Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control

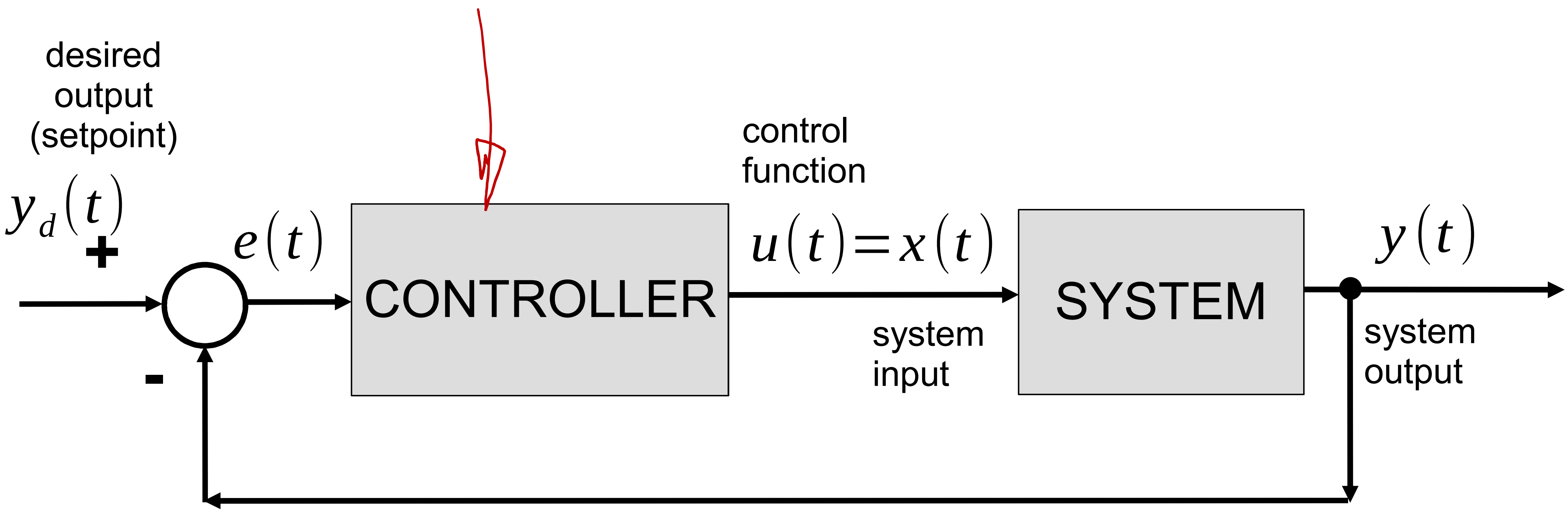
Winter 2019/2020

Lecturer: Sebastian Korczak, PhD Eng.

Lecture 12

PID controller.
Stability.

Closed loop control



P, I and D controllers transfer functions

Controller	Transfer function
Proportional (P)	k_P $\in \mathbb{R}_+$
Integral (I)	$\dfrac{1}{T_i s}$
Ideal derivative (D)	$T_d s$
Real derivative (D)	$\dfrac{T_d s}{T s+1}$

$T_i \in \mathbb{R}_+$ [s]

$T_d \in \mathbb{R}_+$ [s]

$T_d, T \in \mathbb{R}_+$ [s]

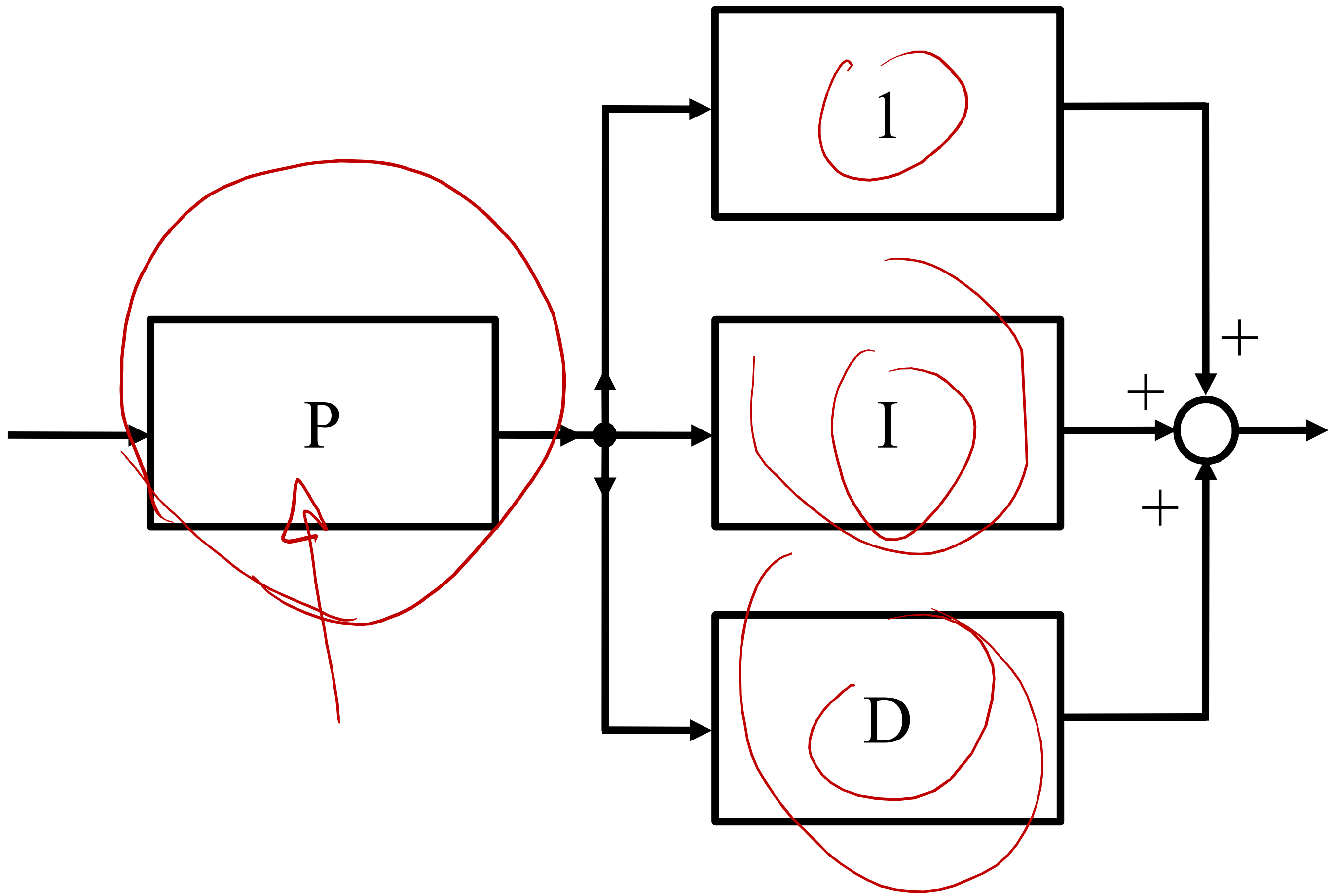
PID controllers transfer functions

Controller	Transfer function
Proportional-integral-derivative (PID) <u>in standard form</u> with ideal derivative	$k_P\left(1+\frac{1}{T_i s}+T_d s\right)$
Proportional-integral-derivative (PID) <u>in parallel form</u> with ideal derivative	$k_P+k_i\frac{1}{s}+k_d s$

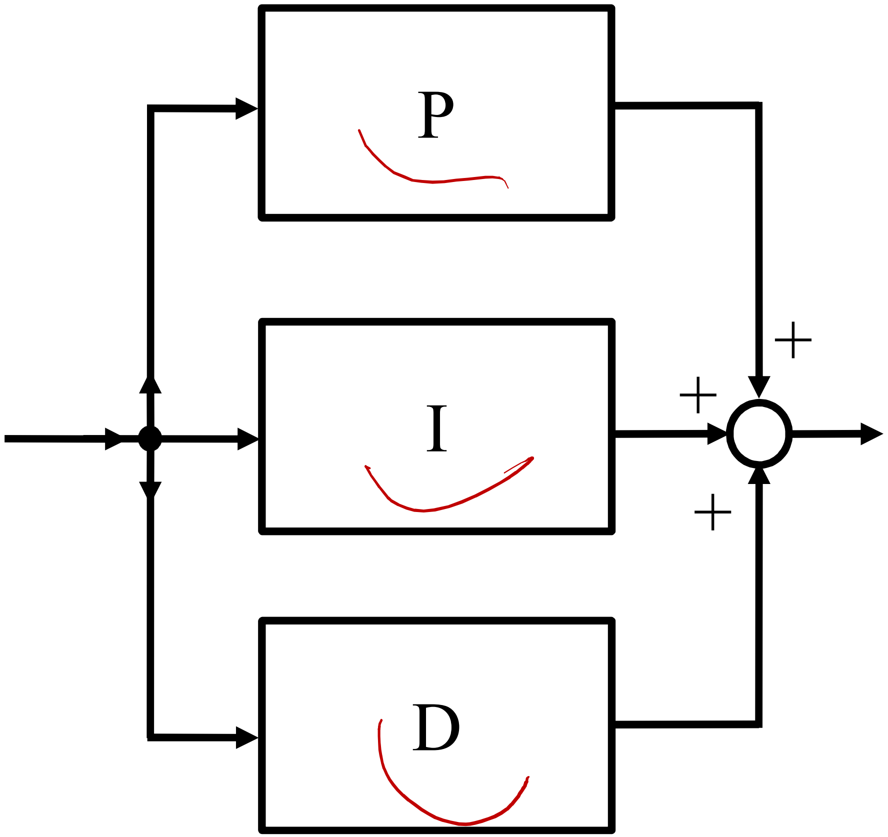
PID controllers transfer functions

Controller	Transfer function
Proportional-integral-derivative (PID) <u>in standard form</u> with real derivative	$k_P\left(1+\frac{1}{T_i s}+\frac{T_d s}{Ts+1}\right)$
Proportional-integral-derivative (PID) <u>in parallel form</u> with real derivative	$k_P+k_i\frac{1}{s}+k_d\frac{s}{Ts+1}$

