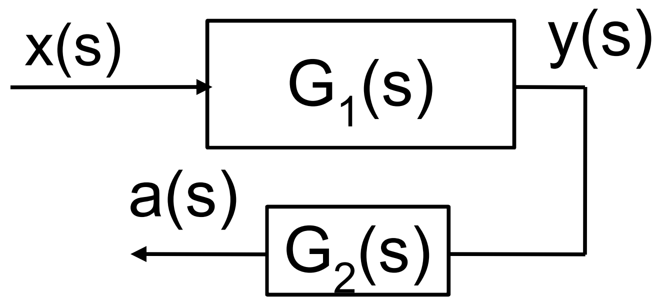


$$G_{open}(s)=\frac{a(s)}{x(s)}=G_1(s)G_2(s)$$

Unstable if: $\boxed{G_1 G_2} = -1$

Nyquist stability criterion – idea

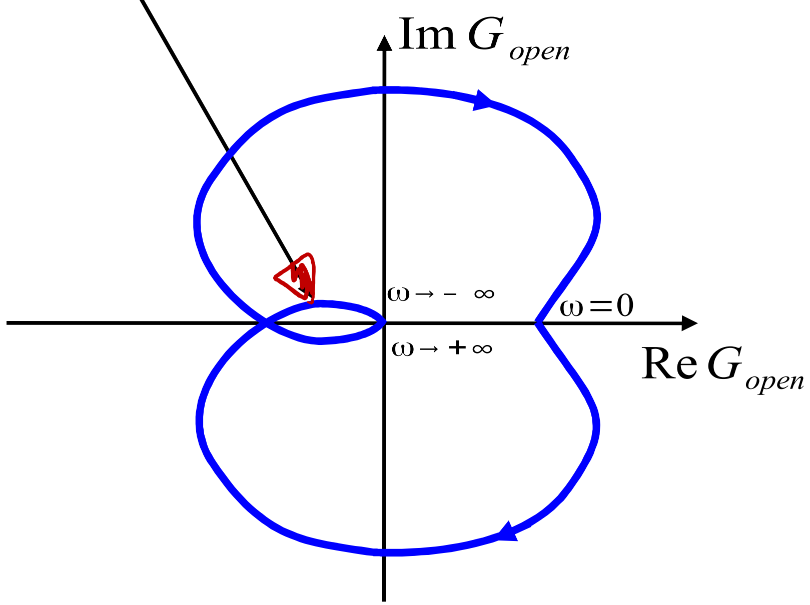




$$G_z(s)=\frac{y(s)}{x(s)}=\frac{G_1(s)}{1+G_1(s)G_2(s)}$$

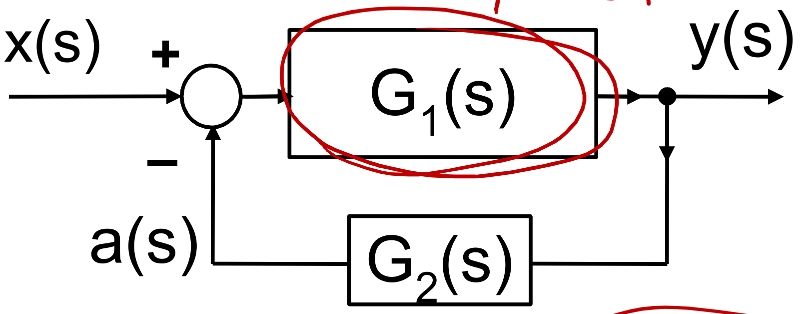
$$G_{open}(s)=\frac{a(s)}{x(s)}=G_1(s)G_2(s)$$

Unstable if: $\boxed{G_1 G_2} = -1$



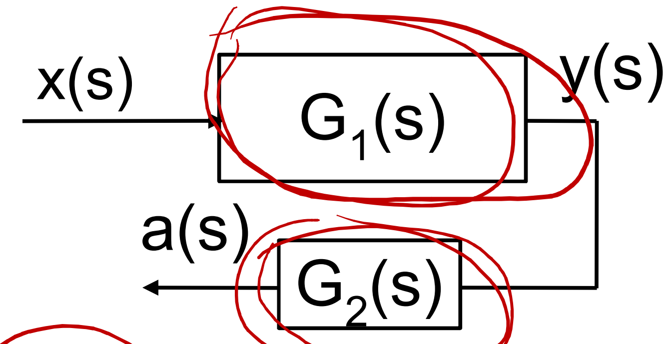
Nyquist stability criterion – idea

CLOSED SYSTEM



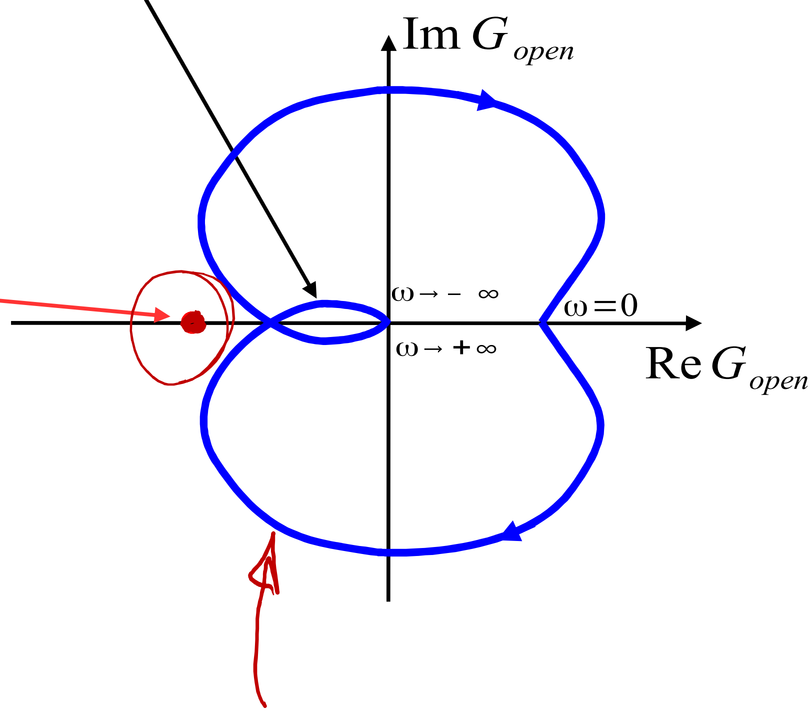
$$G_z(s)=\frac{y(s)}{x(s)}=\frac{G_1(s)}{1+G_1(s)G_2(s)}$$

OPENED SYSTEM



$$G_{open}(s)=\frac{a(s)}{x(s)}=G_1(s)G_2(s)$$

Unstable if: $\boxed{G_1 G_2} = -1$



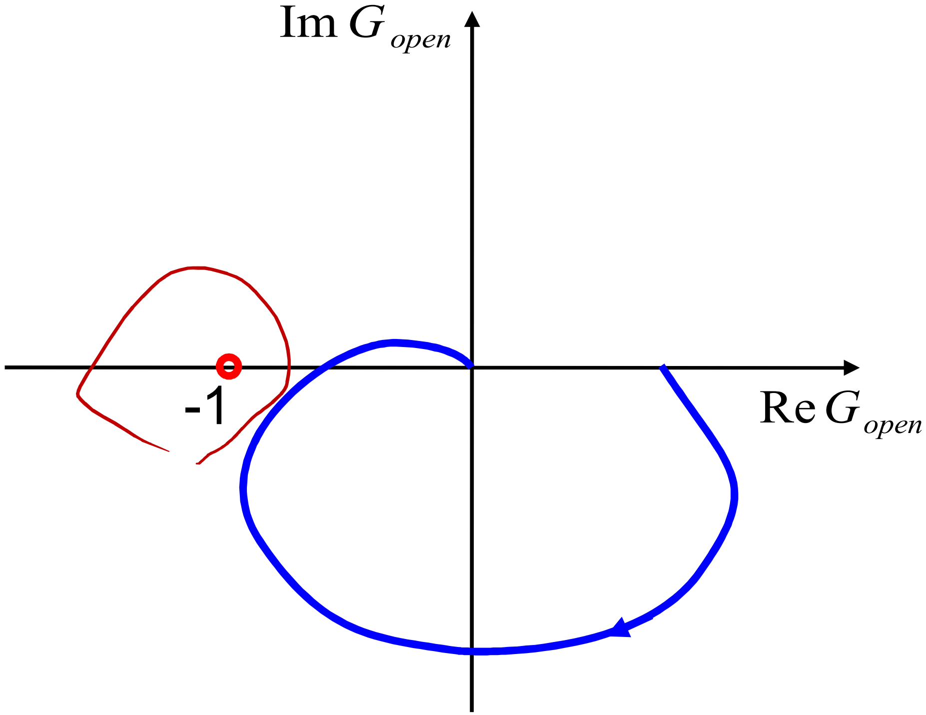
Particular Nyquist stability criterion – definition

The closed-loop system with feedback is stable if:

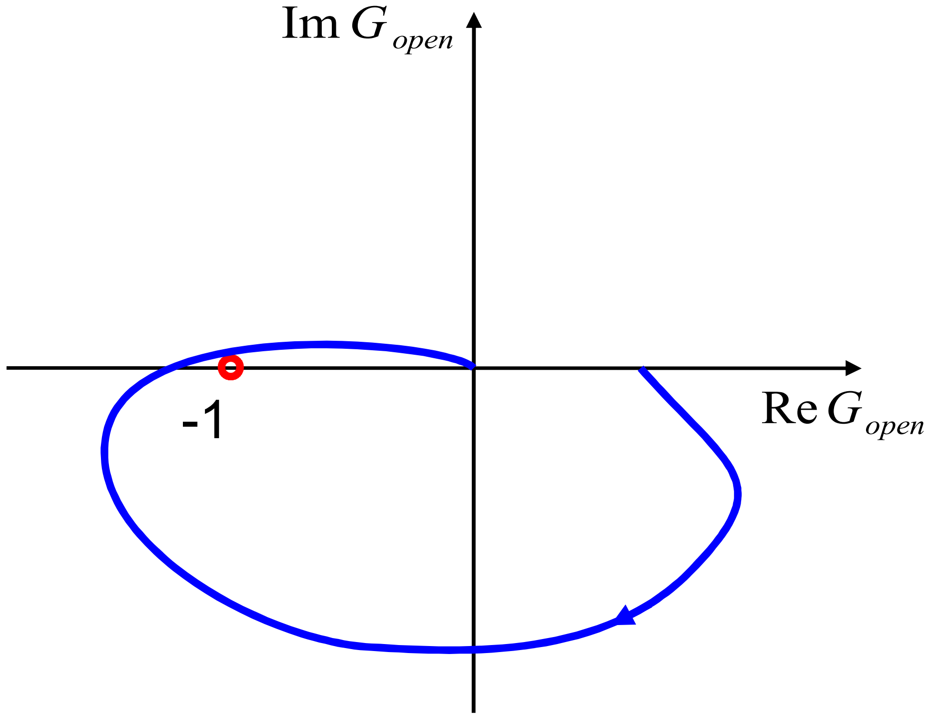
1) open-loop transfer function is stable

AND

2) open-loop transfer function not enclosing the point (-1,j0).



stable

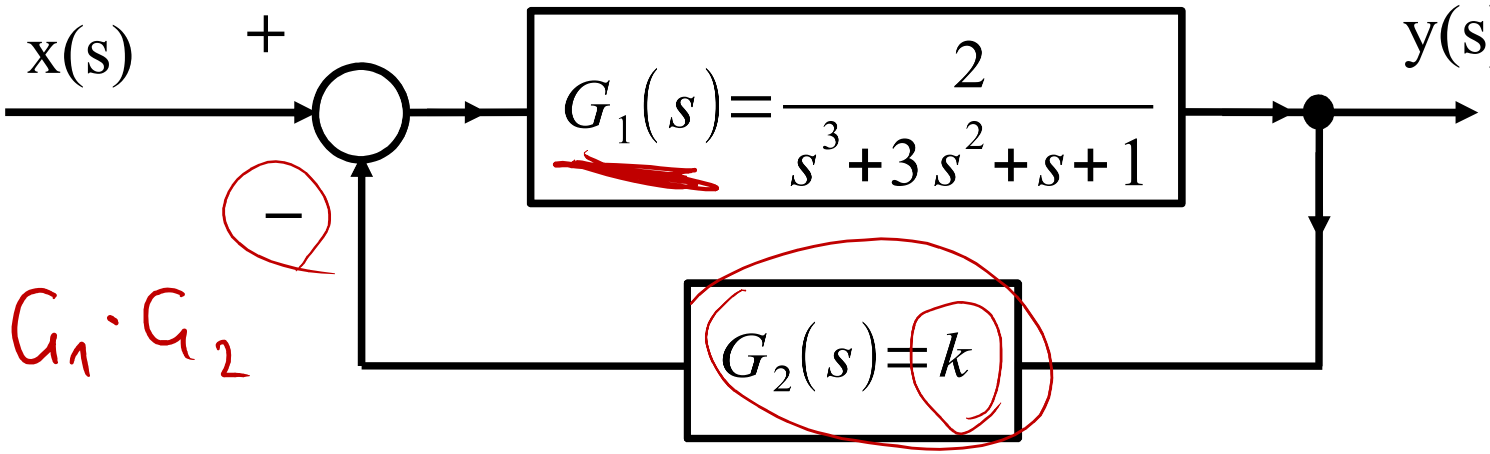


unstable

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



① $G_{open}(s) = G_1 \cdot G_2$

$$= \frac{2k}{s^3+3s^2+s+1}$$

is $G_{OPEN}(s)$ stable?

Hurwitz $n=3$

$a_3=1$ $a_2=3$ $a_1=1$ $a_0=1$

$a_3, a_2, a_1, a_0 > 0$

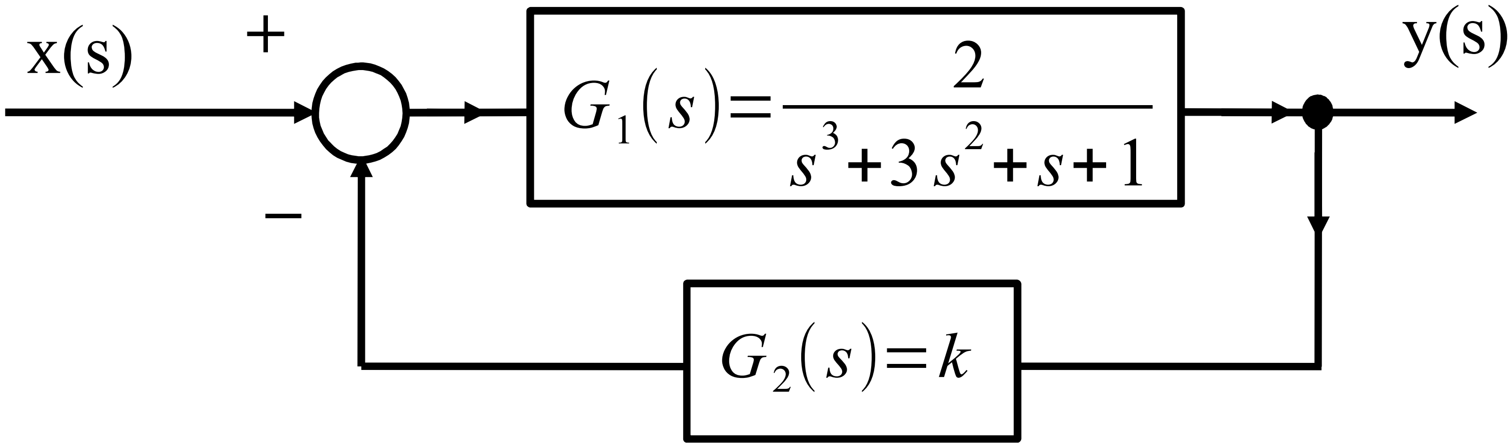
$$\det \Delta_2 = \det\begin{bmatrix} a_2 & a_3 \\ a_0 & a_1 \end{bmatrix} = 3-1 = 2 > 0$$

$G_{OPEN}(s)$ is stable from Hurwitz

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion

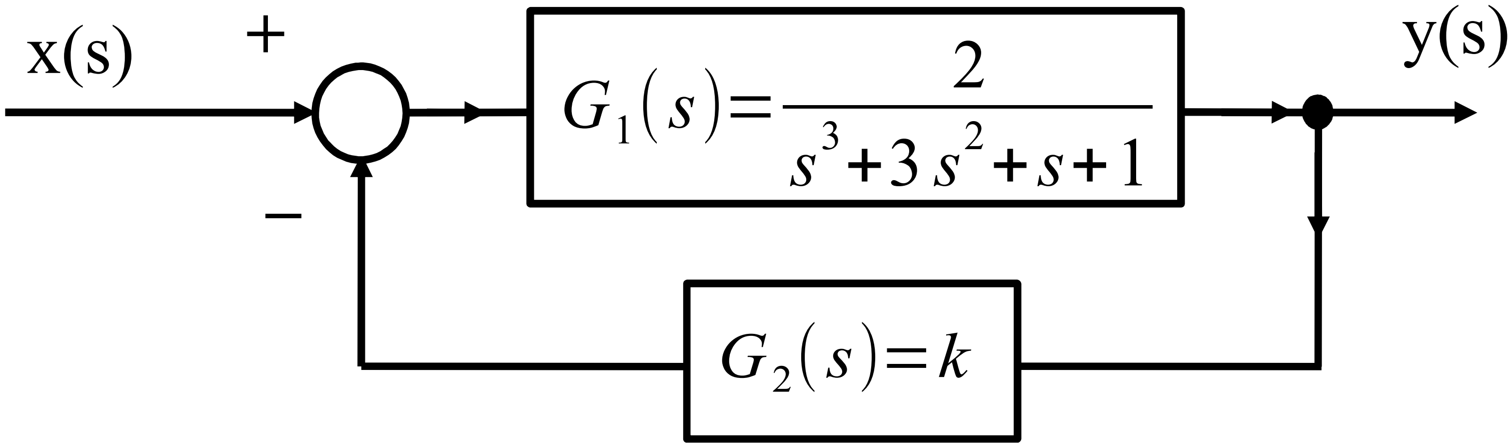


$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



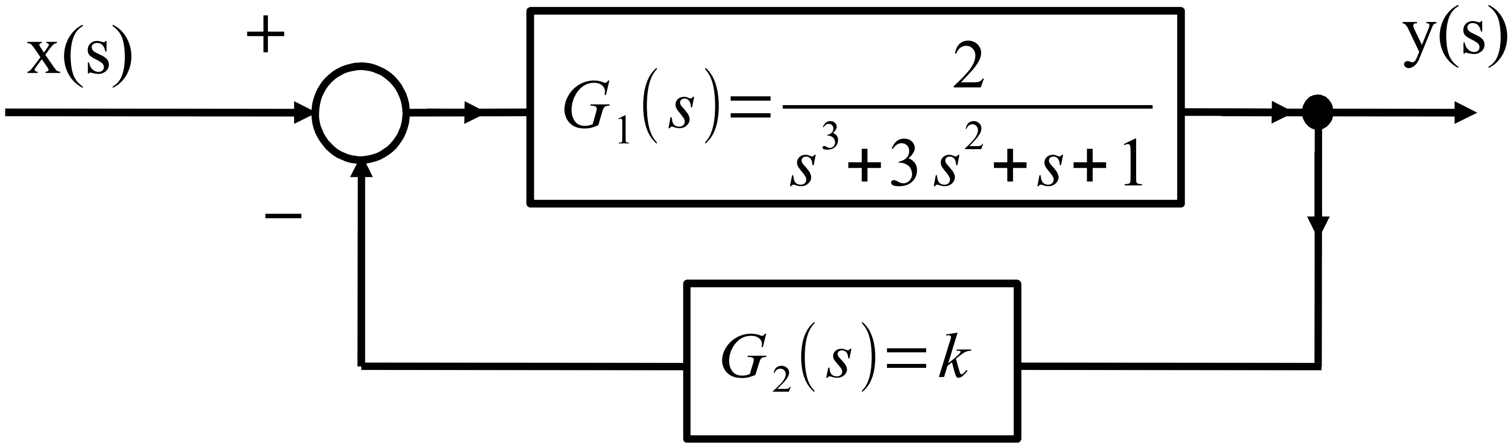
$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$ - stable from Hurwitz

$$G_{open}(j\omega)=\frac{2k}{-j\omega^3-3\omega^2+j\omega+1}=P(\omega)+jQ(\omega)$$

Nyquist criterion

Example 8

Choose k parameter to satisfy Nyquist criterion



$$G_{open}(s)=G_1 G_2=\frac{2k}{s^3+3s^2+s+1}$$ - stable from Hurwitz

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$

Nyquist criterion

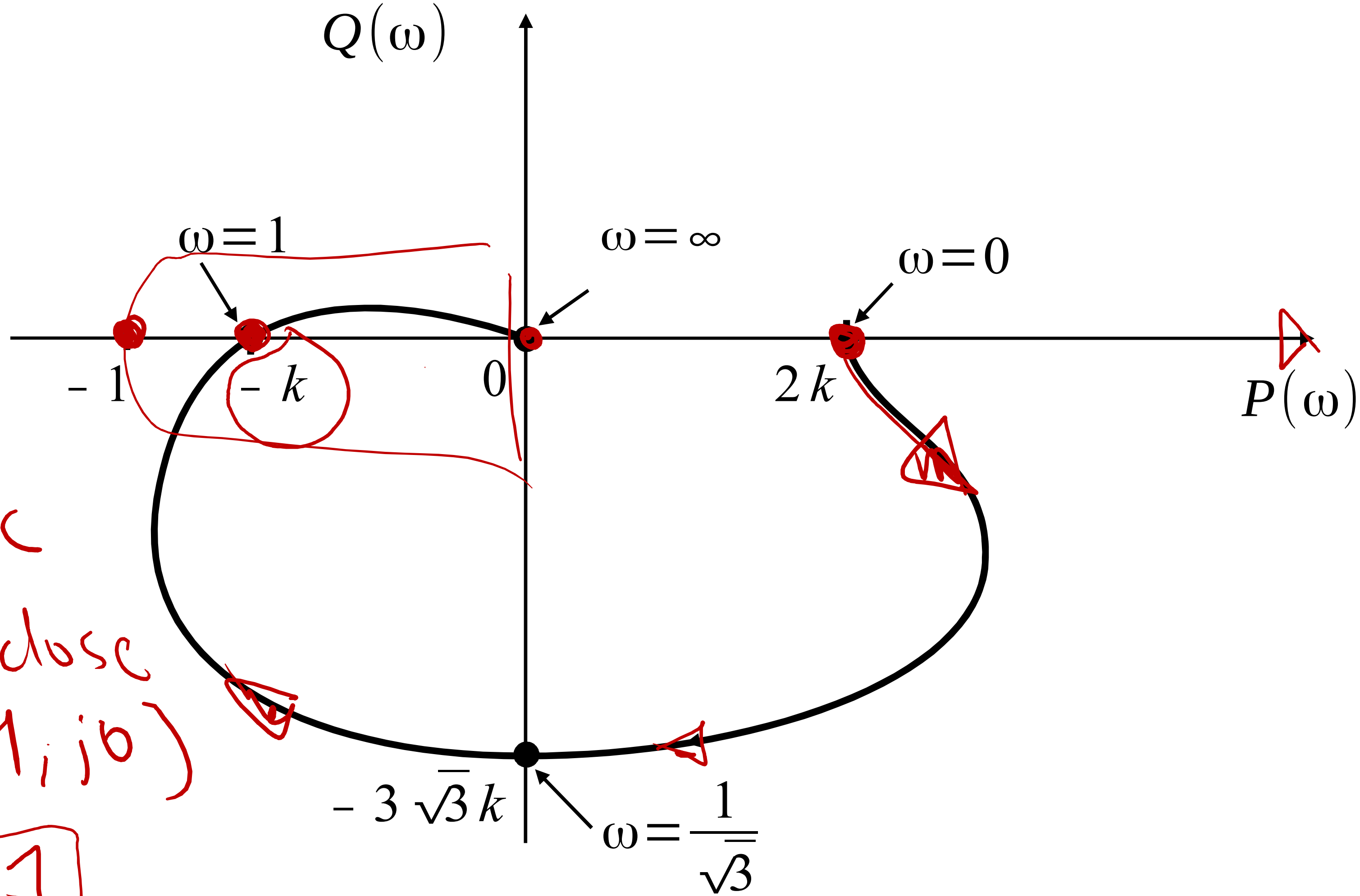
Example 8

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$

Nyquist criterion

Example 8

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$



choose k to not enclose point (-1; j0)

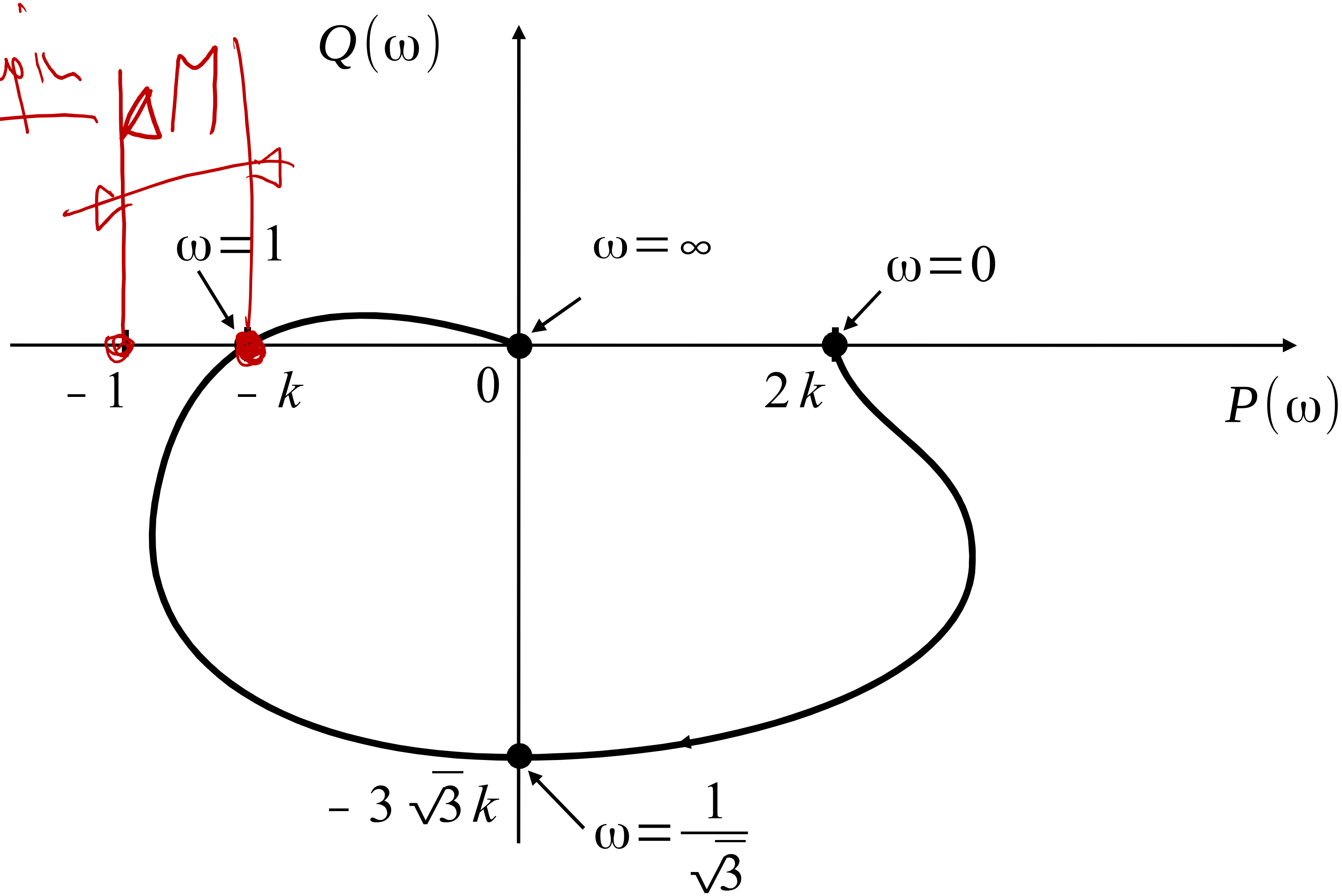
$\boxed{0<k<1}$

Nyquist criterion

Example 8

$$P(\omega)=\frac{2k-6k\omega^2}{(1-3\omega^2)^2+(\omega-\omega^3)^2},\quad Q(\omega)=\frac{2k\omega^3-2k\omega}{(1-3\omega^2)^2+(\omega-\omega^3)^2}$$