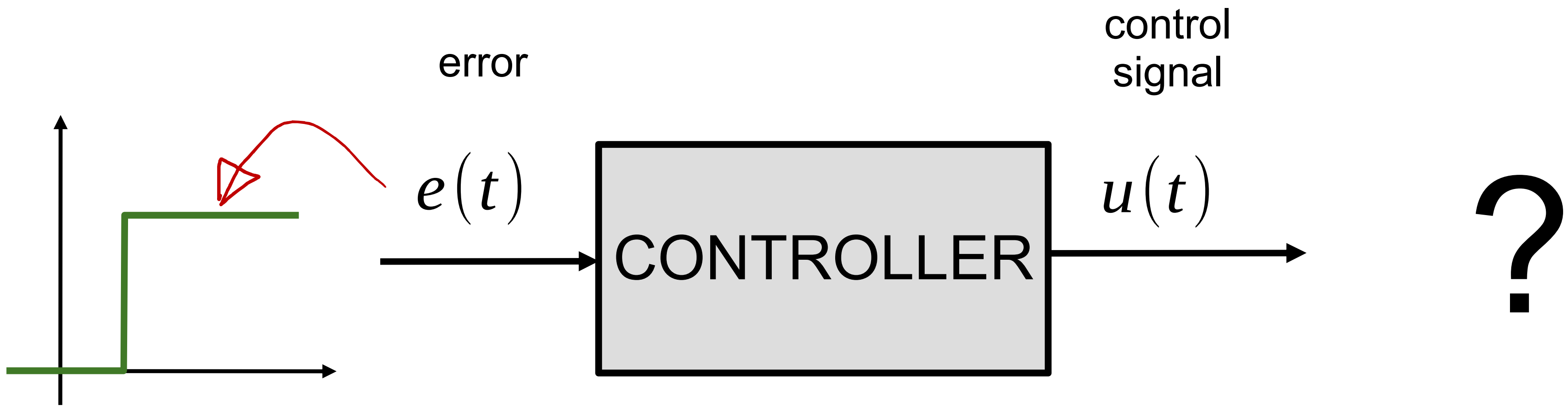
PID CONTROLLER
standard form



PID CONTROLLER
parallel form

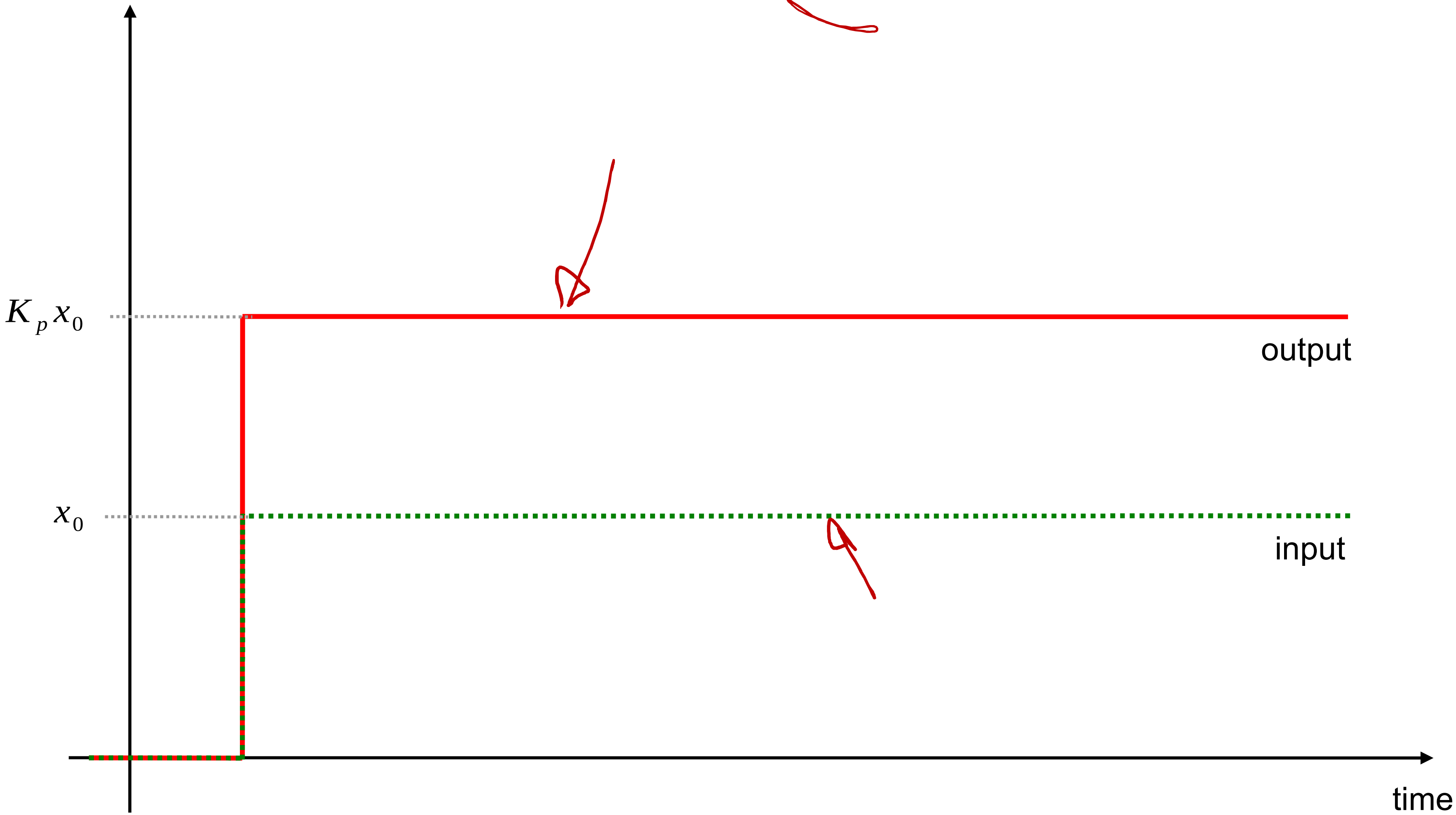


PID CONTROLLER
step responses



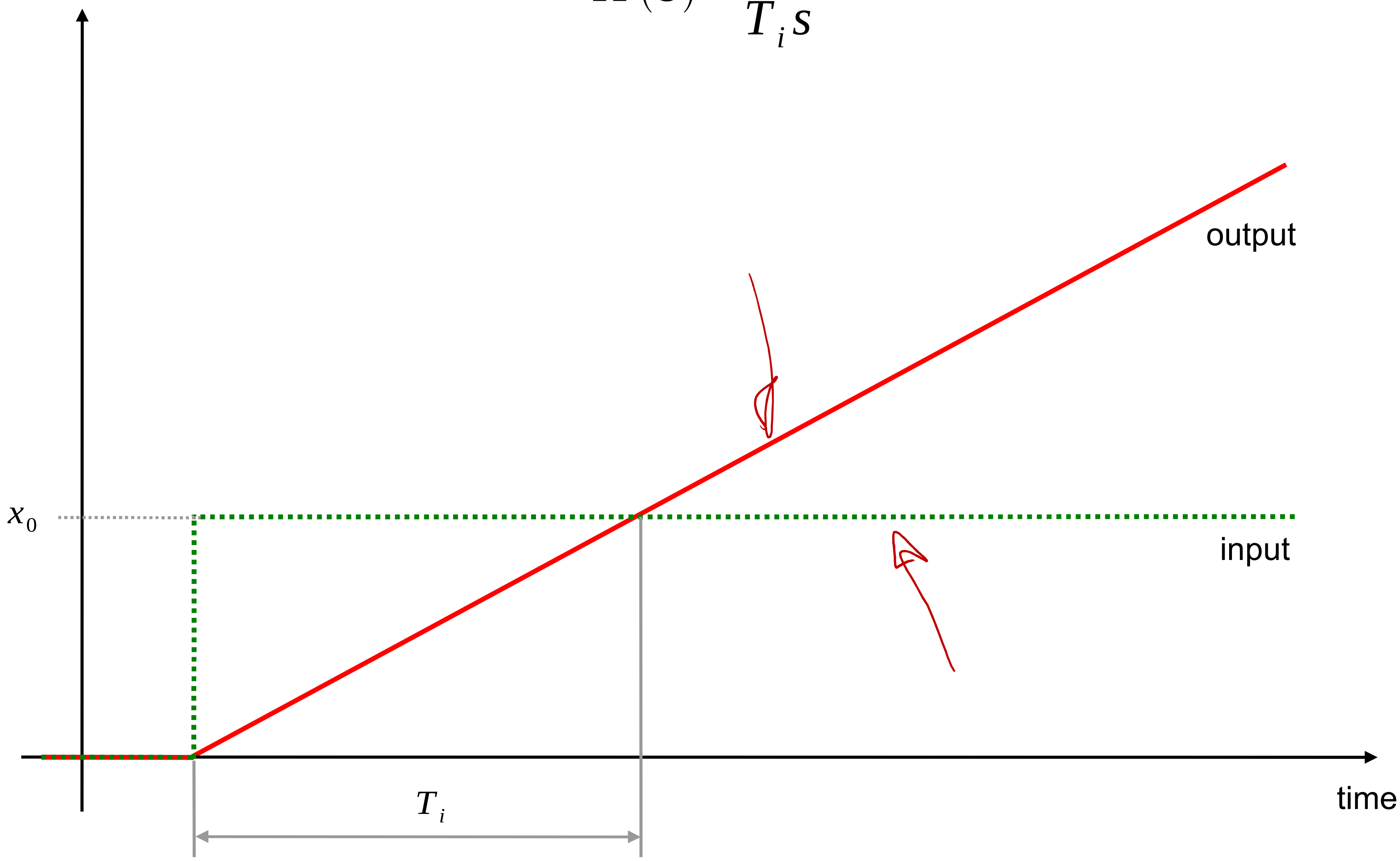
P - CONTROLLER

$$H(s)=K_p$$



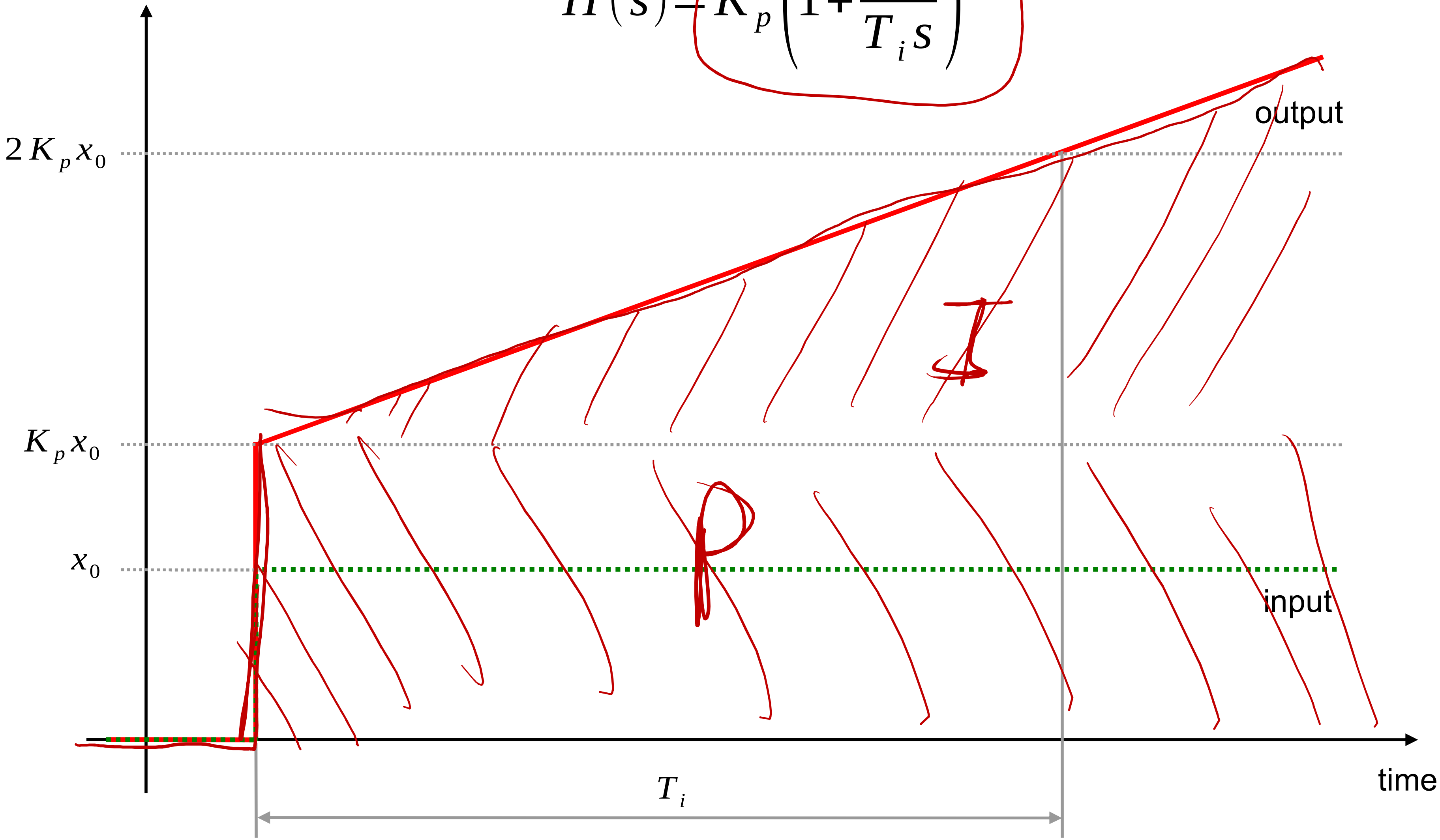
I - CONTROLLER

$$H(s) = \frac{1}{T_i s}$$



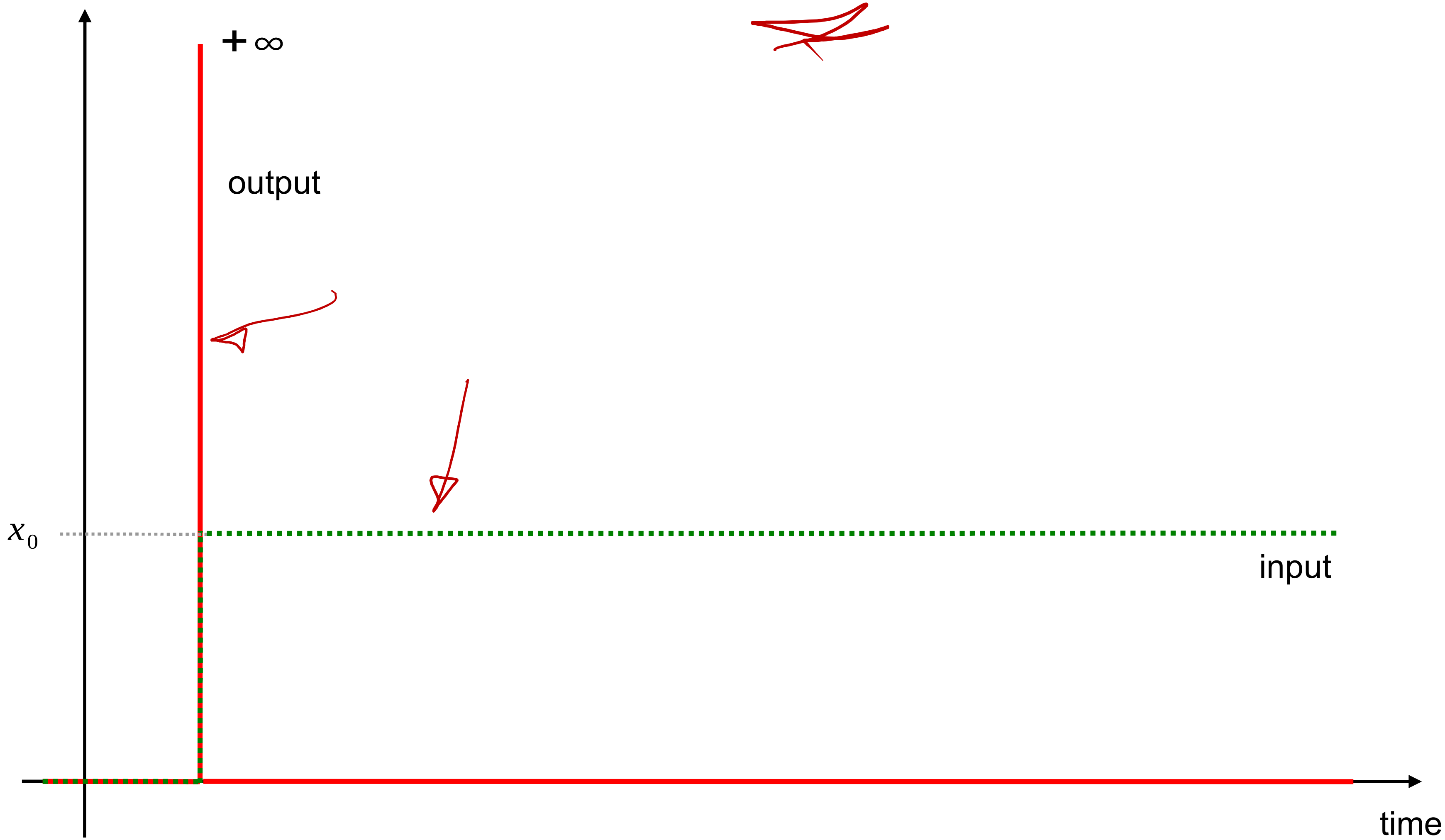
PI - CONTROLLER

$$H(s)=K_p\left(1+\frac{1}{T_i s}\right)$$



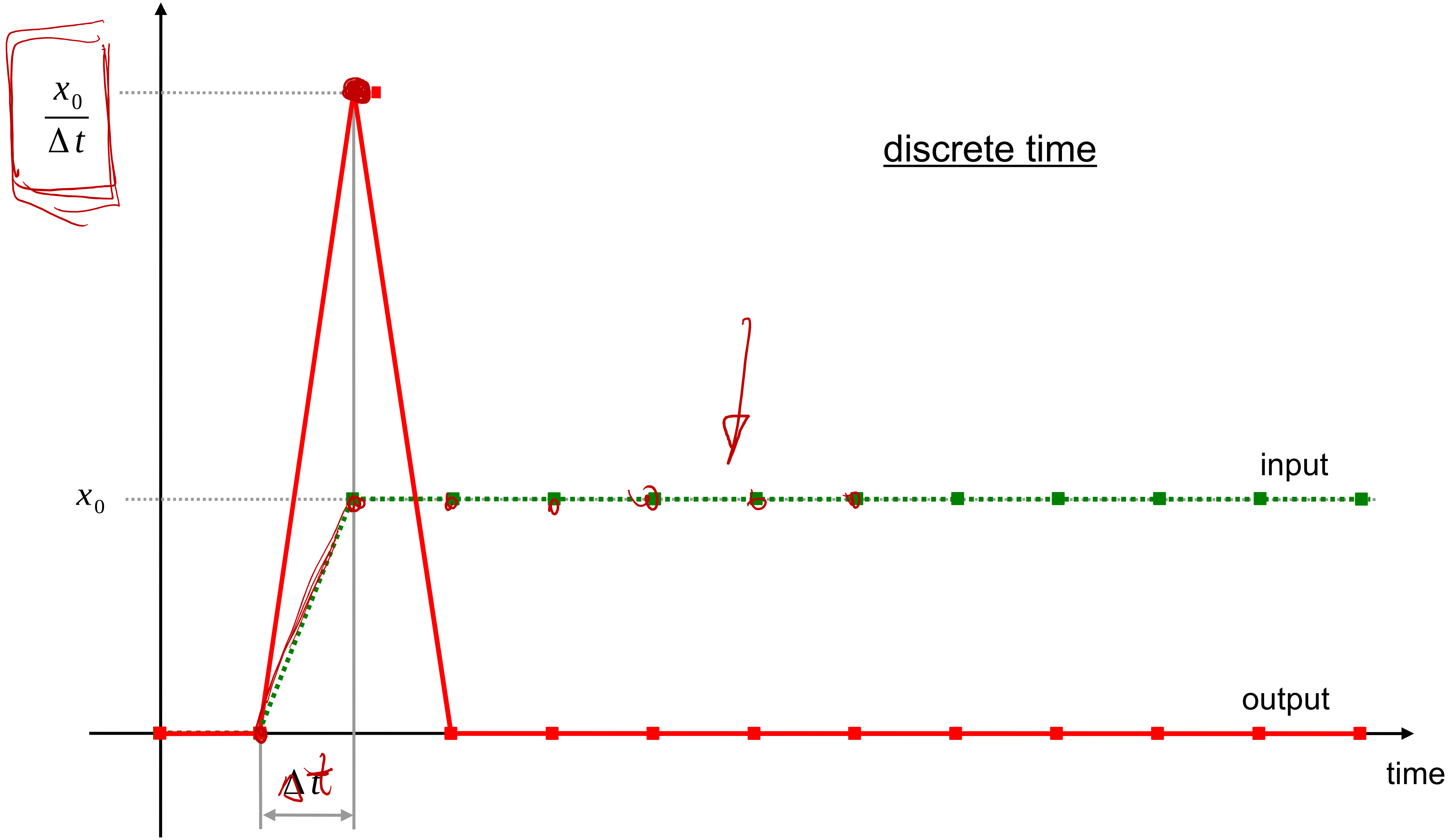
D - CONTROLLER

$$H(s)=T_d s$$



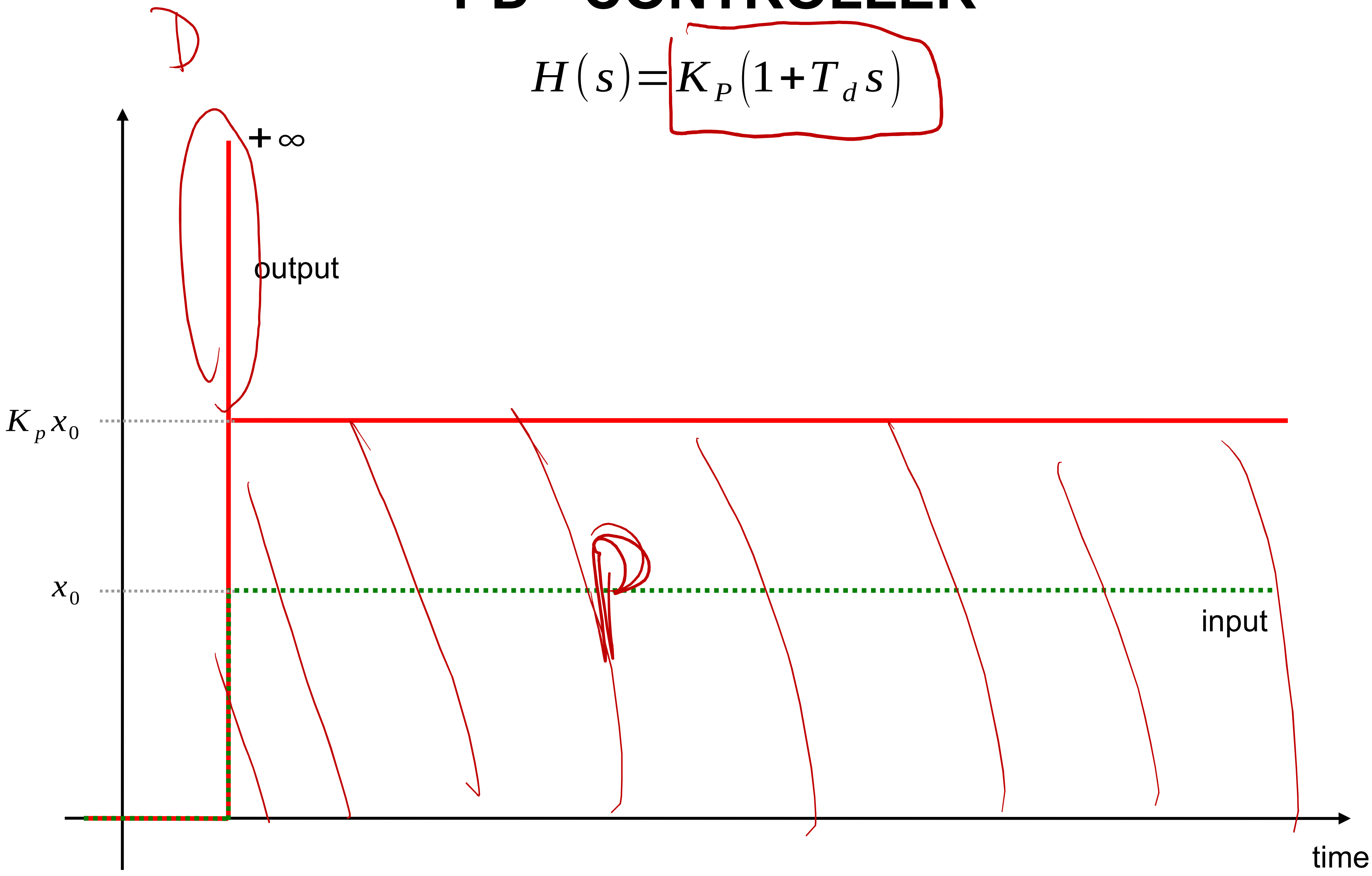
D - CONTROLLER

$$H(s)=T_d s$$



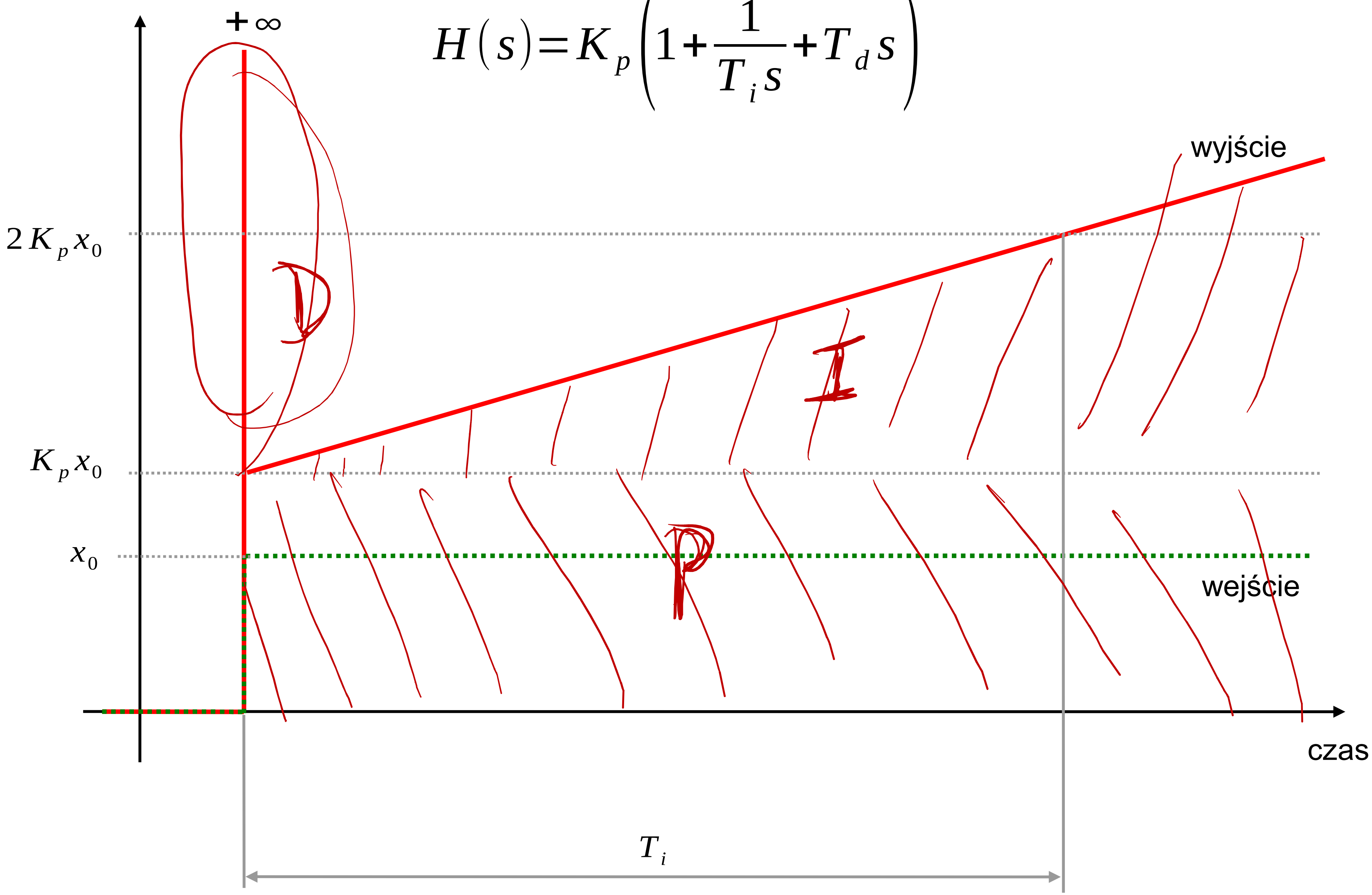
PD - CONTROLLER

$$H(s) = K_P(1 + T_d s)$$



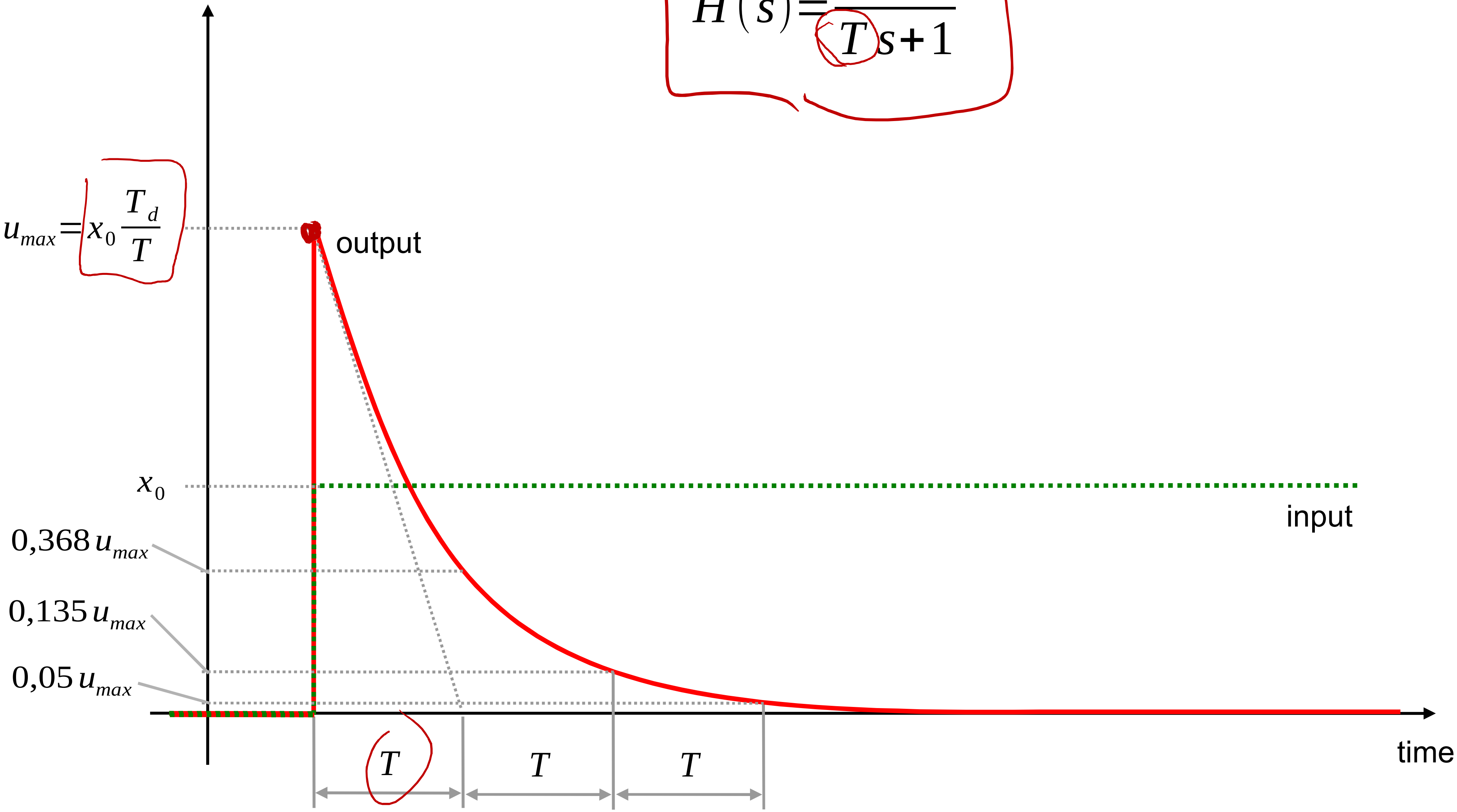
PID – CONTROLLER
standard form, ideal derivative