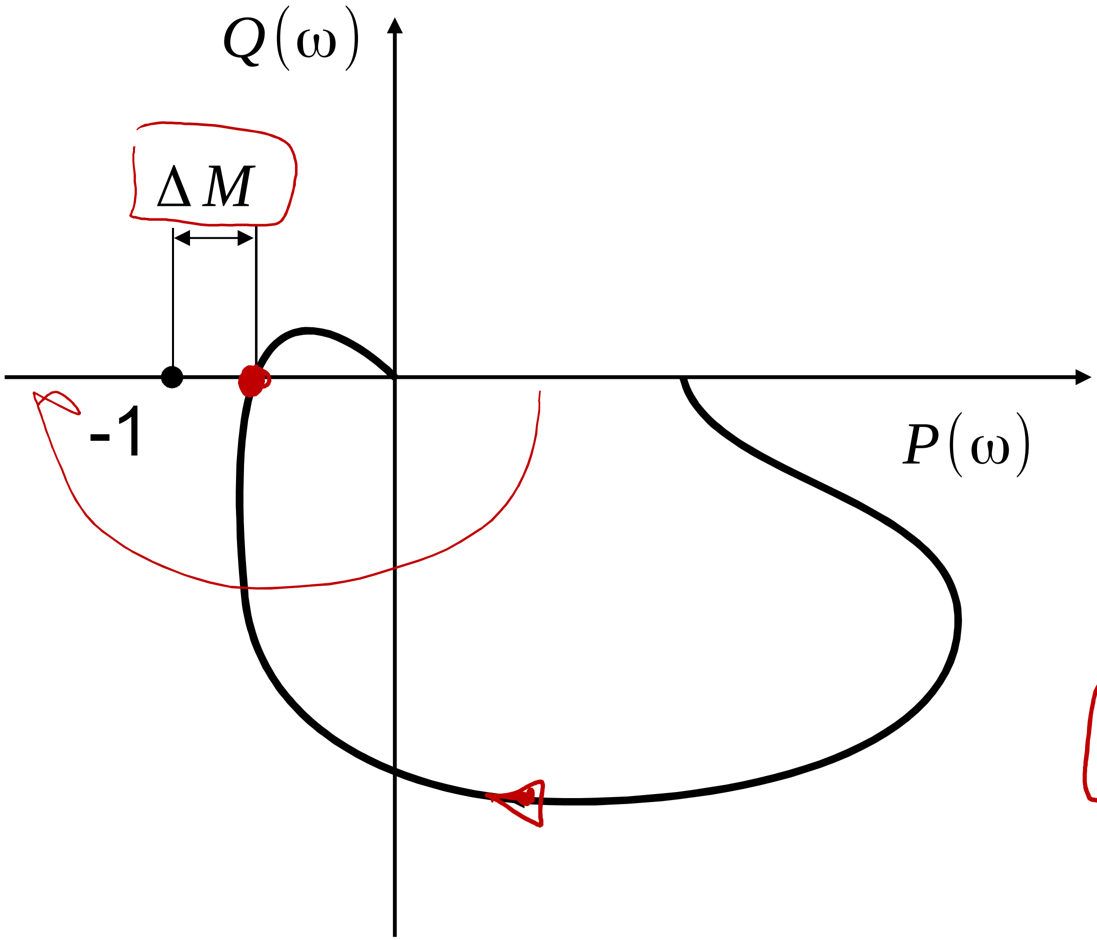
gain margin

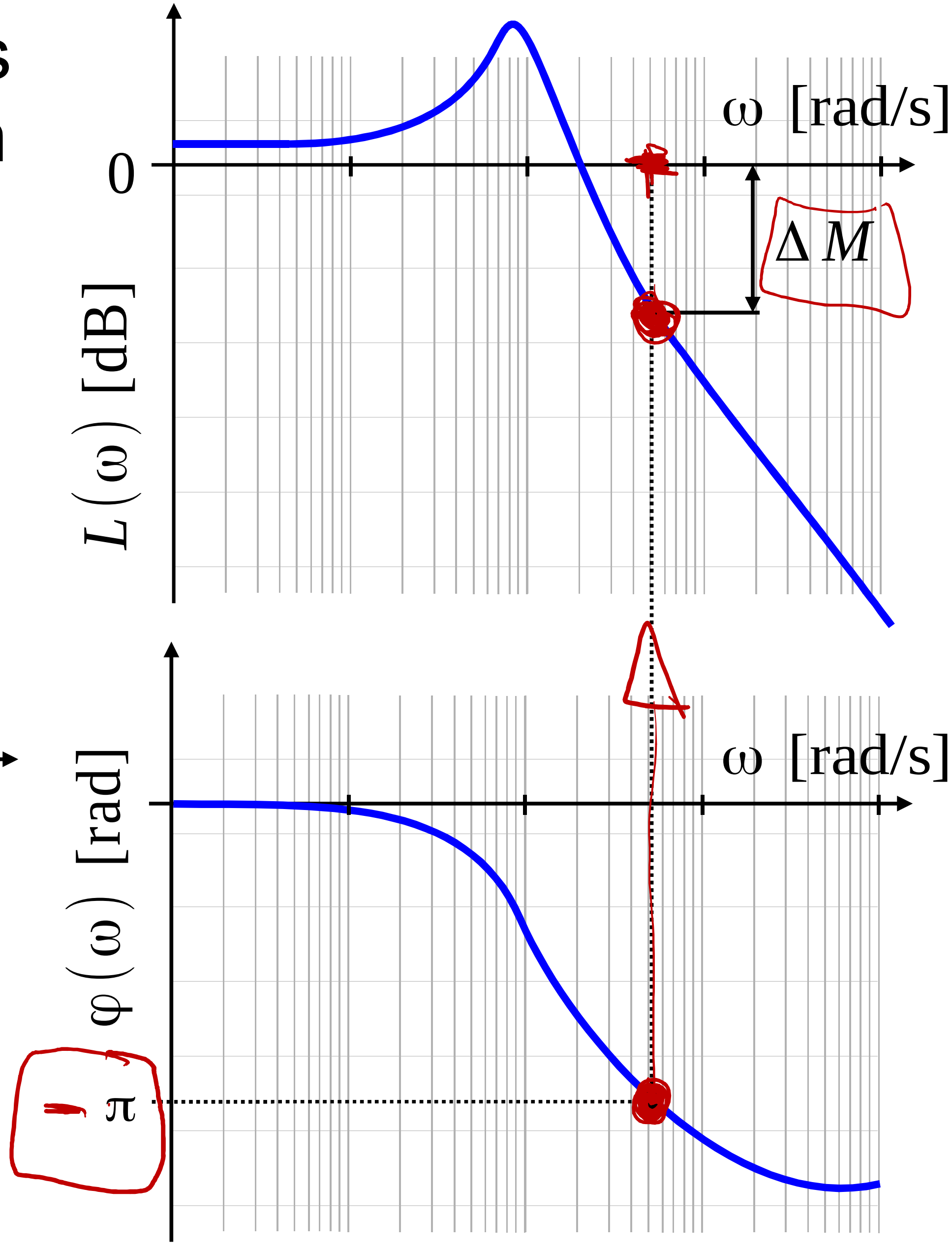


closed-loop system stable for $0 < k < 1$

Gain margin

Closed-loop system will loose its stability if we add additional gain (in serial) greater or equals to gain margin.

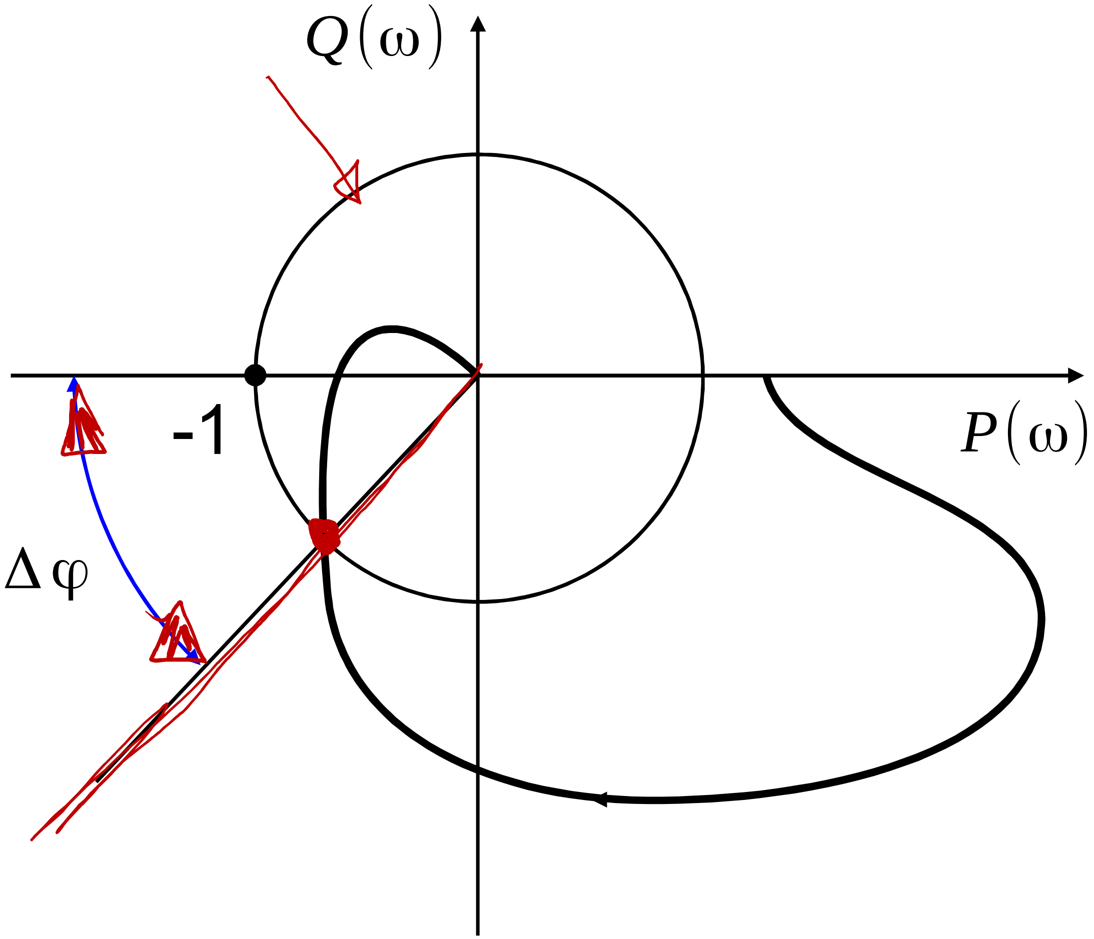


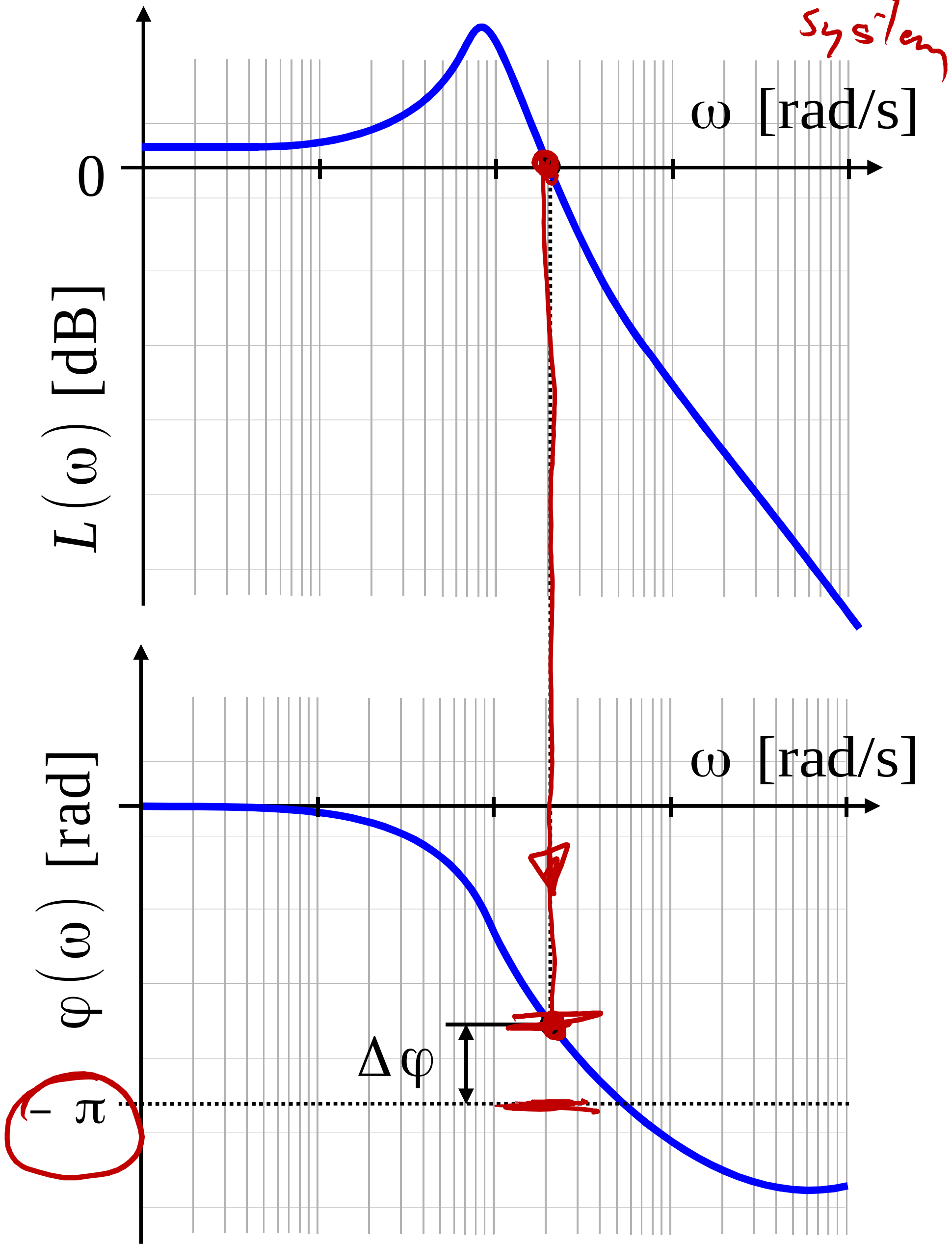


Phase margin

Closed-loop system will loose its stability if we add additional delay (in serial) greater or equals to phase margin.