$$H(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$



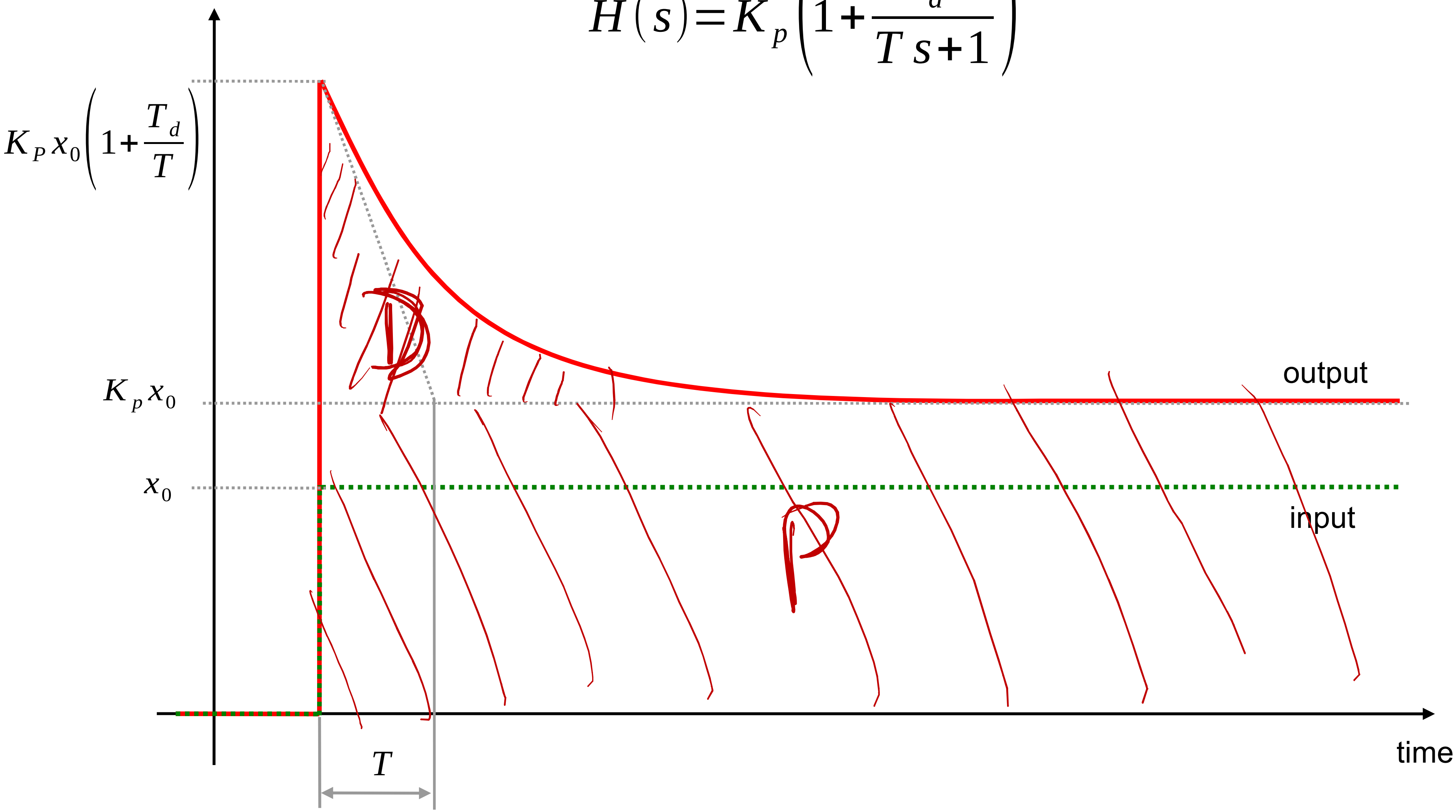
D - CONTROLLER

$$H(s)=\frac{T_d s}{T s+1}$$



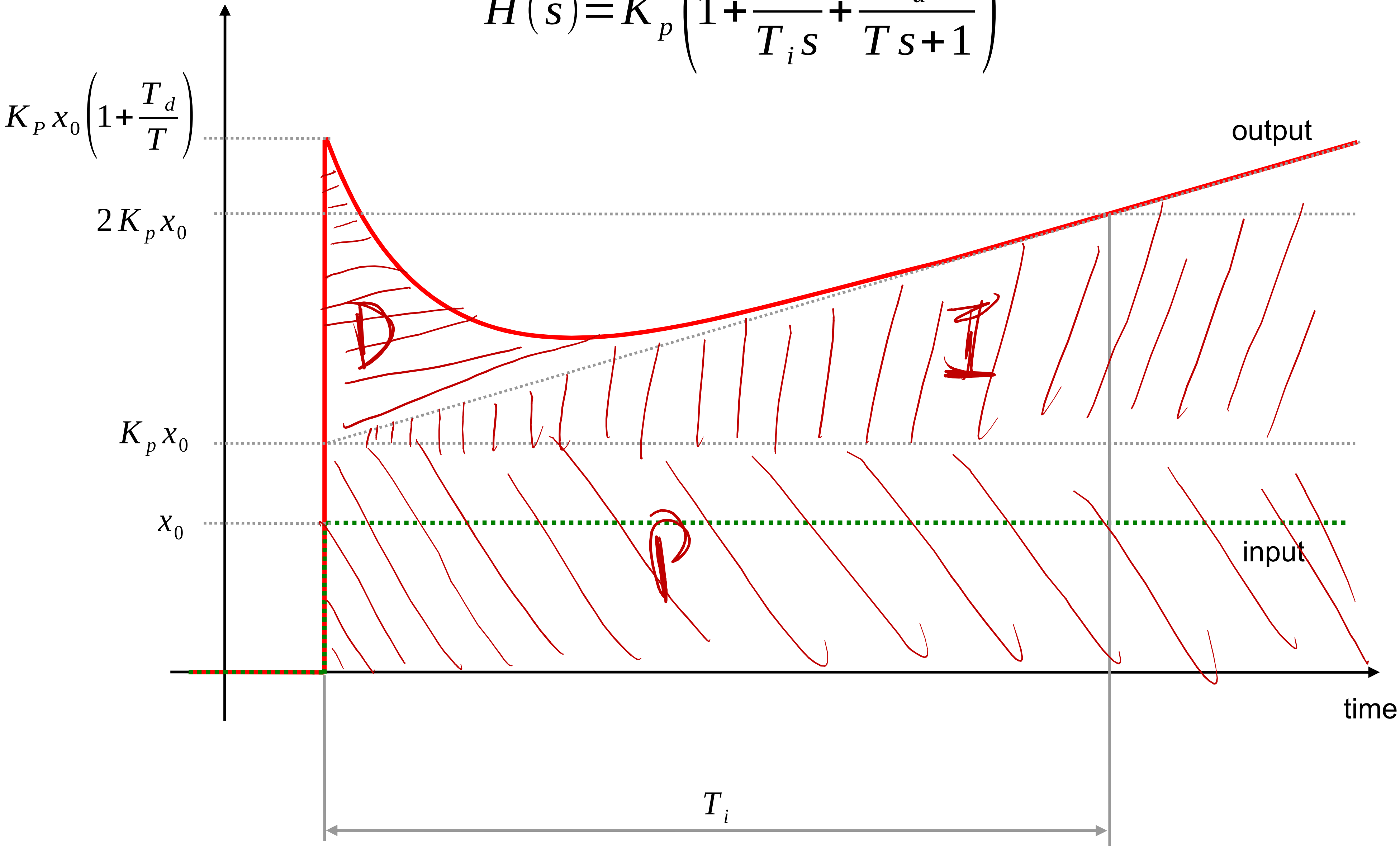
PD - CONTROLLER

$$H(s)=K_p\left(1+\frac{T_d s}{T s+1}\right)$$



PID – CONTROLLER
standard form, real derivative

$$H(s) = K_p\left(1 + \frac{1}{T_i s} + \frac{T_d s}{T s + 1}\right)$$



PID CONTROLLER

important notes

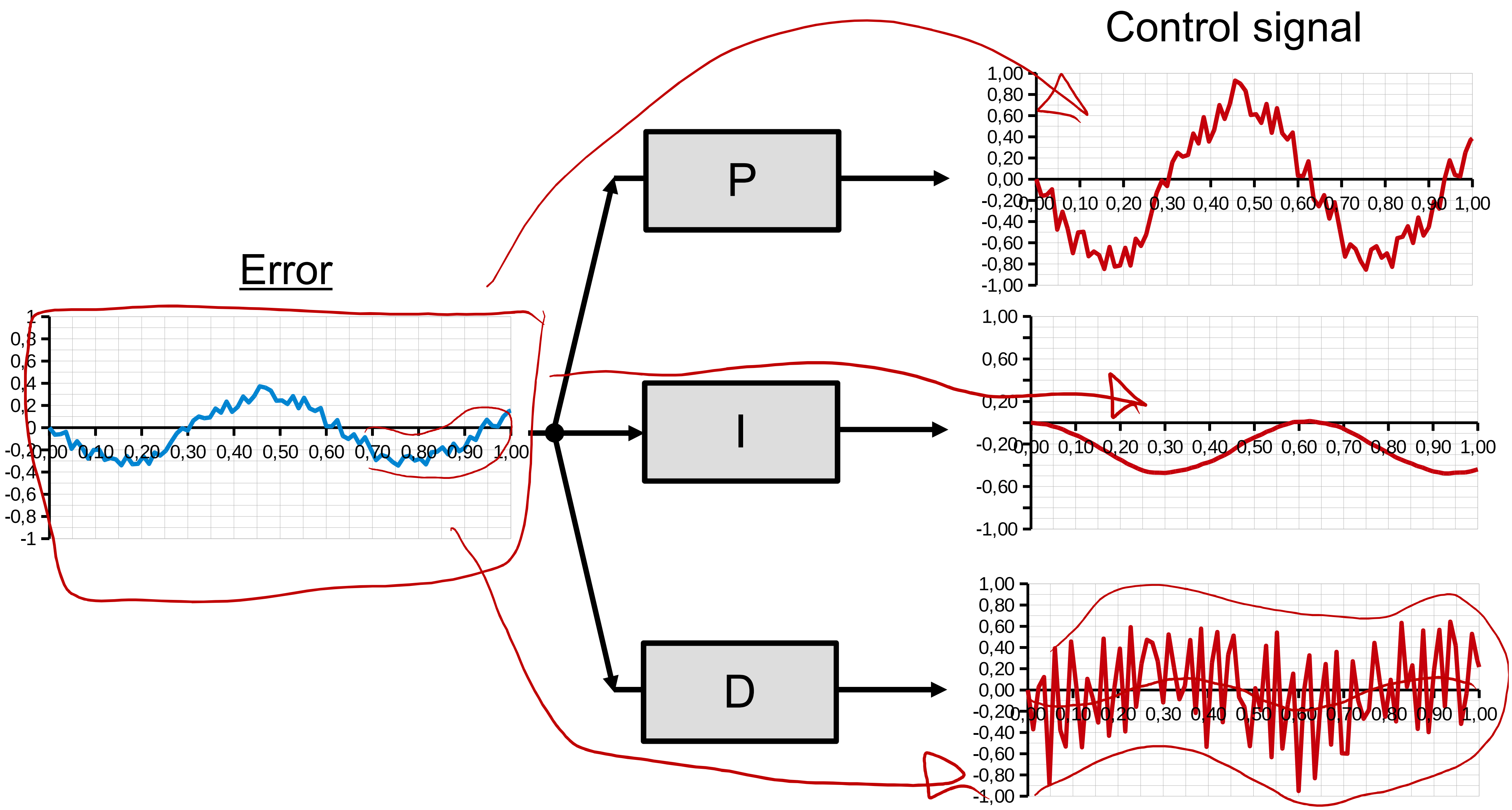
Proportional term – necessary part of the controller, creates a main part of control signal that bring output of the system closer to desired value; higher K_P coefficient gives lover errors; control signal is based on present error;

Integral term – this part of the controller accumulates error; for nonzero error control signal increases that helps to achieve zero error; control signal is based on past error values; “integral windup” problem;

Derivative term – this part of the controller reacts on error changes; for constant error control signal is zero; control signal is based on the trend of future error; this term is very sensitive to noise;

PID CONTROLLER

Influence of errors onto control signal



PID CONTROLLER

integral windup problem

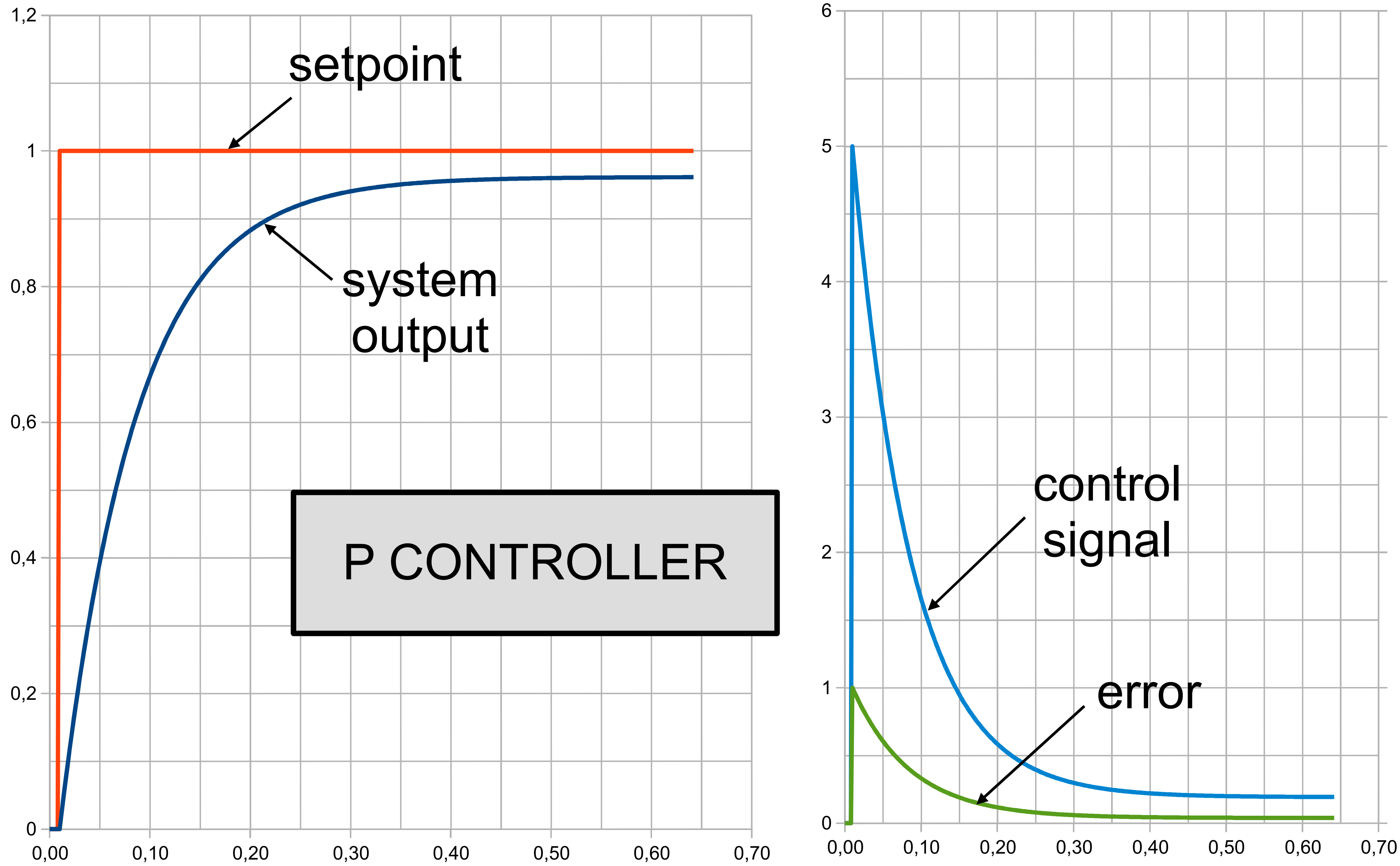
After a large change in a setpoint the integral term can produce very large control signal (higher than maximum possible) – system input is very height until accumulated error goes back close to zero.

Possible solution: disabling and zeroing integral term outside the small region around the setpoint.

Additional reading: google: "integral anti-windup for pi controllers"

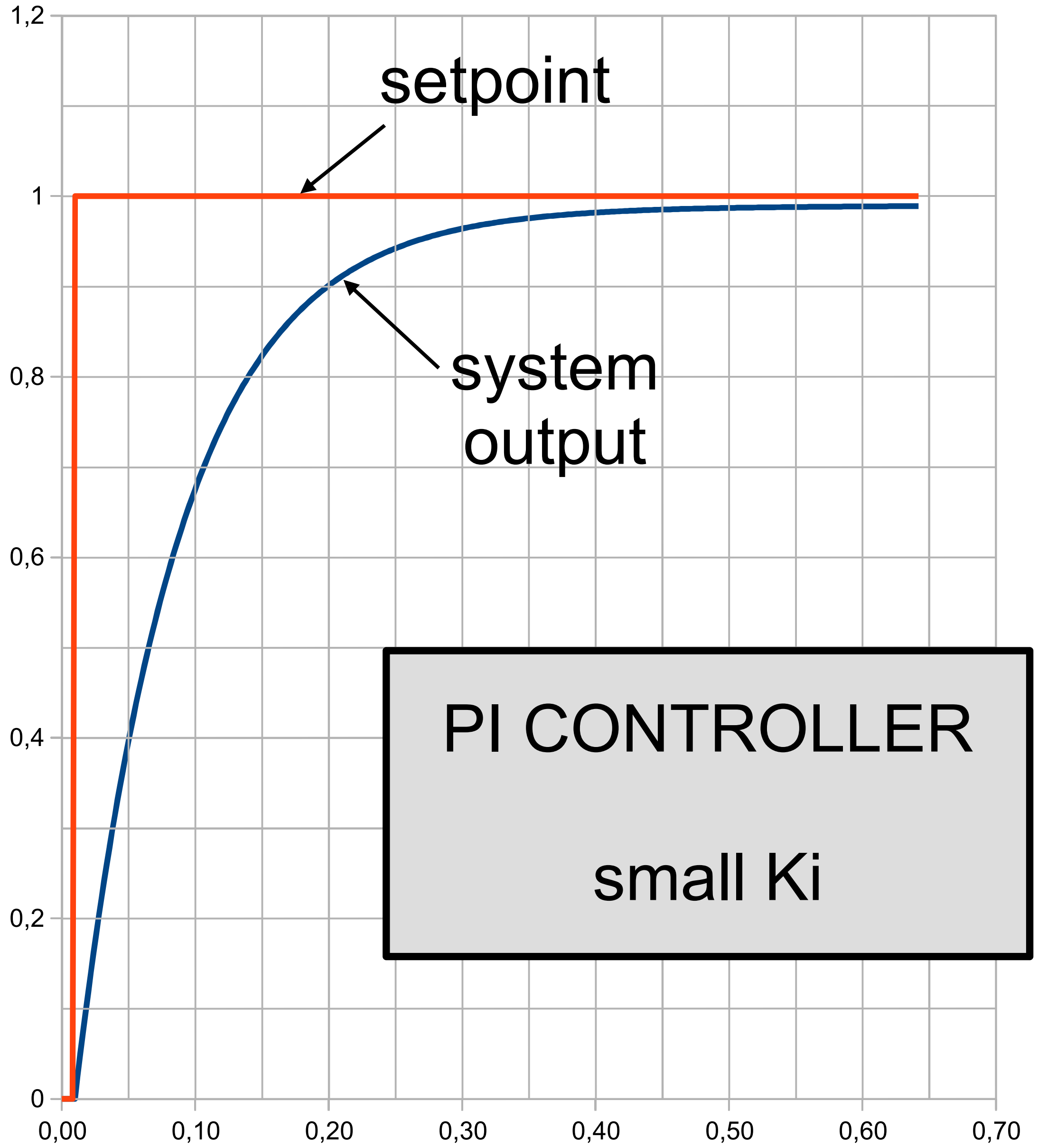
PID CONTROLLER

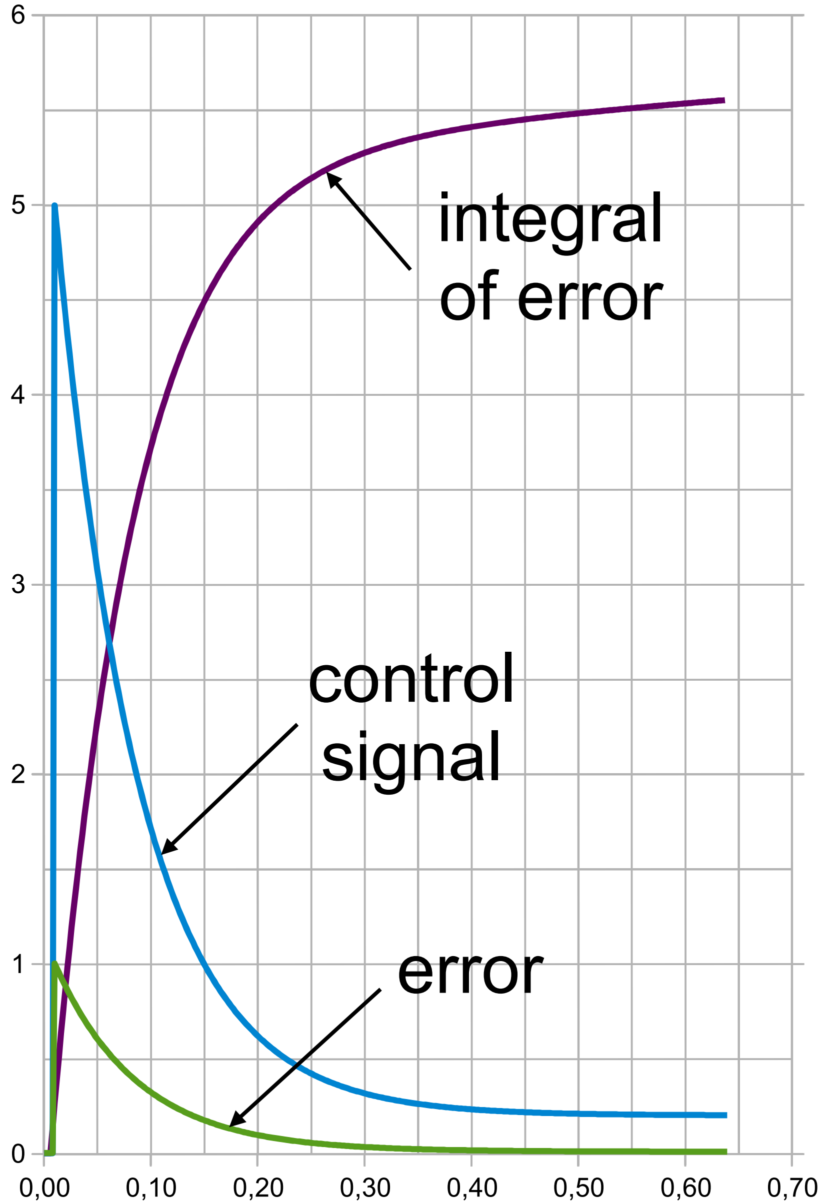
integral windup problem example (1st order inertial system + parallel PID)



PID CONTROLLER

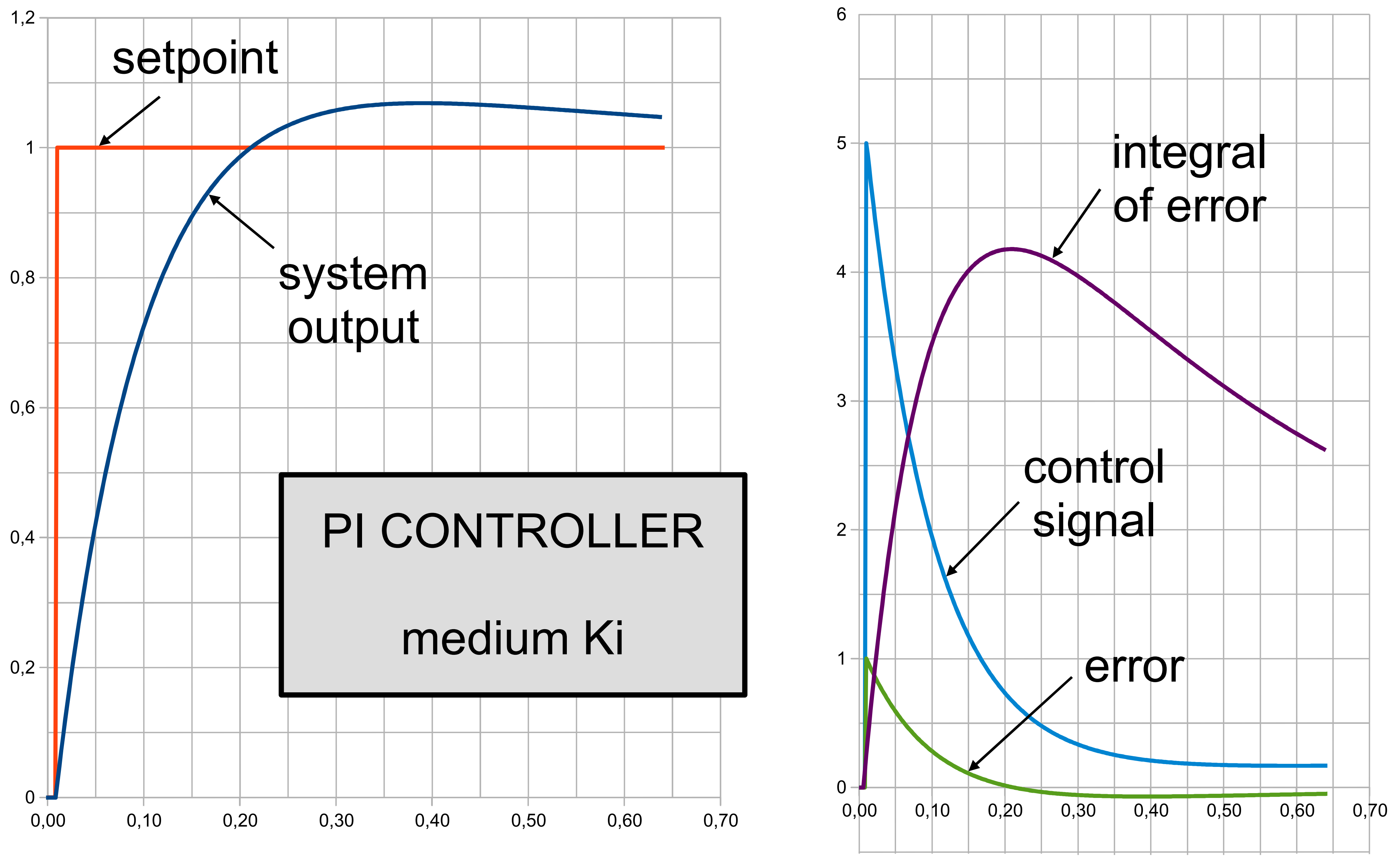
integral windup problem example (1st order inertial system + parallel PID)





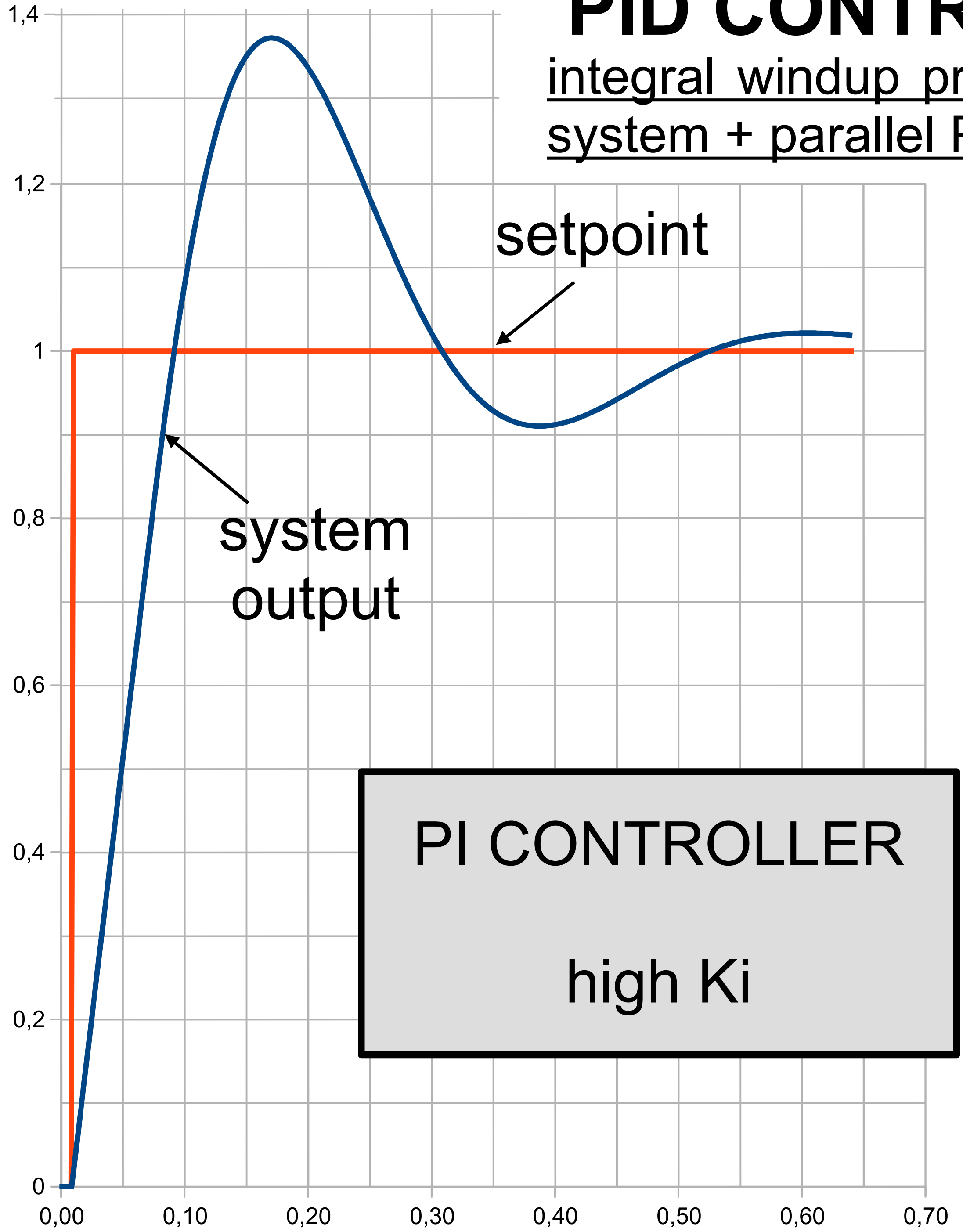
PID CONTROLLER

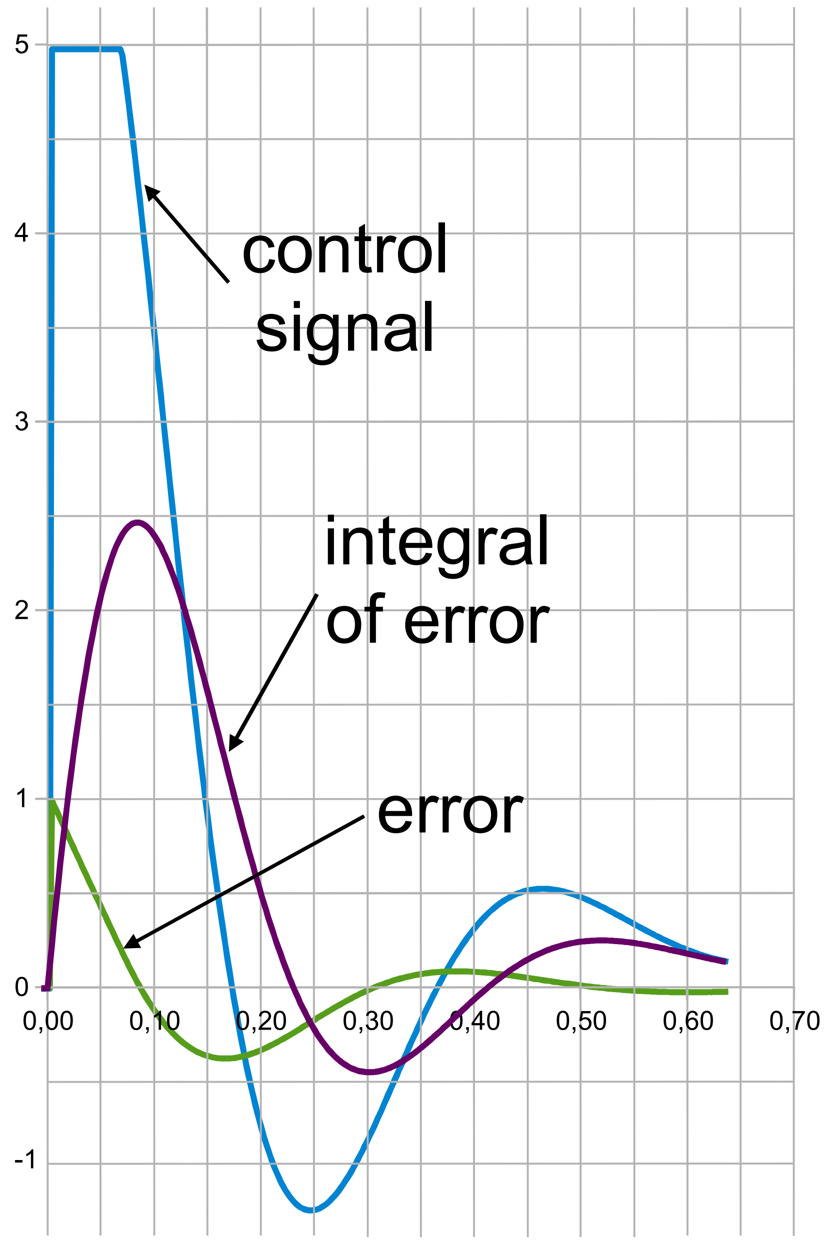
integral windup problem example (1st order inertial system + parallel PID)



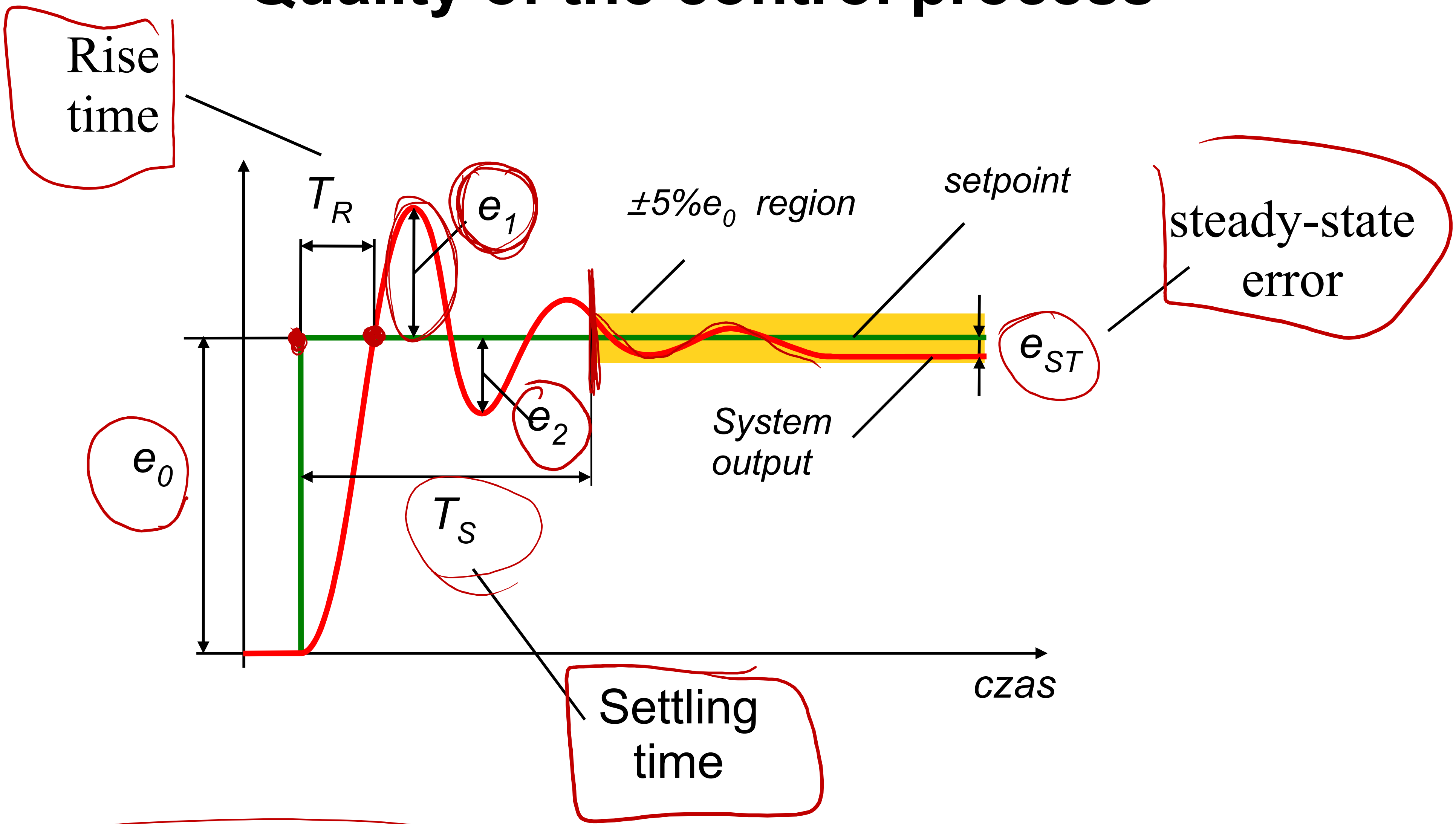
PID CONTROLLER

integral windup problem example (1st order inertial system + parallel PID)