Bode plot of open-ed system



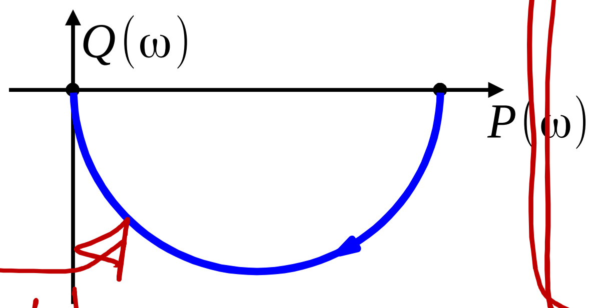


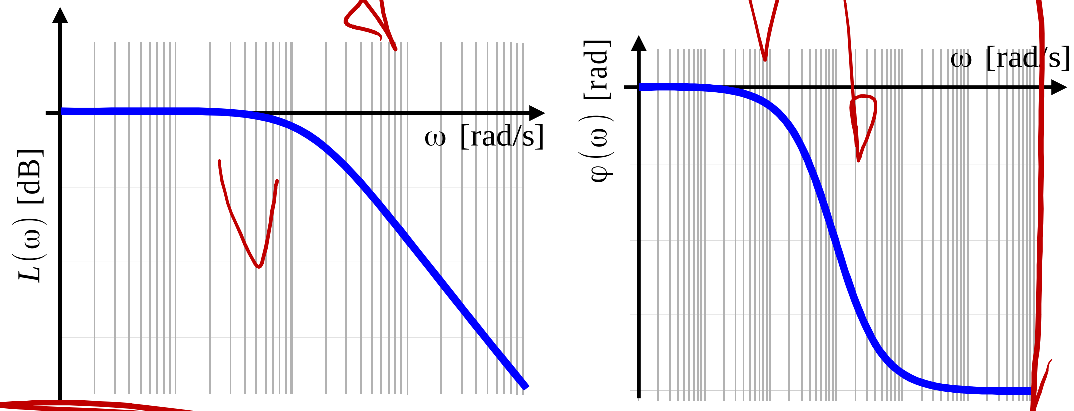
Stability vs Bode plot

<u>Bode plot (gain + delay) has no physical meaning if the system is unstable!</u>

Example:

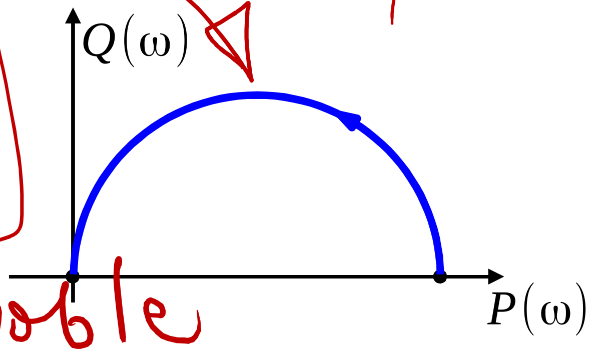
$$G(s)=\frac{1}{s+1}$$

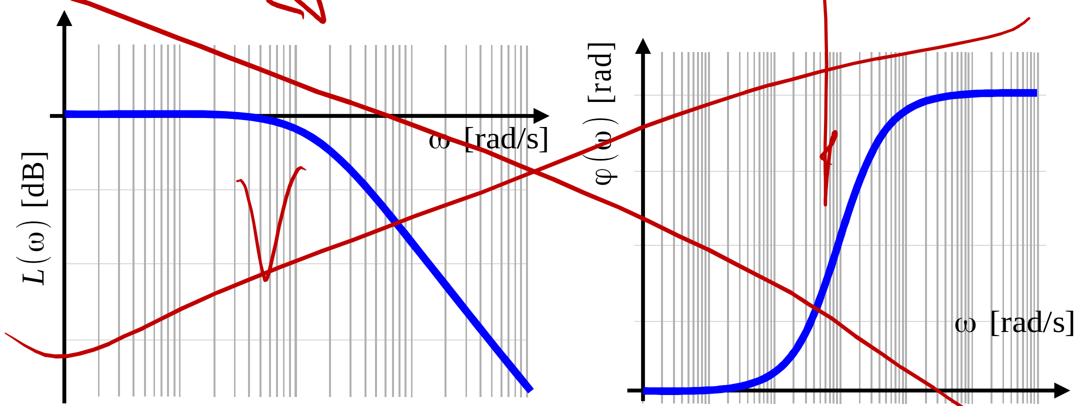




p=-1 stable

$$G(s)=\frac{1}{s-1}$$





p=1 unstable

Summing of Bode plots

$\log(a \cdot b \cdot c) =$

$\log a + \log b + \log c$

$$H(s) = H_1(s)\,H_2(s)\,H_3(s)$$

Gain: $|H(j\omega)| = |H_1(j\omega)| \cdot |H_2(j\omega)| \cdot |H_3(j\omega)|$

↓

gain [dB]

$20\log|H(j\omega)| = 20\log|H_1(j\omega)| + 20\log|H_2(j\omega)| +$

$+ 20\log|H_3(j\omega)|$

Summing of Bode plots

$$H(s) = H_1(s)\,H_2(s)\,H_3(s)$$

$$H(j\omega) = H_1(j\omega)\,H_2(j\omega)\,H_3(j\omega)$$

$$\text{Gain: } |H(j\omega)| = |H_1(j\omega)|\cdot|H_2(j\omega)|\cdot|H_3(j\omega)|$$

$$\text{Gain [dB]: } 20\log|H(j\omega)| = 20\log\left(\ |H_1(j\omega)|\cdot|H_2(j\omega)|\cdot|H_3(j\omega)|\ \right)$$

$$\text{Gain [dB]: } 20\log\left(|H_1(j\omega)|\right) + 20\log\left(|H_2(j\omega)|\right) + 20\log\left(|H_3(j\omega)|\right)$$

Summing of Bode plots

$$H(s) = \boxed{H_1(s)\,H_2(s)\,H_3(s)}$$

$$H(j\omega) = H_1(j\omega)\,H_2(j\omega)\,H_3(j\omega)$$

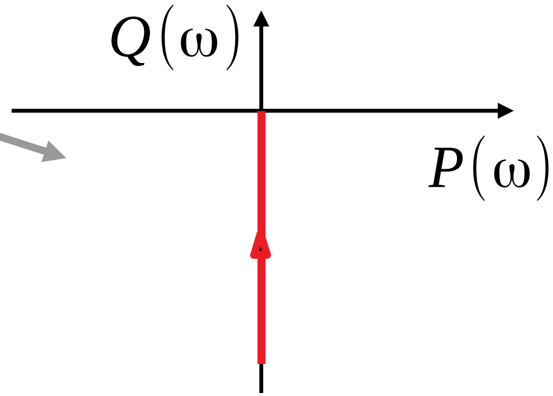
Phase: $\operatorname{Arg} H(j\omega) = \operatorname{Arg} H_1(j\omega) + \operatorname{Arg} H_2(j\omega) + \operatorname{Arg} H_3(j\omega)$

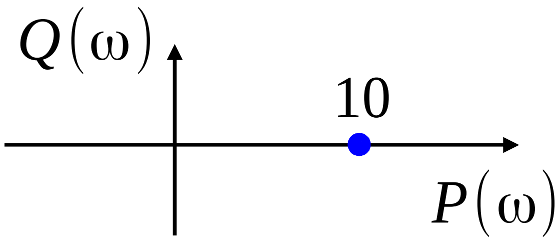
Summing of Bode plots – example

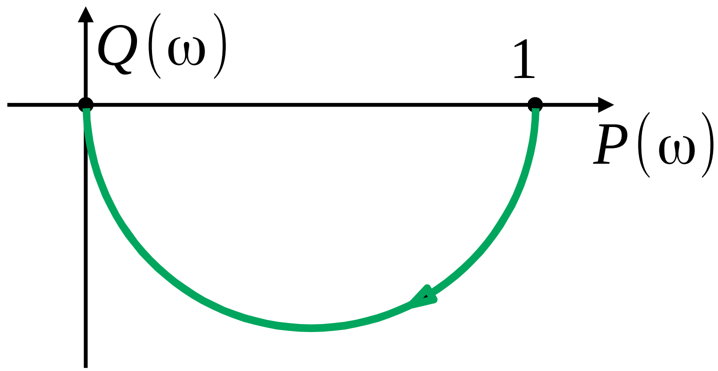
$$G(s) = \boxed{\frac{10}{s^2+s}} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$