Quality of the control process



Overshoot: $w = \frac{e_1}{e_0} 100\%$

Damping: $d = \frac{e_2}{e_1} 100\%$

PID CONTROLLER

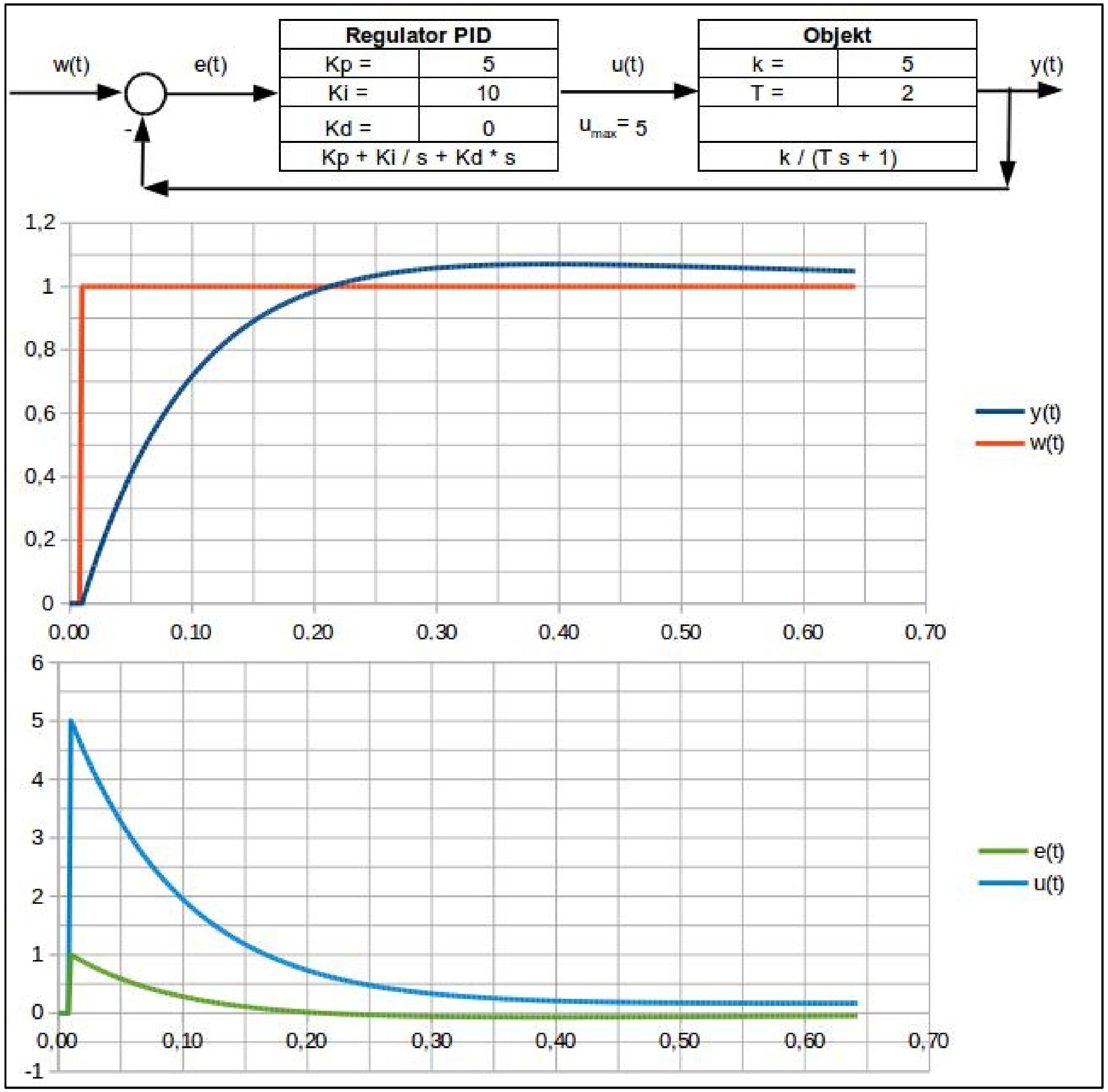
tuning methods

Analytical	With a simulation	Experimental
1st step: calculation of the system's reduced transfer function 2nd step: calculation of the system's step response 3rd step: tuning of the Kp, Ki and Kd coefficients to obtain desired shape of step response	1st step: calculation of the system's reduced transfer function 2nd step: numerical implementation of the system's reduced transfer function 3rd step: tuning of the Kp, Ki and Kd coefficients to obtain desired shape of the system's simulated outputs	Manual tuning or methods: • Ziegler-Nichols • Pessen • Cohen-Coon • Åström–Hägglund

PID CONTROLLER

<u>interactive simulation and tuning</u>

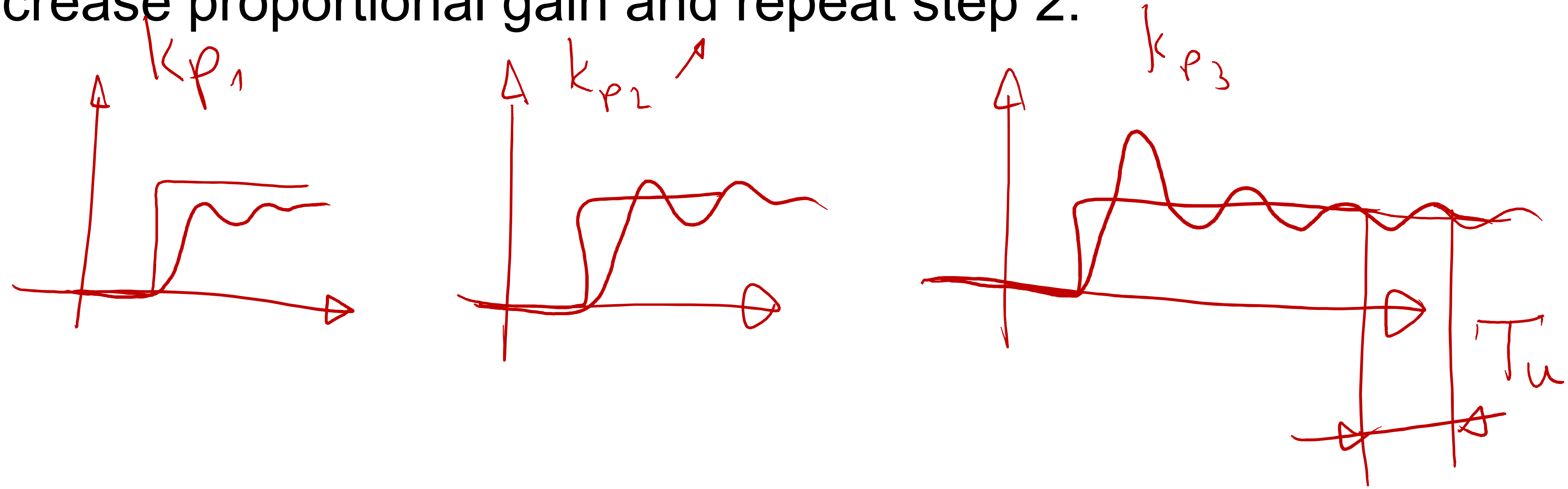
Download spreadsheet file from the website



PID CONTROLLER

Ziegler-Nichols tuning method (PID in standard form)

1. Disable integral and derivative terms of the controller. Set proportional gain to small value.
2. Observe a step response of the output of control loop. Go to point 3, if you observe stable and consistent oscillations. If not, increase proportional gain and repeat step 2.



PID CONTROLLER

Ziegler-Nichols tuning method (PID in standard form)

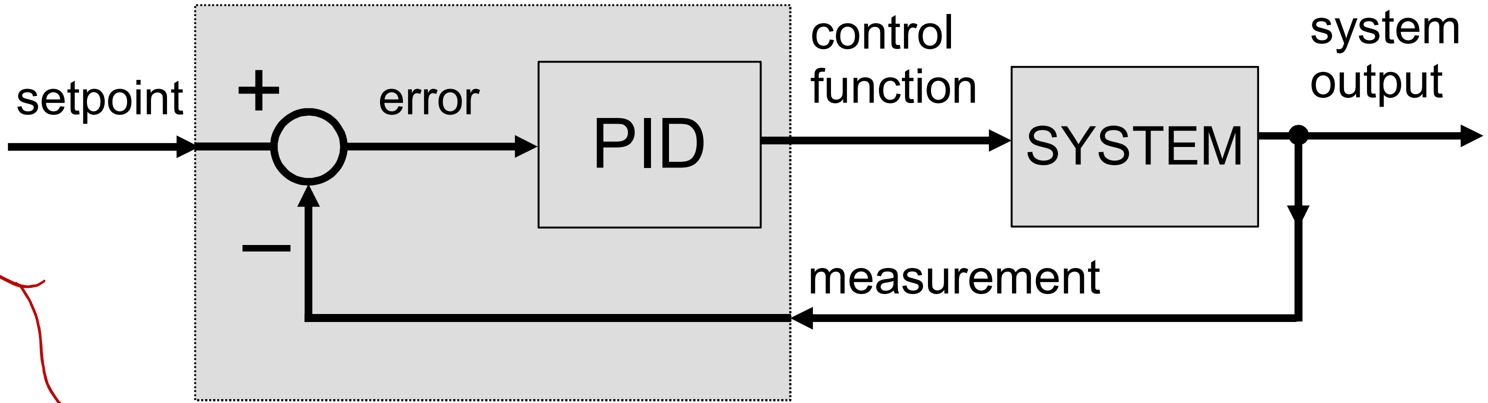
1. Disable integral and derivative terms of the controller. Set proportional gain to small value.
2. Observe a step response of the output of control loop. Go to point 3, if you observe stable and consistent oscillations. If not, increase proportional gain and repeat step 2.
3. For the ultimate gain K_u from step 2 and oscillation period T_u calculate parameters of the controller according to the table:

	k_p	T_i	T_d
Classic Ziegler-Nichols	$0.6\ K_u$	$0.5\ T_u$	$0.125\ T_u$
Pessen	$0.7\ K_u$	$0.4\ T_u$	$0.15\ T_u$
no overshoot	$0.2\ K_u$	$0.5\ T_u$	$0.333\ T_u$

PID CONTROLLER

programming

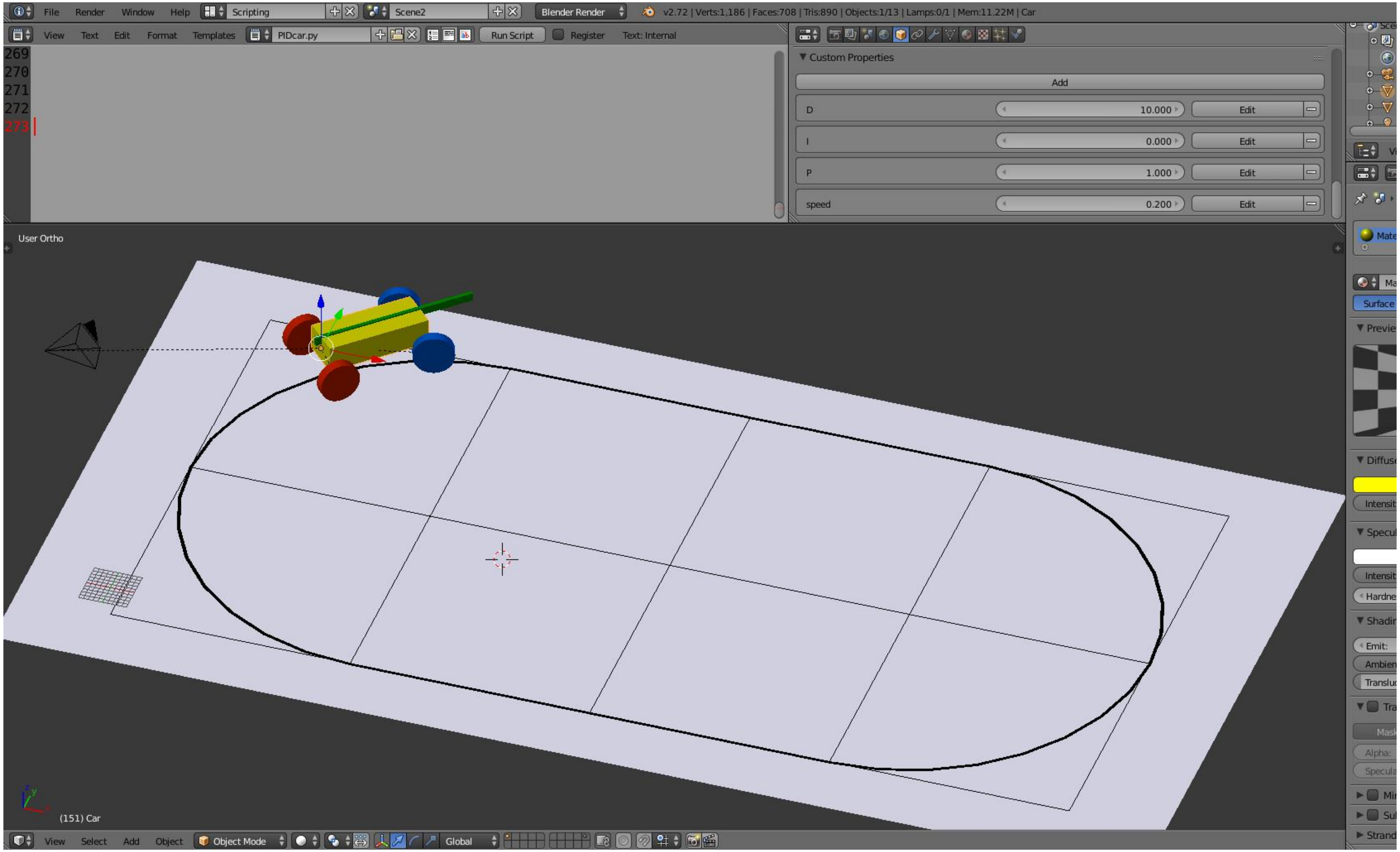
```
dt = 0.1
p_error = 0.
sum = 0.
Kp = 2.
Ki = 0.5
Kd = 0.01
start:
   setpoint = …
   measurement = …
   error = setpoint – measurement
   sum = sum + error * dt
   derivative = (error – p_error) / dt
   output = Kp*error + Ki*sum + Kd*derivative
   p_error = error
   wait(dt)
   goto start
```