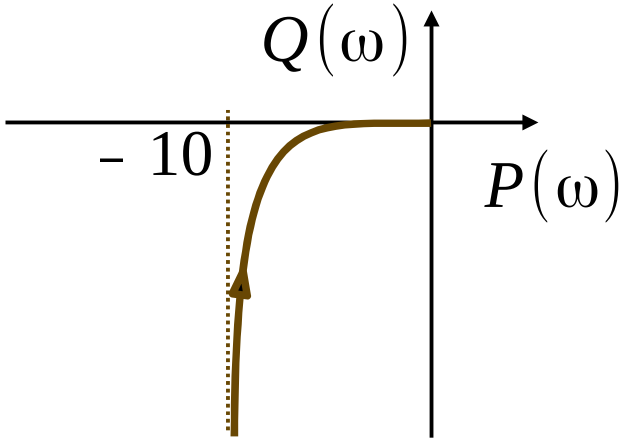
Summing of Bode plots – example

$$G(s) = \frac{10}{s^2+s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$



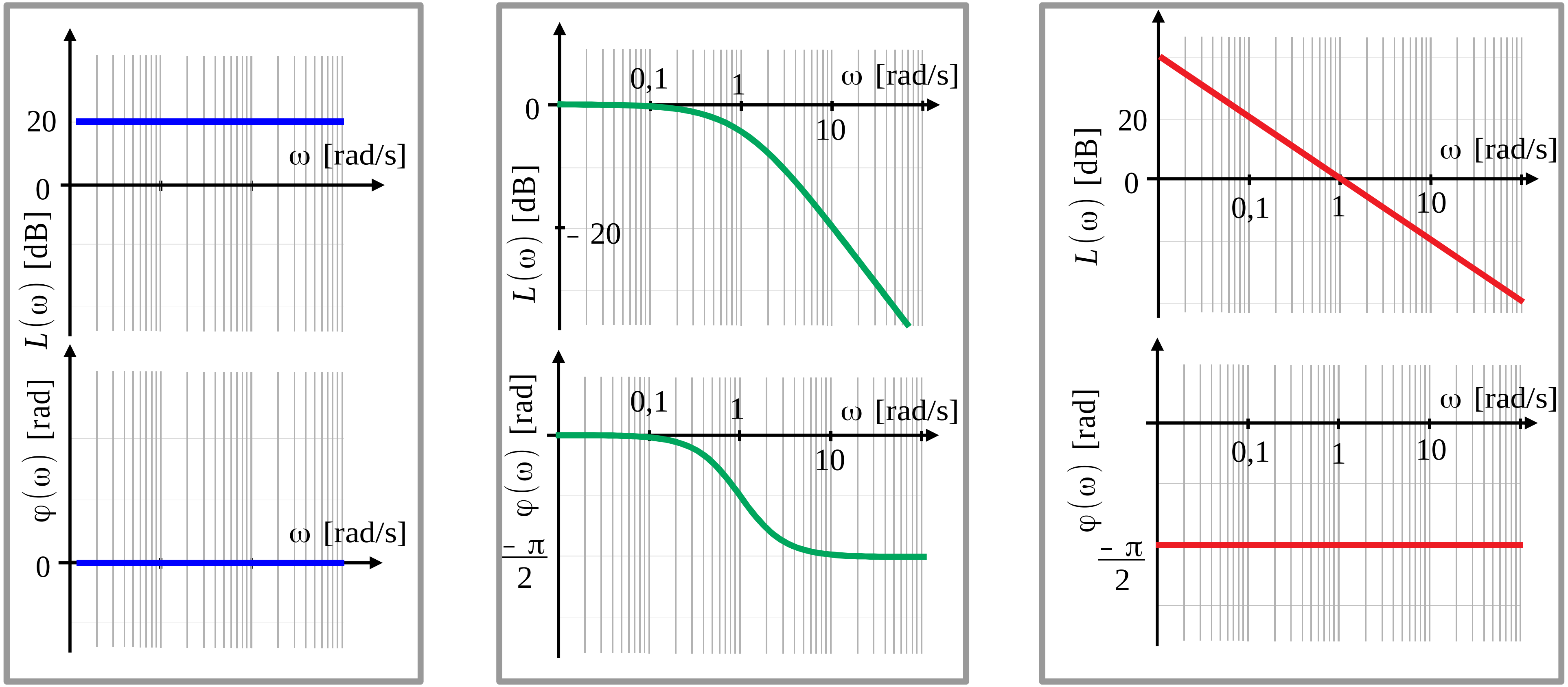






Summing of Bode plots – example

$$G(s) = \frac{10}{s^2+s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$



Summing of Bode plots – example

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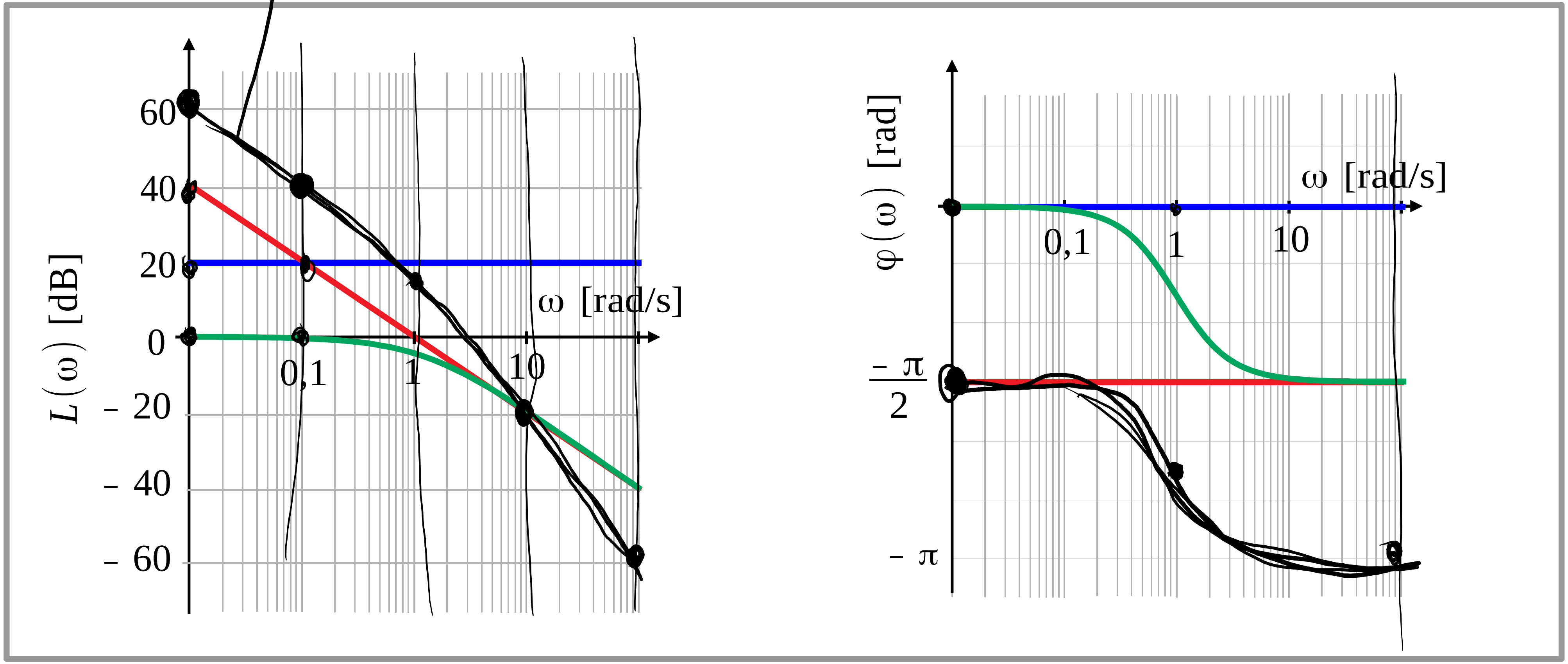
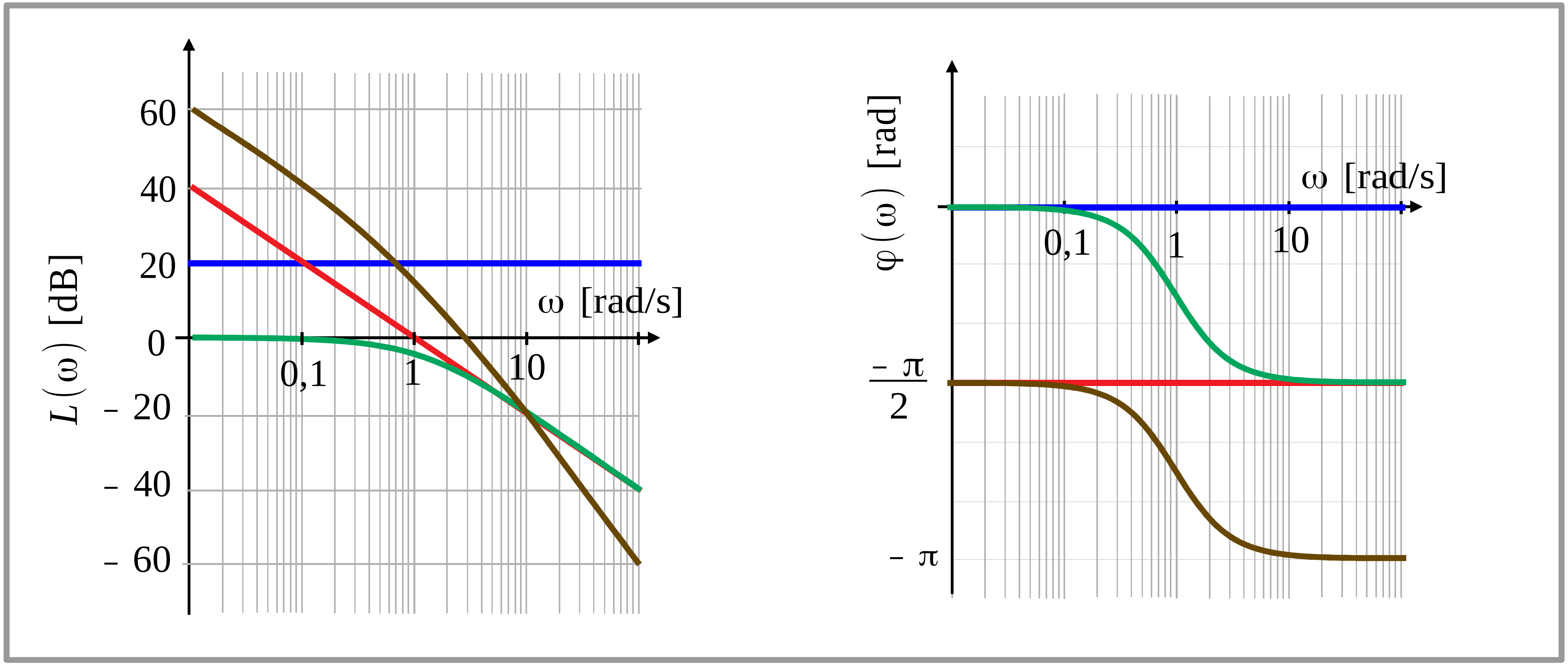


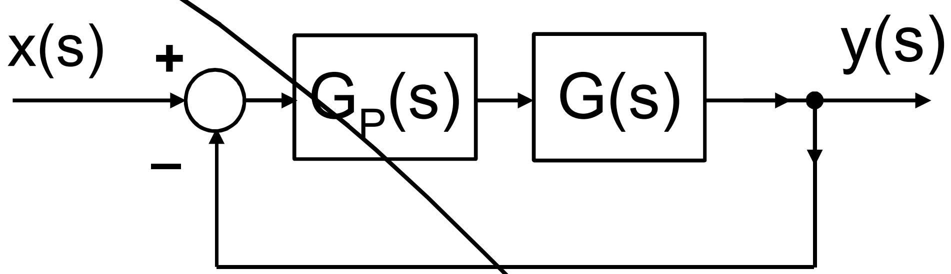
Summing of Bode plots – example

$$G(s) = \frac{10}{s^2+s} = 10 \cdot \frac{1}{s+1} \cdot \frac{1}{s}$$



Nyquist stability criterion

control loop with P controller

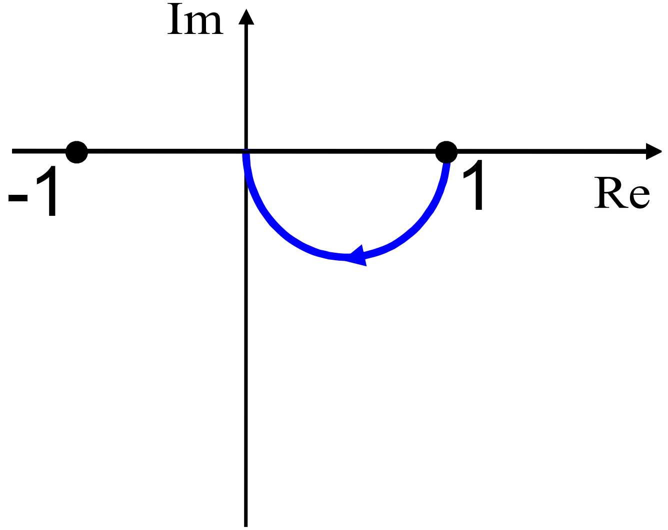


$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

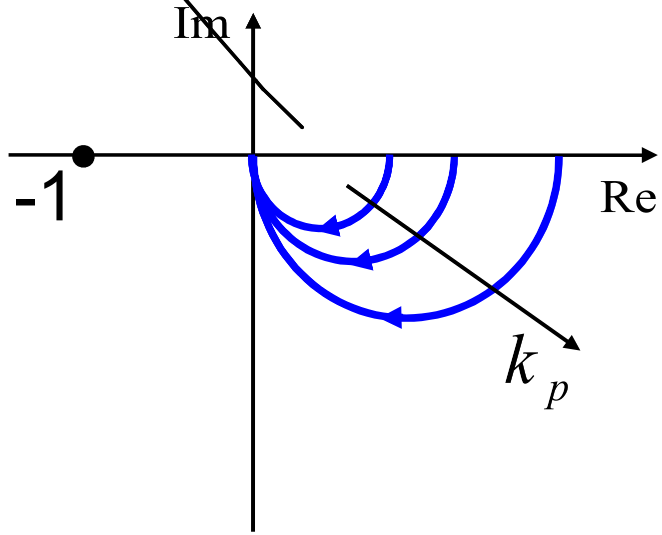
$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

$$G(s)=\frac{1}{Ts+1}$$



$$G_{opened}(s)=k_p\frac{1}{Ts+1}$$



G_{opened} is always stable

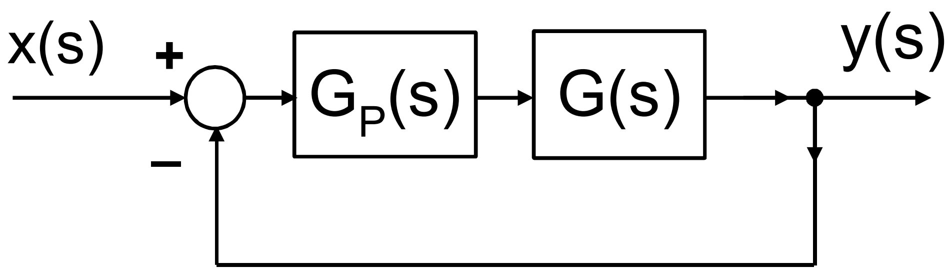
G_{closed} is always stable

steady state error ratio:

$$\frac{k_P}{k_P+1}$$

Nyquist stability criterion

control loop with P controller

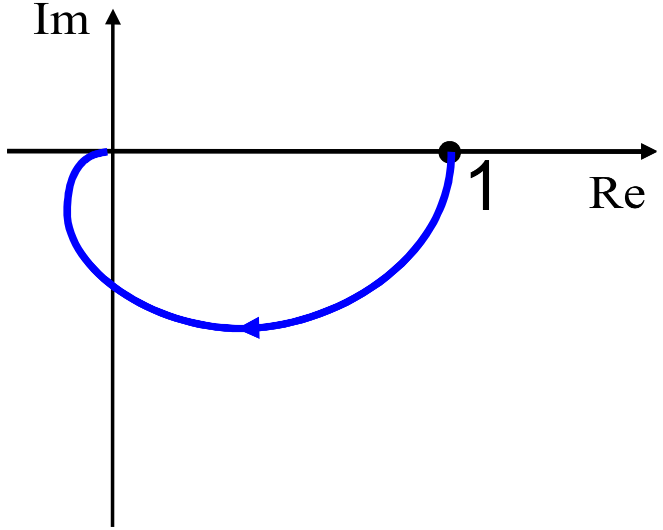


$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

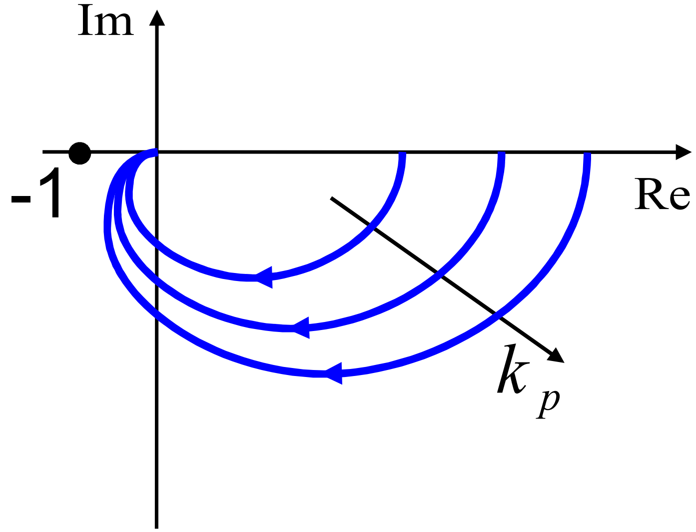
$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

$$G(s)=\frac{1}{T_1^2s^2+T_2s+1}$$



$$G_{opened}(s)=\frac{k_p}{T_1^2s^2+T_2s+1}$$



G_{opened} is always stable

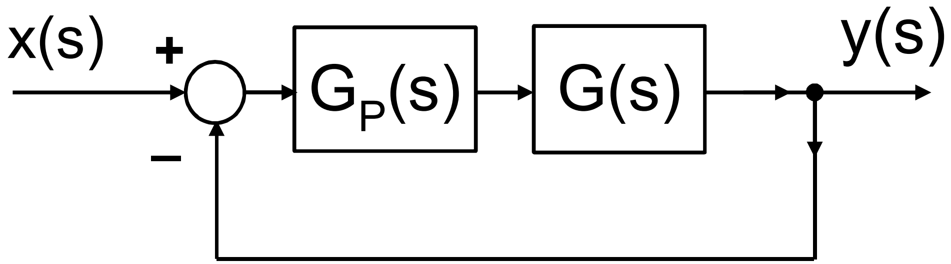
G_{closed} is always stable

steady state error ratio:

$$\frac{k_P}{k_P+1}$$

Nyquist stability criterion

control loop with P controller

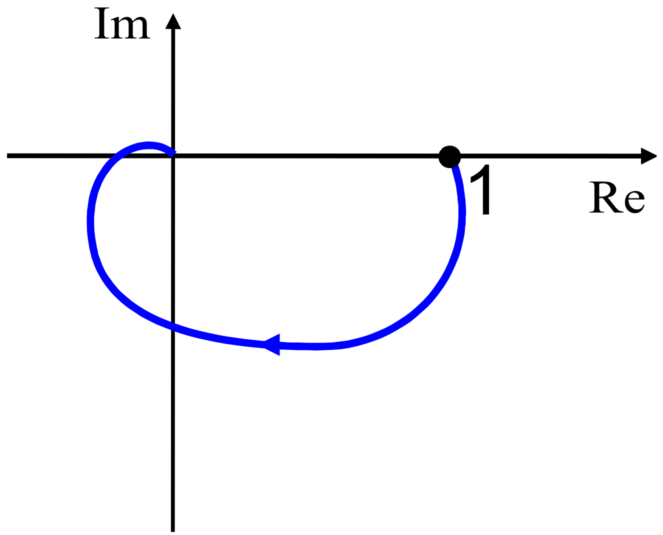


$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

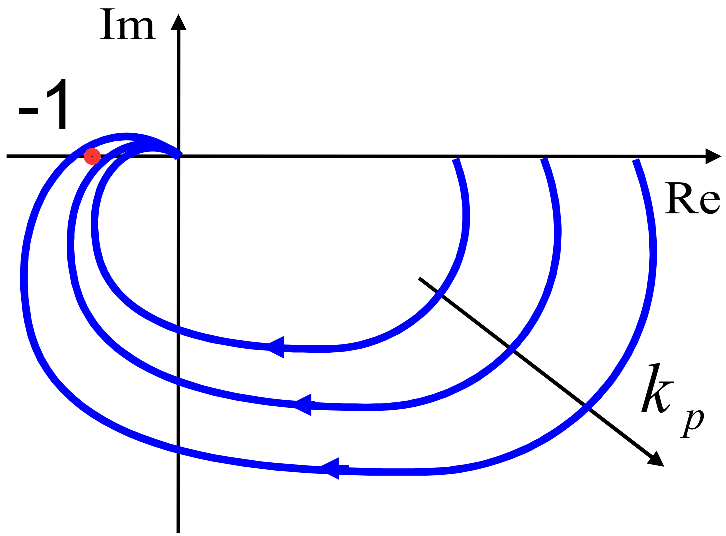
$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

$$G(s)=\frac{1}{T_3^2 s^3+T_2^2 s^2+T_1 s+1}$$



$$G_{opened}(s)=\frac{k_p}{T_3^2 s^3+T_2^2 s^2+T_1 s+1}$$



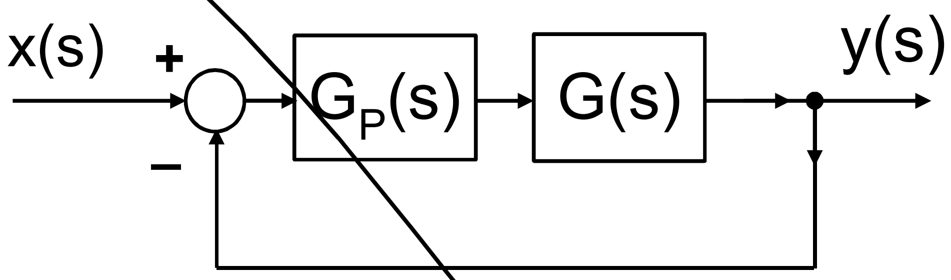
G_{closed} is not always stable

steady state error ratio:

$$\frac{k_P}{k_P+1}$$

Nyquist stability criterion

control loop with P controller



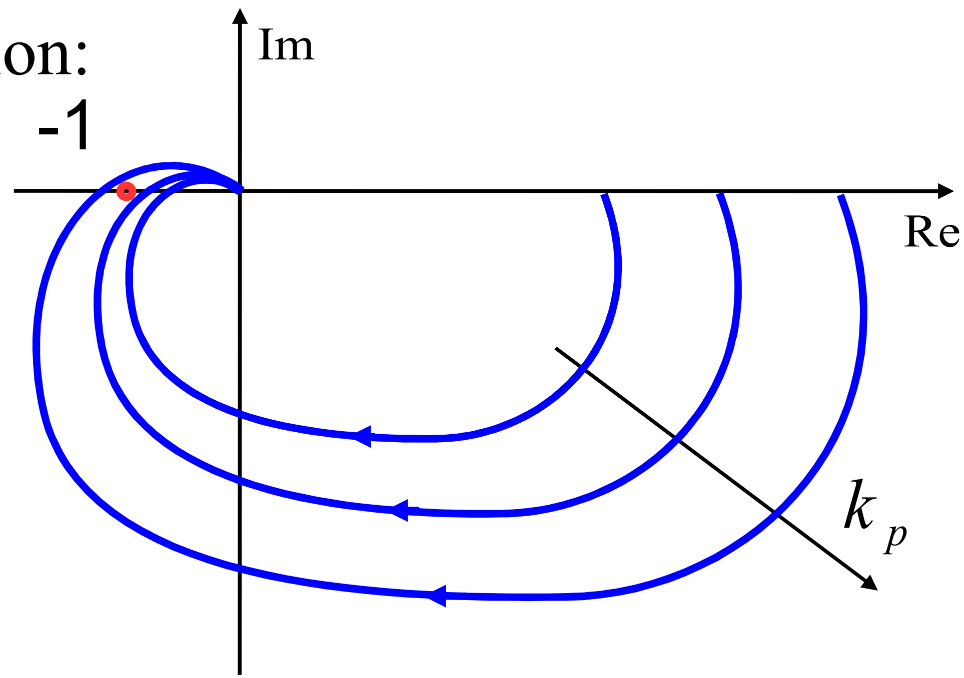
$$G_{closed}(s)=\frac{G_P(s)G(s)}{1+G_P(s)G(s)}$$

$$G_{opened}(s)=G_P(s)G(s)$$

$$G_P(s)=k_P$$

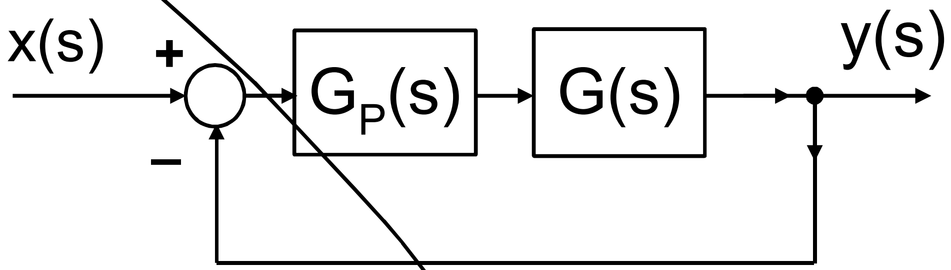
$$G(s)=\frac{1}{T_3^2 s^3+T_2^2 s^2+T_1 s+1}$$

conclusion for open-loop transfer function:
higher kp → lower steady state error
lower kp → better stability
(higher gain margin)



Nyquist stability criterion

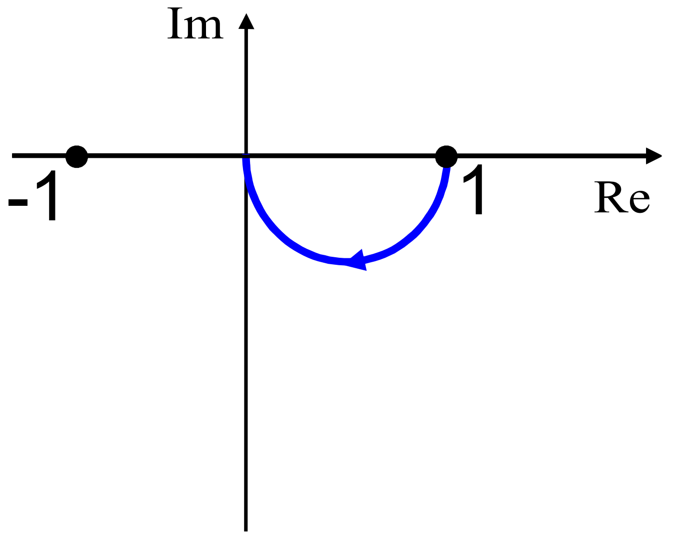
control loop with PI controller



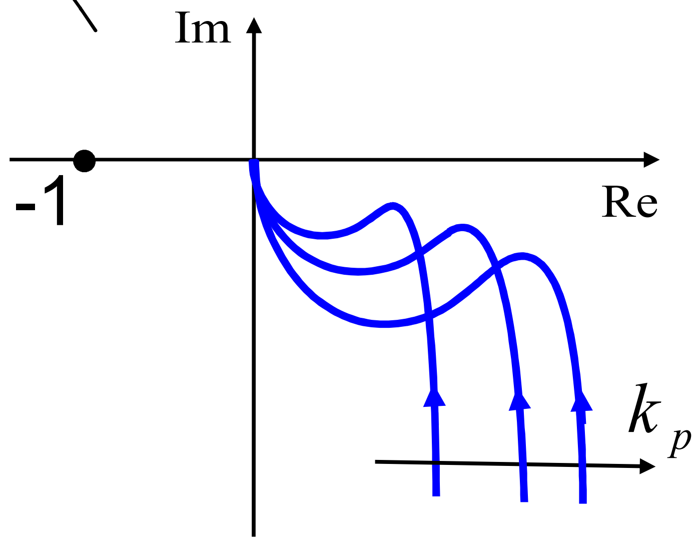
$$G_P(s) = k_P\left(1 + \frac{1}{T_i s}\right)$$