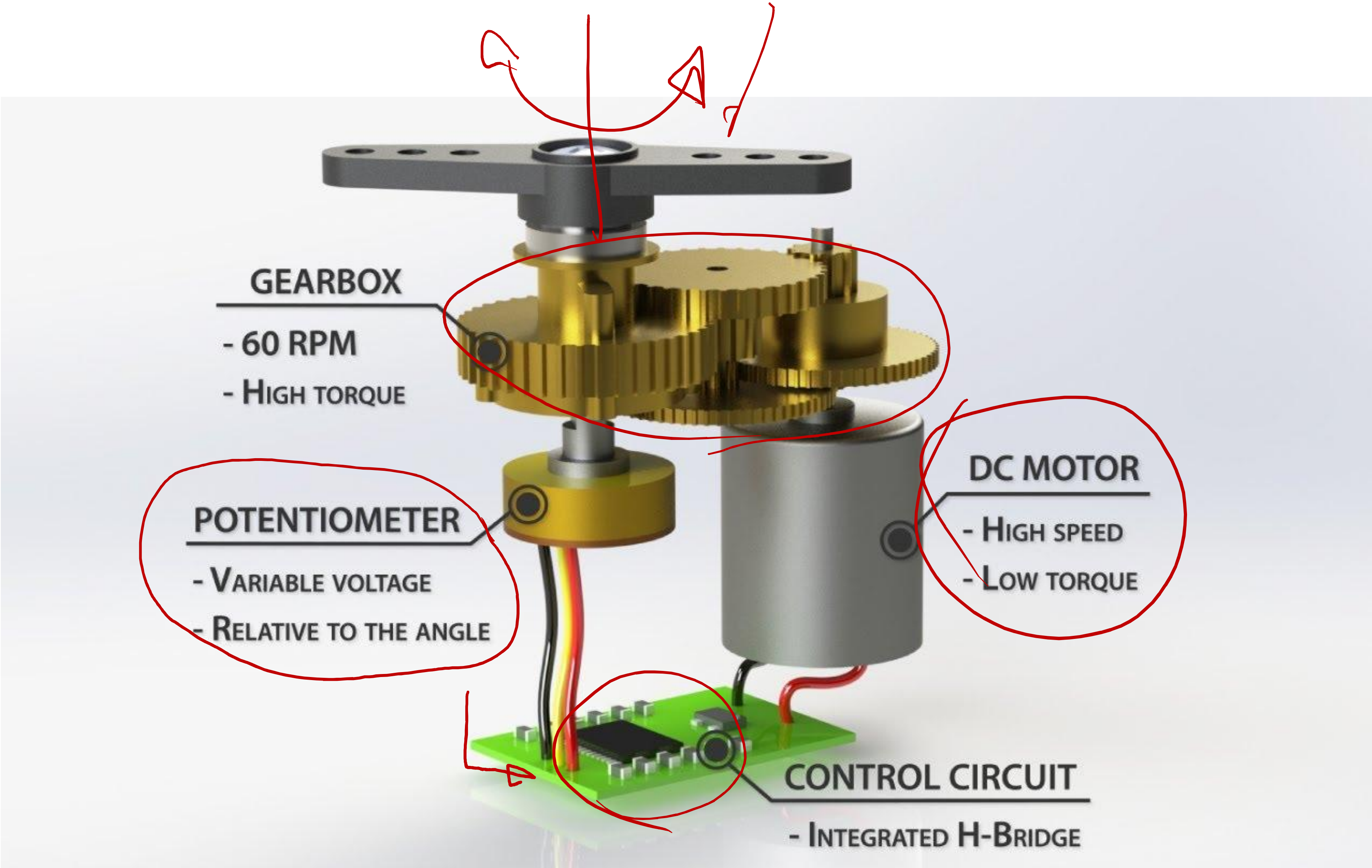
PID CONTROLLER

<u>interactive simulation</u>

PID for a car position control – real-time simulation

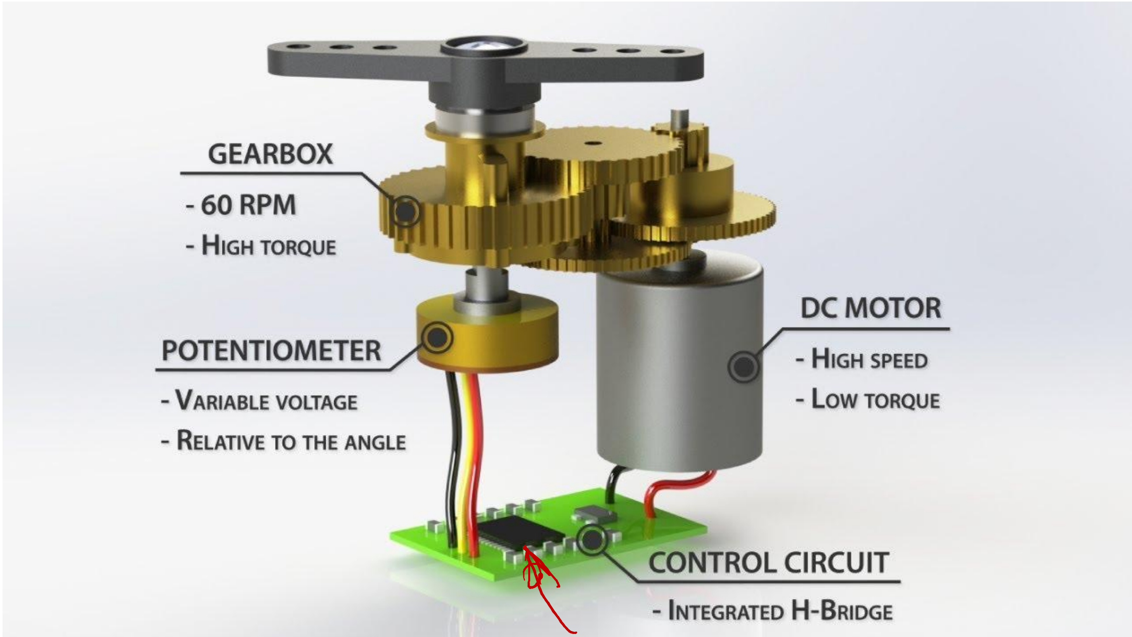


Position control (servomotor)



Source: https://howtomechatronics.com/how-it-works/how-servo-motors-work-how-to-control-servos-using-arduino/

Position control (servomotor)



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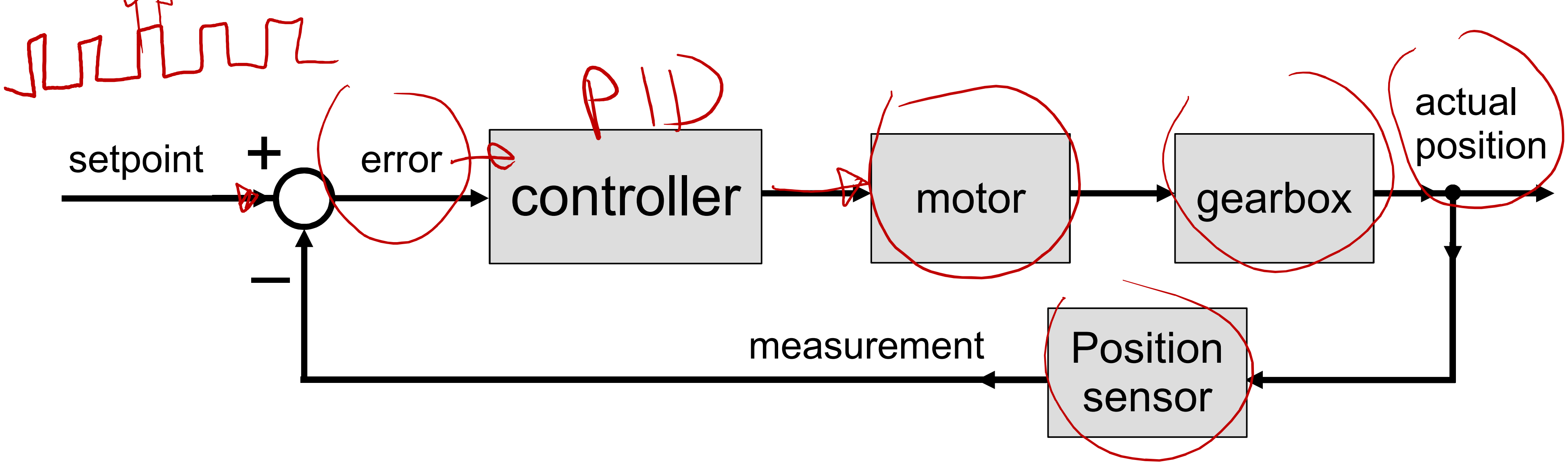


Stability

Stability

In mathematics:

- stability theory
- numerical stability
- stability in geometric theory

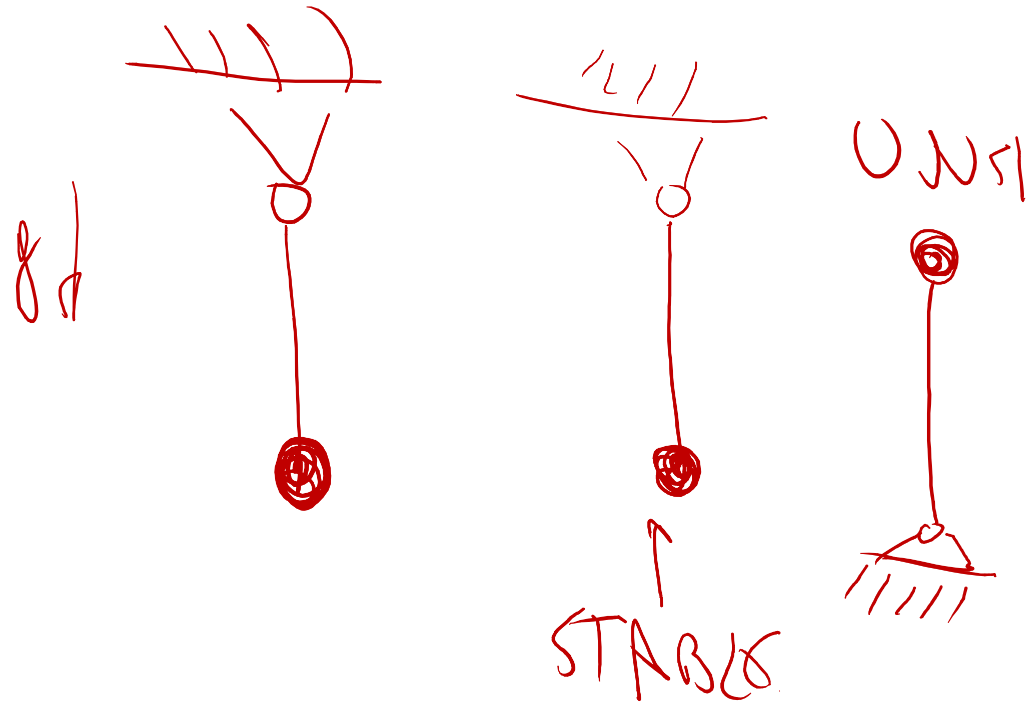
In engineering:

- BIBO stability
- stability in flight dynamics
- ship stability

Stability

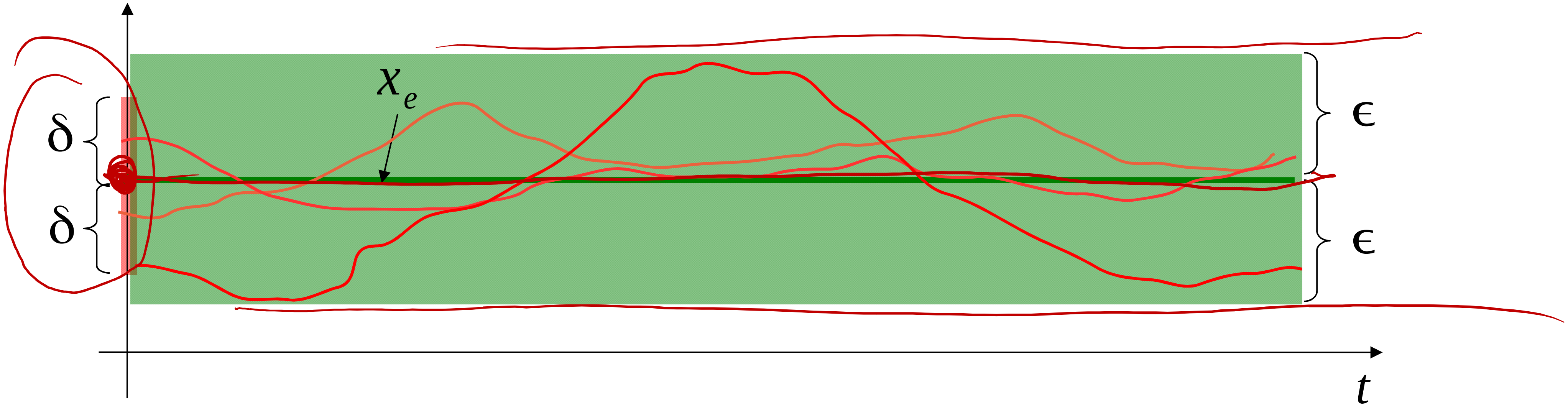
Stability theory (math) - study of the stability of differential equations' and dynamical systems' trajectories under small perturbations of initial conditions

- Lyapunov stability
- asymptotic stability
- orbital stability
- structural stability



Lyapunov stability

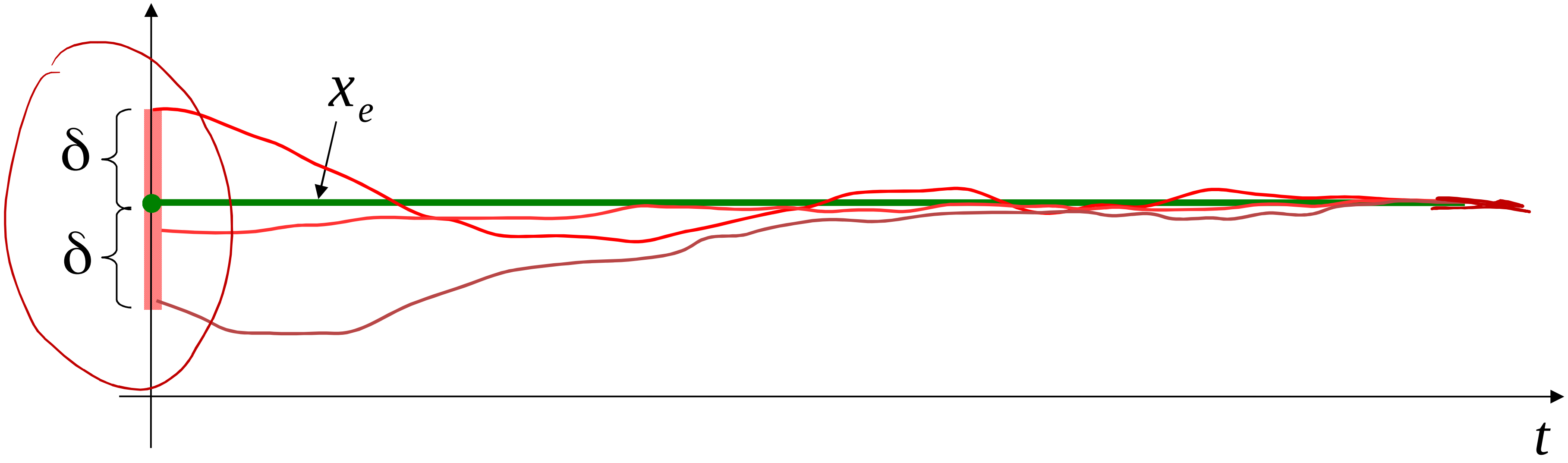
$$\dot{x}(t) = f(x(t)) \qquad f(x_e) = 0,\ x_e - \ equilibrium$$



$$\underset{t \geqslant 0}{\forall} \quad \underset{\epsilon > 0}{\forall} \quad \underset{\delta > 0}{\exists} \quad if\ \| x(0) - x_e \| < \delta,\ then\ \| x(t) - x_e \| < \epsilon$$

Asymptotic stability

$\dot{x}(t)=f(x(t))$ $f(x_e)=0,\ x_e-\ equilibrium$

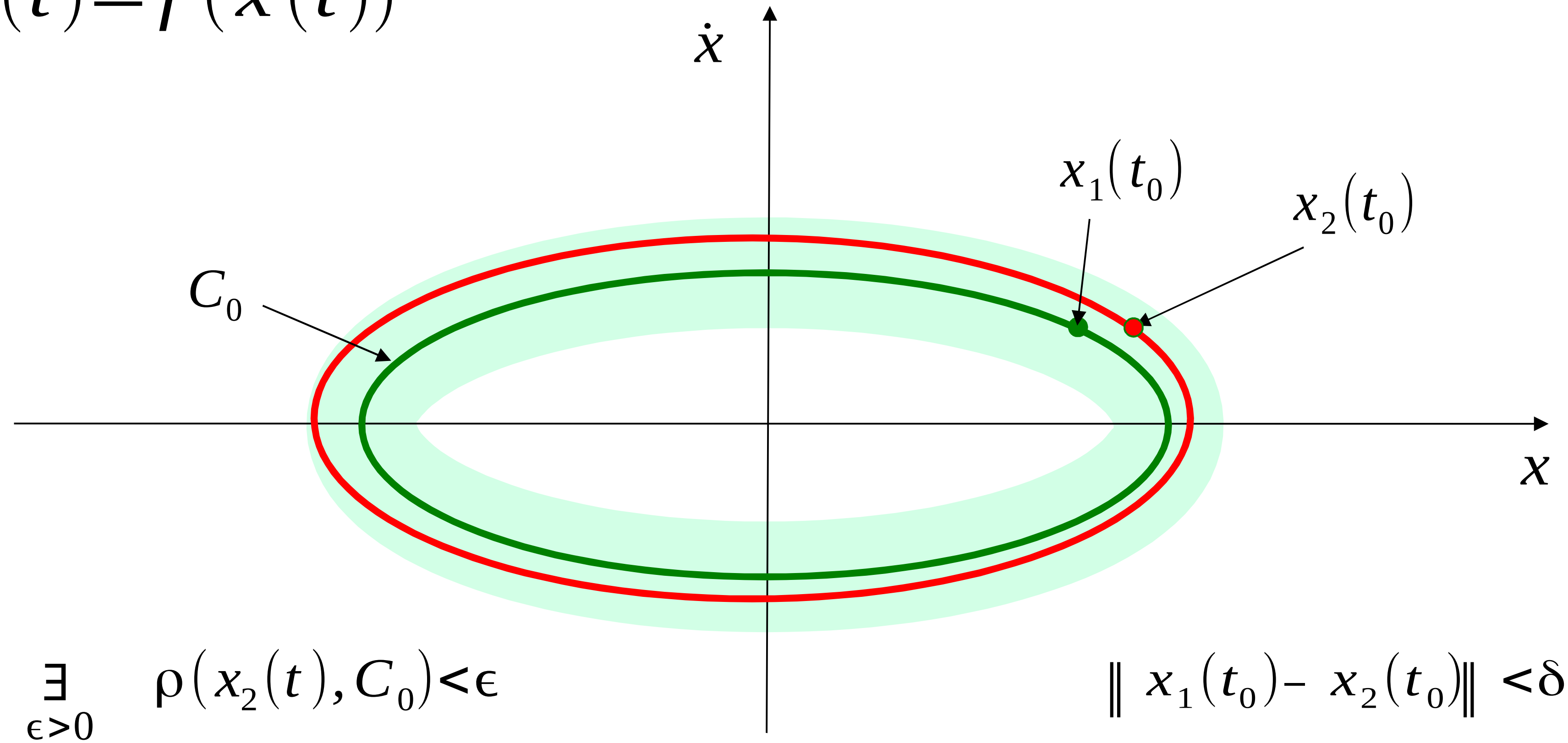


$$\underset{t\geq 0}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\quad if\ \|x(0)-x_e\|<\delta,\ then\ \lim_{t\to\infty}\|x(t)-x_e\|=0$$

Orbital stability

phase plane

$$\dot{x}(t) = f(x(t))$$



$$\forall_{t \geqslant 0} \; \exists_{\epsilon > 0} \quad \rho(x_2(t), C_0) < \epsilon$$

$$\| x_1(t_0) - x_2(t_0) \| < \delta$$

Structural stability

$$\dot{x}(t)=f(x(t)) \longrightarrow \dot{x}(t)=f(x(t))+\Delta f(x(t))$$

$\downarrow$ stable $\longrightarrow$ $\downarrow$ stable?