$$G_{opened}(s) = G_P(s)\,G(s)$$

$$G(s) = \frac{1}{Ts+1}$$



$$G_{opened}(s) = k_p^2 \frac{s T_i^2 + 2 T_i}{T_i^3 T s^2 + T_i^2 s}$$



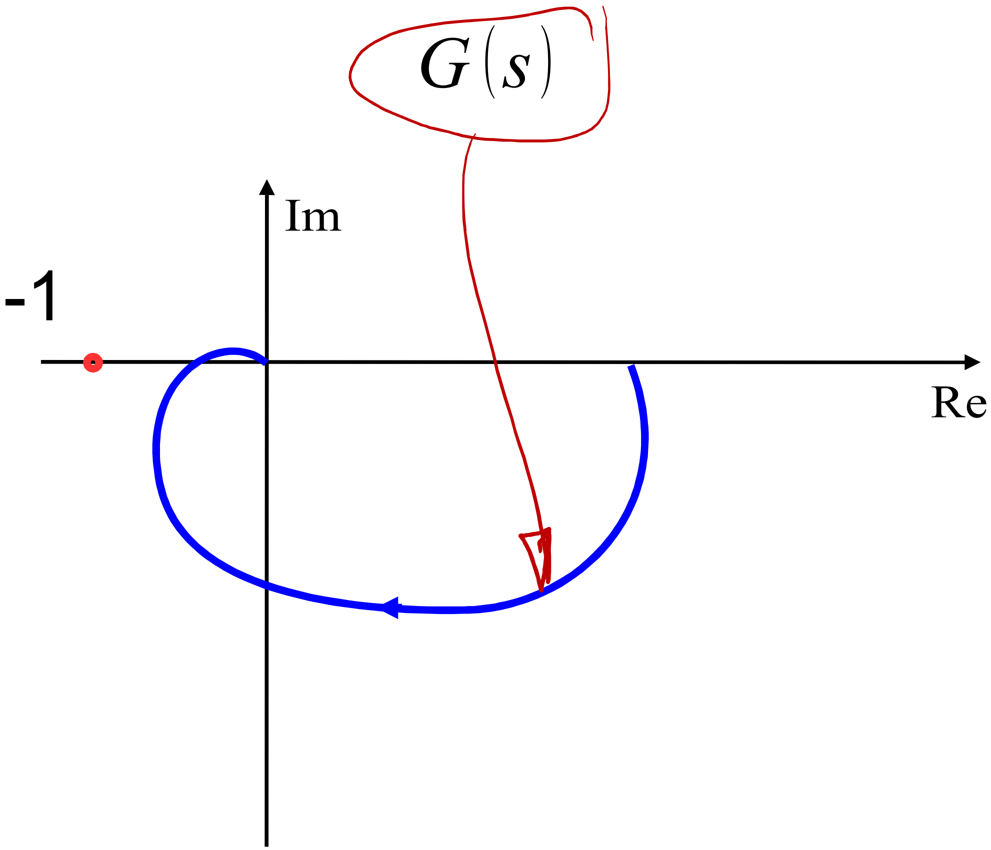
G_{opened} is stable,

so G_{closed} is stable

$G_{opened}(\omega=0) \rightarrow \infty$
so steady state
error $\rightarrow 0$

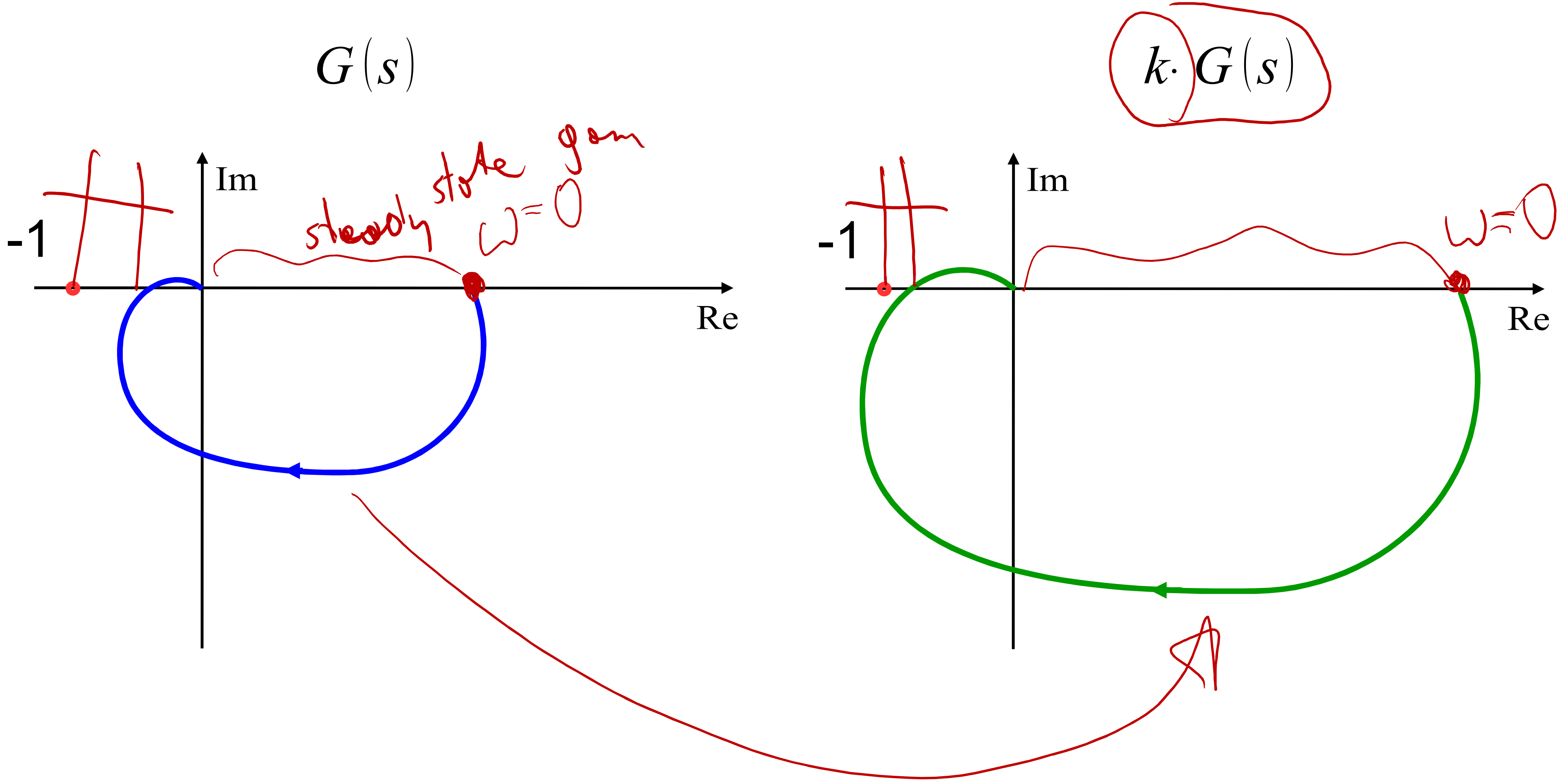
Correction of the system

Correction by proportional term



Correction of the system

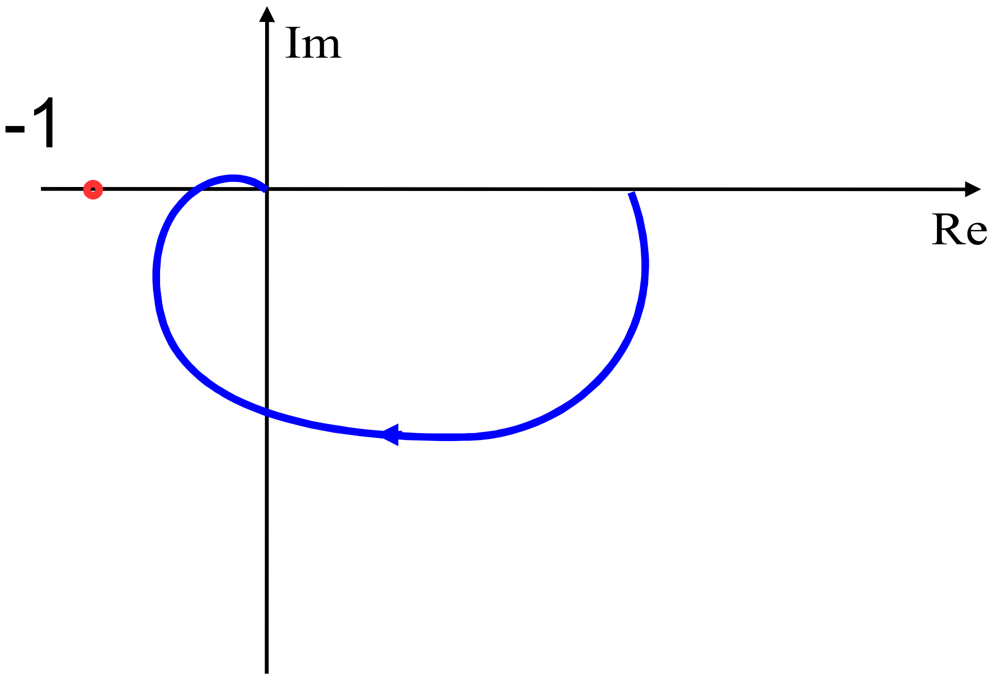
Correction by proportional term



Correction of the system

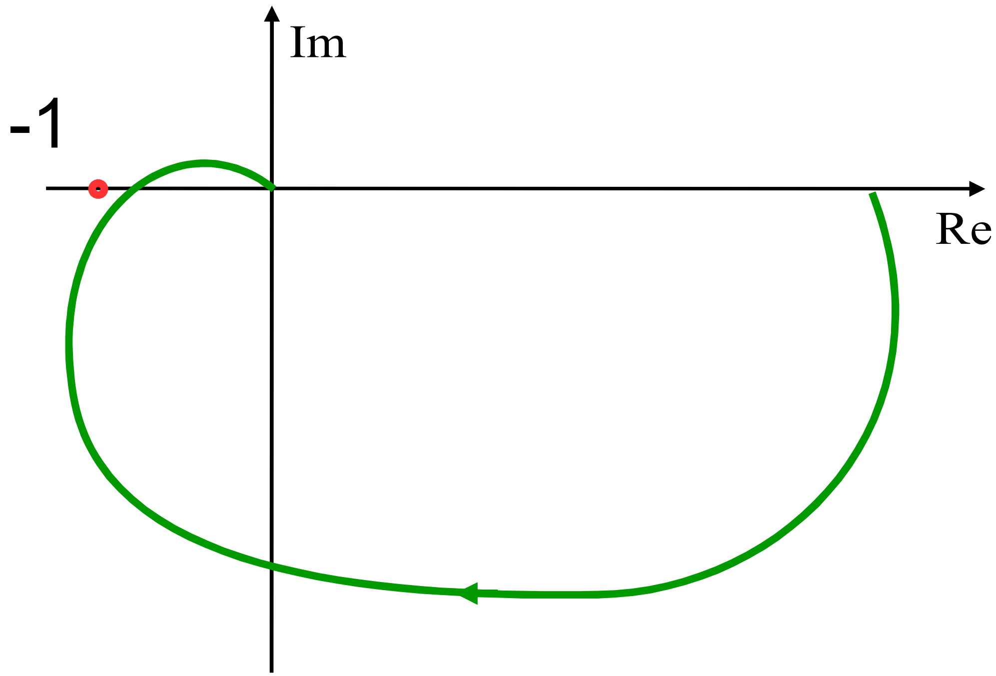
Correction by proportional term

$$G(s)$$



Higher gain margin,
higher phase margin,
higher steady state error

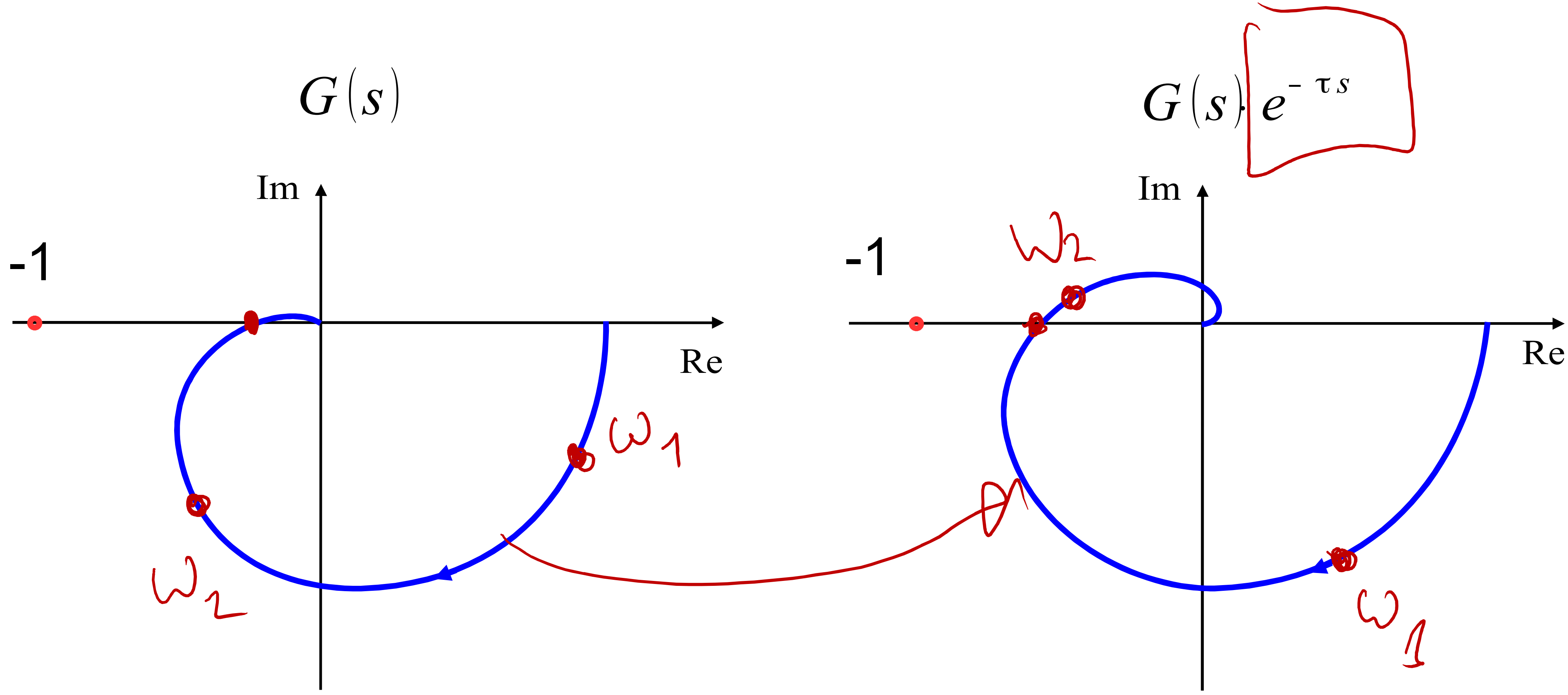
$$k \cdot G(s)$$



Lower gain margin,
lower phase margin,
lower steady state error

Correction of the system

Correction by delay



Higher gain and phase margins

Lower gain and phase margins

Correction of the system

Derivative $$K(s)=\frac{1+Ts}{1+as+bs^2}$$

Proportional-derivative $$K(s)=k_P\frac{Ts+1}{\alpha Ts+1},\ \alpha<1$$

Integral $$K(s)=1+\frac{k}{1+Ts}$$

Proportional-integral $$K(s)=\alpha\frac{Ts+1}{\alpha Ts+1},\ \alpha>1$$

Proportional-integral-derivative $$K(s)=k(T_d s+1)\left(1+\frac{1}{T_i s}\right)$$

Modern control theory

Types of controllers

<u>Robust control</u> – controller is designed to work assuming that certain system parameters will be not constant but bounded. Control law is not changing.

<u>Adaptive control</u> – controller adapts its parameters or changes it’s control law for varying parameters of the system. Parametric estimation of the system is used.

<u>Intelligent control</u> – control techniques that uses e.g. neural networks, fuzzy logic, machine learning or genetic algorithms.

Control tasks

<u>Stabilization</u> – control system setpoint is steady in time (e.g. constant temperature, constant speed, fixed position).

<u>Trajectory tracking</u> – system's desired position is described with a function of time, so time is restricted (e.g. control of a robotic arm)

<u>Path following</u> – system's desired position is described by a parametric path, time is not restricted so controller decides about the velocity (e.g. autonomous platforms)

State-space representation

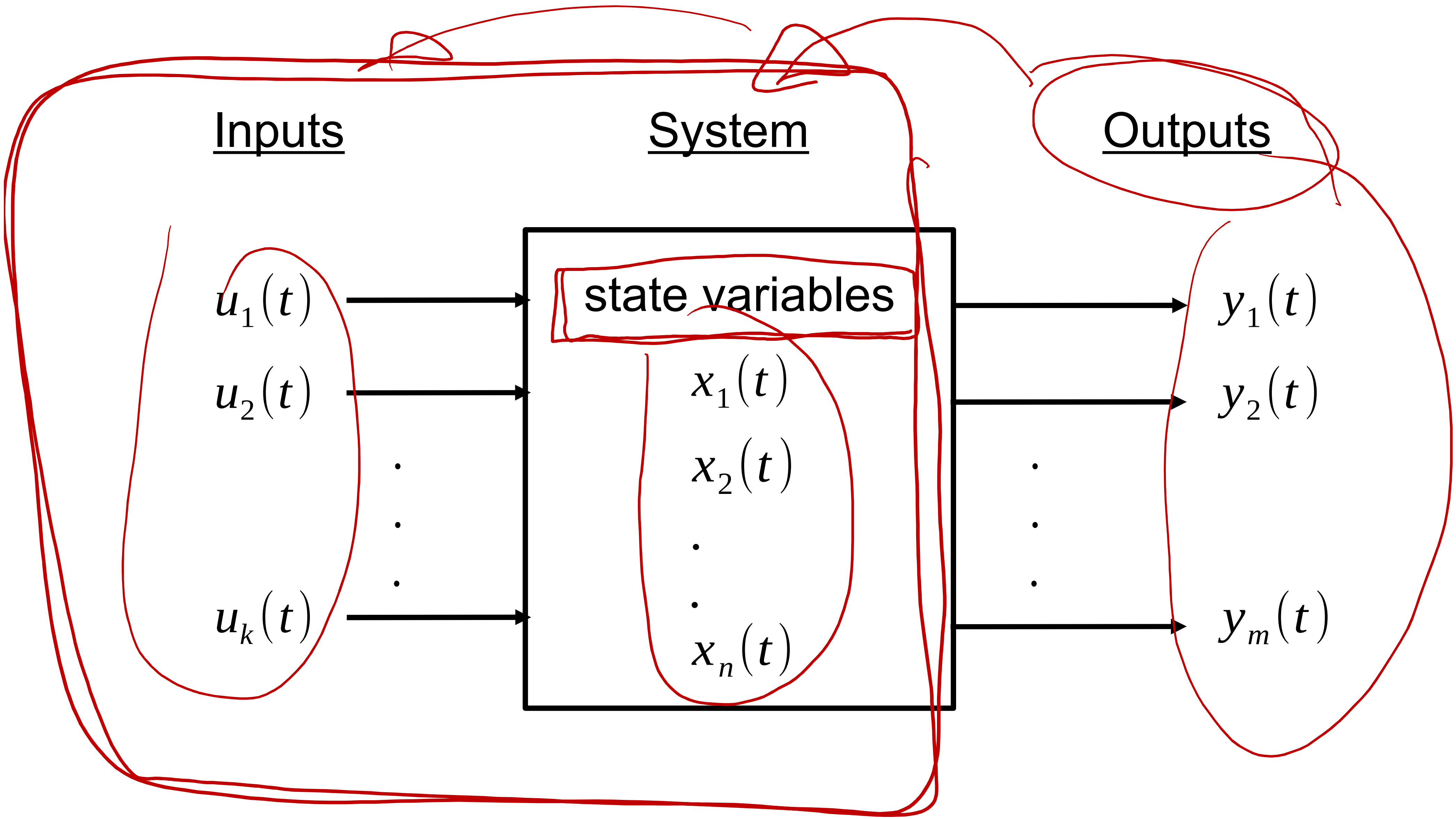
State-space representation is an alternative to transfer function form for writing system models.

State variable or set of state variables – representation of status of a system at any time.

Typical state variables: position, velocity, temperature, pressure, volume flow, current, voltage.

There are many different state variable representations for the same system, but input-output relation does not depend on its' selection.

State-space representation



State-space representation

For continouse, linear and time-invariant system

State-space equation: $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t)$

$\boldsymbol{x}_{n\,x\,1}(t)$ - state variables vector

$\boldsymbol{A}_{n\,x\,n}$ - state matrix (system matrix)

$\boldsymbol{B}_{n\,x\,k}$ - input matrix

$\boldsymbol{u}_{k\,x\,1}(t)$ - control inputs

External outputs equation: $\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + \boldsymbol{D}\,\boldsymbol{u}(t)$

$\boldsymbol{y}_{m\,x\,1}(t)$ - outputs vector

$\boldsymbol{C}_{m\,x\,n}$ - output matrix

$\boldsymbol{D}_{m\,x\,k}$ - transmition matrix (direct feedthrough matrix)

$\boldsymbol{u}_{k\,x\,1}(t)$ - control inputs

State-space representation

Example for n = 2, k = 4, m = 3

State-space equation: $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ B_{21} & B_{22} & B_{23} & B_{24} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}$$

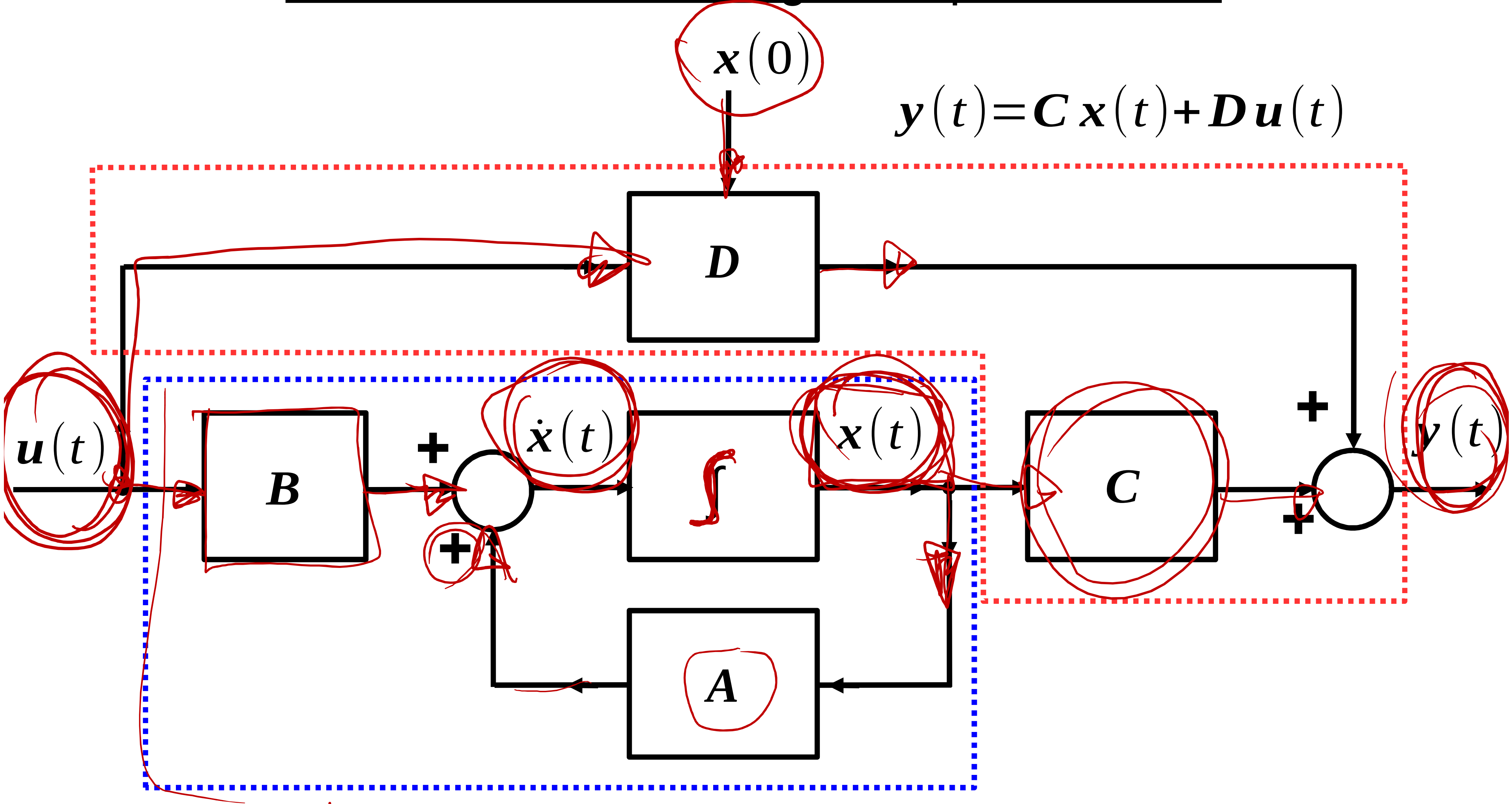
Outputs equation: $\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + \boldsymbol{D}\,\boldsymbol{u}(t)$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix}$$

State-space representation

Time-domain block diagram representation

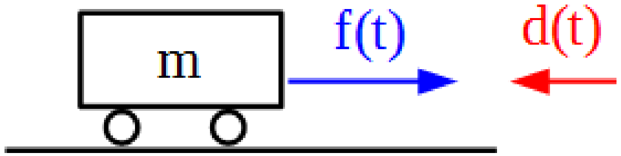
$$\boldsymbol{y}(t) = \boldsymbol{C}\,\boldsymbol{x}(t) + \boldsymbol{D}\,\boldsymbol{u}(t)$$



$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\,\boldsymbol{x}(t) + \boldsymbol{B}\,\boldsymbol{u}(t)$$

State-space representation – example 1

Car on a flat surface
m – mass,
f(t) – driving force,
d(t)=c*v(t) – air ressistance,
x(t) – displacement



state: $\mathbf{x} = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$

$$\boxed{m\frac{d^2x(t)}{dt^2} = f(t) - d(t)}$$

$$\begin{cases} m \cdot \dfrac{dv}{dt} = f(t) - c \cdot v(t) \\ \dfrac{dx}{dt} = v \end{cases} \Rightarrow \begin{cases} \dot{v} = (f(t) - c\,v(t))/m \\ \dot{x} = v \end{cases}$$

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} [f(t)]$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B \cdot u$$

State-space representation – example 1

Car on a flat surface
m – mass,
f(t) – driving force,
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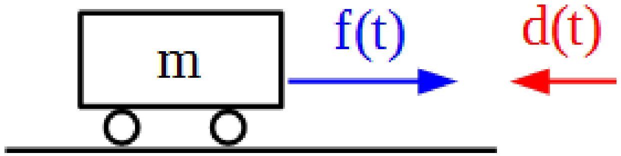
x(t) – displacement



$$m\frac{d^2x(t)}{dt^2}=f(t)-d(t)$$

State-space representation – solution

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{cases}$$

Solution:

$$\boldsymbol{y}(t) = \boldsymbol{C} e^{\boldsymbol{A}t} \boldsymbol{x}_0 + \boldsymbol{C} \int_0^t e^{\boldsymbol{A}(t-\tau)} \boldsymbol{B}\boldsymbol{u}(\tau)\, d\tau + \boldsymbol{D}\boldsymbol{u}(t)$$

satisfying initial conditions (free response)

convolution of an input with system impulse responses (forced response)

State-space representation to transfer function conversion

$$\boxed{\dot{x}(t)=\boldsymbol{A}\,\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{u}(t) \qquad \boldsymbol{y}(t)=\boldsymbol{C}\,\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{u}(t)}$$

$\mathcal{L}$ + zero IC

$$s\boldsymbol{X}(s)=\boldsymbol{A}\boldsymbol{X}(s)+\boldsymbol{B}\boldsymbol{U}(s)$$

$$s\boldsymbol{X}(s)-\boldsymbol{A}\boldsymbol{X}(s)=\boldsymbol{B}\boldsymbol{U}(s)$$

$$(s\boldsymbol{I}-\boldsymbol{A})\boldsymbol{X}(s)=\boldsymbol{B}\boldsymbol{U}(s)$$

for $\det(s\boldsymbol{I}-\boldsymbol{A})\neq 0$

$$\boldsymbol{X}(s)=(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)$$

$\mathcal{L}$ + zero IC

$$\boldsymbol{Y}(s)=\boldsymbol{C}\boldsymbol{X}(s)+\boldsymbol{D}\boldsymbol{U}(s)$$

$$\boldsymbol{Y}(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{U}(s)+\boldsymbol{D}\boldsymbol{U}(s)$$

$$\boldsymbol{Y}(s)=(\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D})\boldsymbol{U}(s)$$

$$\boxed{\boldsymbol{H}(s)=\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D}}$$

State-space representation – software

MATLAB & Simulink

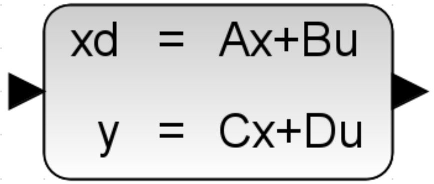
$$x' = Ax + Bu$$
$$y = Cx + Du$$

State space

State-space representation to transfer function conversion:

[num,den] = ss2tf(A,B,C,D,iu)

Scilab & Xcos

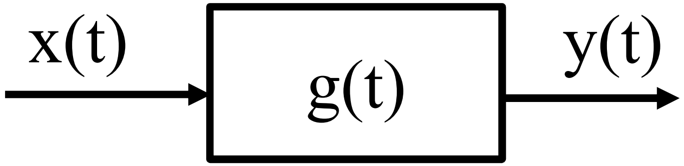


State-space representation to transfer function conversion:

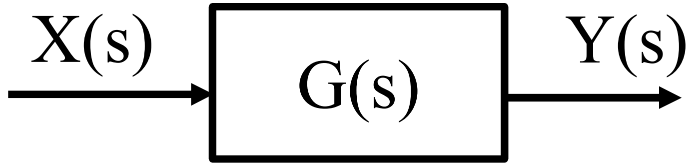
[h] = ss2tf(sl)

Block diagrams

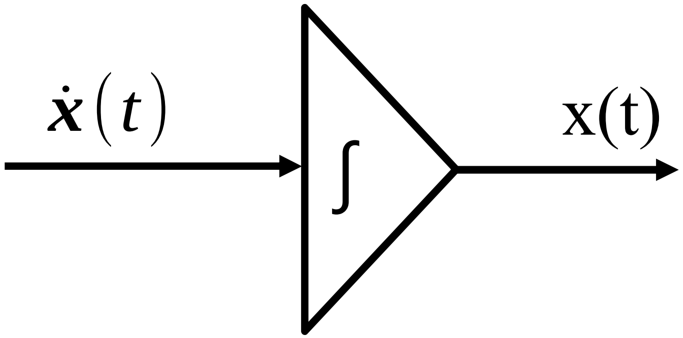
transfer function in time domain



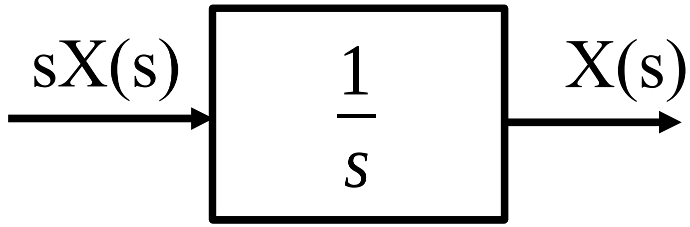
transfer function in complex domain



inegration in time domain



integration in complex domain



Transfer function to state-space representation conversion

Direct method (controllable canonical form) – direct construction of the block diagram with coefficients from transfer function. We can read state-space representation from the diagram.

Parallel method (diagonal canonical form) – partial fraction decomposition of a transfer function is needed, then we can create block diagram. State matrix A will be then diagonal.

Iterative method – factored form of a transfer function is needed (poles and zeroes visible).

Materials for exam – lectures from 1 to 14 (>1000 slides...)

Lecture 14 – material repeat, supplementary info,
information about the exam,
WUT questionnaires,
consultations.

Lecture 15 – modern control theory overview,
experiment with control system,
Consultations.

Exam: Wednesday, 5th February, 12:00-13:30, room 2.19
Wednesday, 12th February, 12:00-13:30, room 2.19

EXAM – IMPORTANT NOTES

- You have to pass the project class to attend the exam.
- Student card or erasmus paper is needed on the exam.
- Please write the exam clearly on the A4 paper.
- Everyone must to return the exam.
- You can not use any electronic devices during the exam (mobile phones, smart watches, calculators).
- Table of Laplce transform will be displayed on the screen.
- Additional persons are delegated to help during the exam.
- Any cheating behaviors will cause exam failure.
- Topics will be distributed in printed form or displayed.

EXAM – IMPORTANT NOTES

- Your answers will be rated with points.
- Exam mark will be based on the total number of points achieved with the rules:
  - < 50% - mark 2 (exam failed)
  - 51%-60% - mark 3,0
  - 61%-70% - mark 3,5
  - 71%-80% - mark 4,0
  - 81%-90% - mark 4,5
  - >90% - mark 5,0
- If marks from project class and exam are positive, then

Final_mark = 0.5 * project_mark + 0.5 * exam_mark