BIBO stability

Bounded Input, Bounded Output stability
(in signal processing and control theory)

a LTI SISO system is called BIBO stable if its output will stay bounded for any bounded input.

$$x(t) \text{ - input}$$

$$y(t) \text{ - output}$$

$$\underset{0<A<\infty}{\exists} \quad \underset{0<B<\infty}{\exists} \quad \underset{t \geqslant 0}{\forall} \qquad \text{if } |x(t)| \leqslant A \text{, then } |y(t)| \leqslant B$$

STABILITY CRITERIA

General stability criterion

Hurwitz criterion

Nyquist stability criterion

General stability criterion

$$H(s) = \frac{1}{s - p_1}$$

$p_1 \in \mathbb{C}$ $\quad p_1 = a_1 + j b_1$

STEP RESPONSE INPUT: $u(t) = u_0 1(t)$

OUTPUT $Y(s) = H(s) \cdot U(s) = \frac{1}{s-p_1} \cdot u_0 \frac{1}{s} = \frac{u_0}{s(s-p_1)}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{u_0}{s(s-p_1)} \right\} = \frac{u_0}{p_1}\left(1 - e^{p_1 t}\right) =$$

$$= \frac{u_0}{p_1}\left(1 - e^{(a_1 + j b_1)t}\right)$$

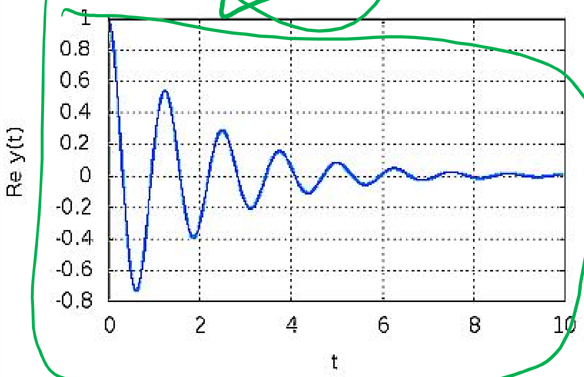
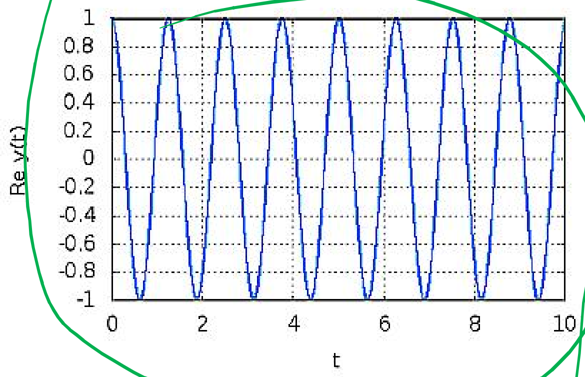
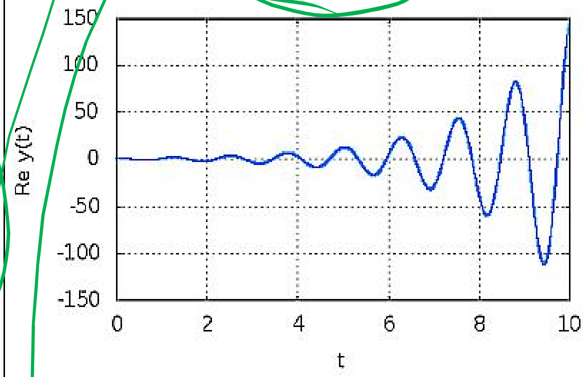
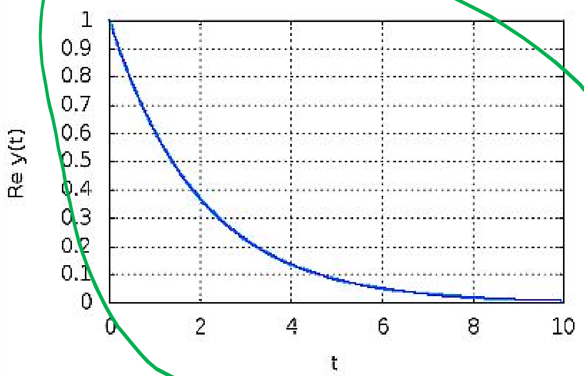
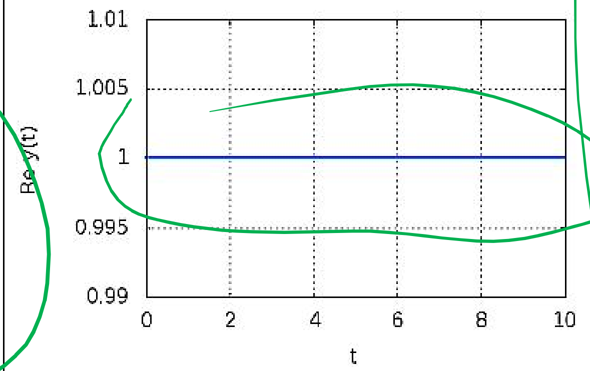
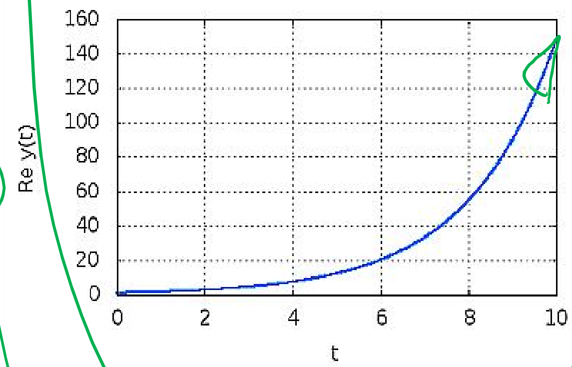
$$e^{a_1 t} e^{j b_1 t}$$

$$e^{a_1 t}(\cos b_1 t + j \sin b_1 t)$$

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s-p_1}$$

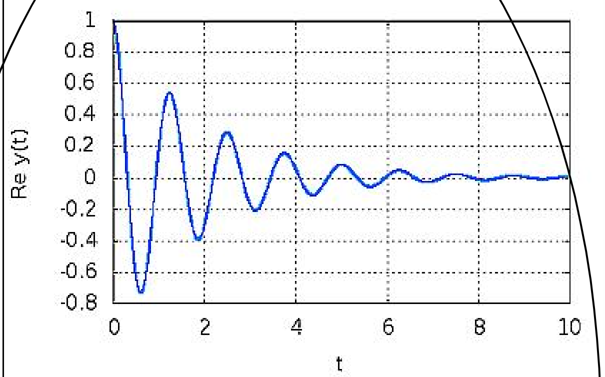
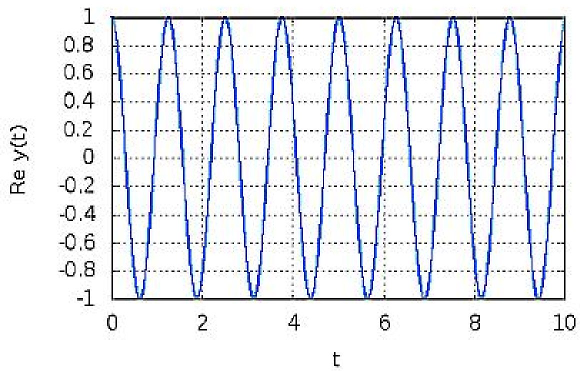
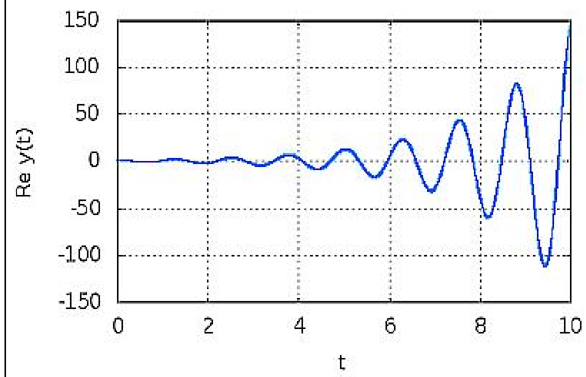
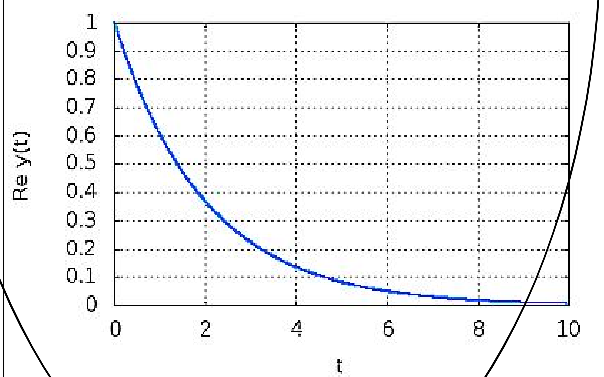
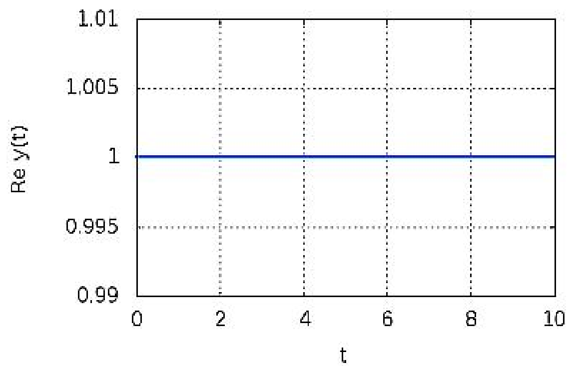
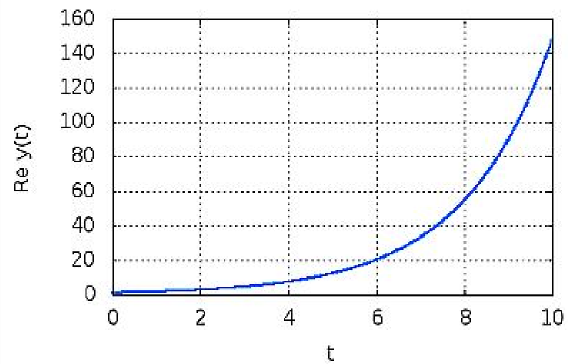
$$\operatorname{Re} y(t)=e^{a_1 t}\cos b_1 t$$

$$\operatorname{Re}(p_1)=a_1$$
$$\operatorname{Im}(p_1)=b_1$$

	$a_1<0$	$a_1=0$	$a_1>0$
$b_1>0$ $b_1<0$			
$b_1=0$			

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s-p_1} \qquad \operatorname{Re} y(t)=e^{a_1 t}\cos b_1 t$$

$$\operatorname{Re}(p_1)=a_1$$
$$\operatorname{Im}(p_1)=b_1$$

	$a_1<0$	$a_1=0$	$a_1>0$
$b_1>0$ $b_1<0$			
$b_1=0$			

H(s) asymptotically stable, if $\operatorname{Re}(p_1)<0$

General stability criterion (definition)

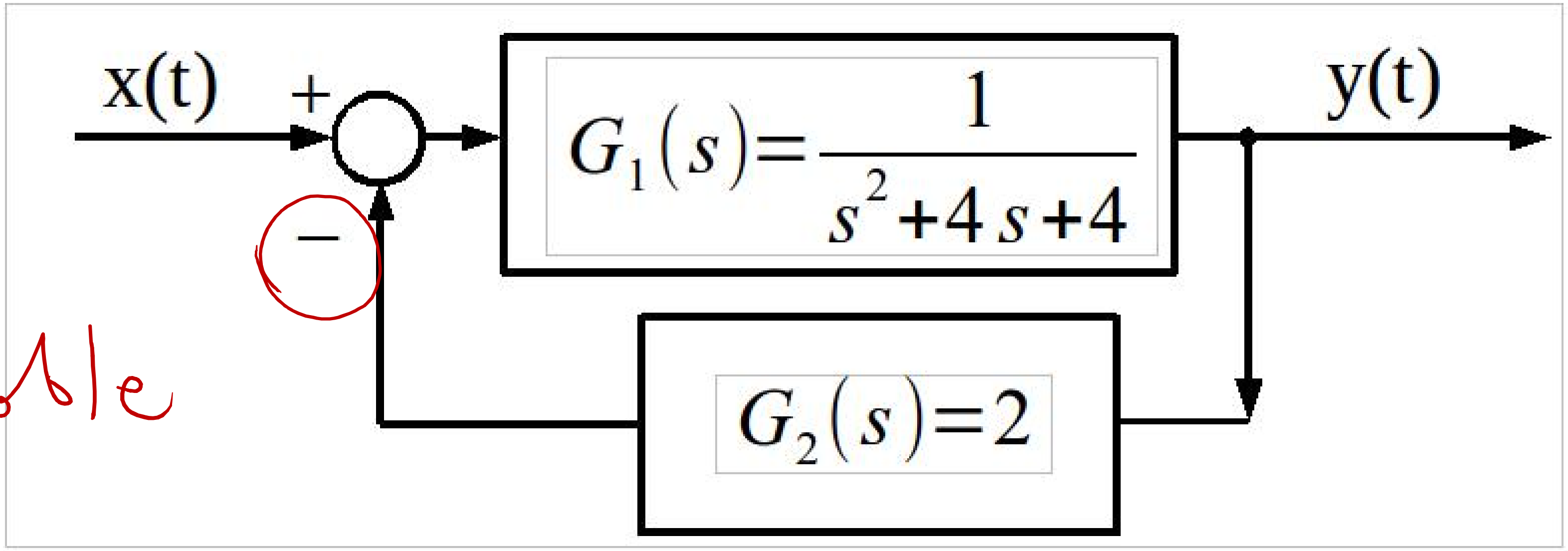
LTI SISO system is stable, if real parts of all transfer function's poles are less than zero.

$$H(s) = \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)}$$

$$H(s) \text{ is stable if: } \operatorname{Re} p_1 < 0 \;\wedge\; \operatorname{Re} p_2 < 0 \;\wedge \dots \wedge\; \operatorname{Re} p_n < 0$$

Example 1

Check stability of the presented system using the general stability criterion.



$Re(p_1) < 0$

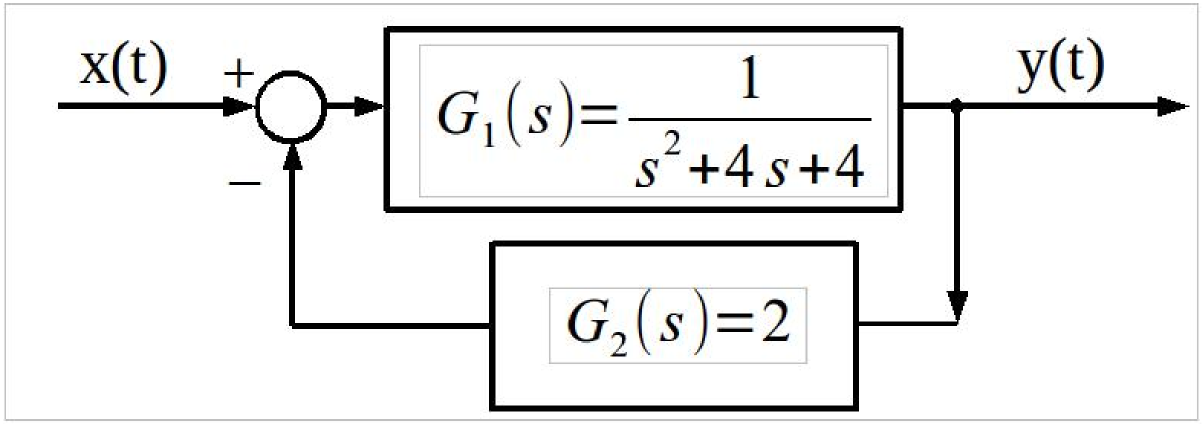
$Re(p_2) < 0$

→ system is stable

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G_1}{1 + G_1 \cdot G_2} = \frac{\frac{1}{s^2+4s+4}}{1 + \frac{2}{s^2+4s+4}} = \frac{1}{s^2+4s+6}$$

Example 1

Check stability of the presented system using the general stability criterion



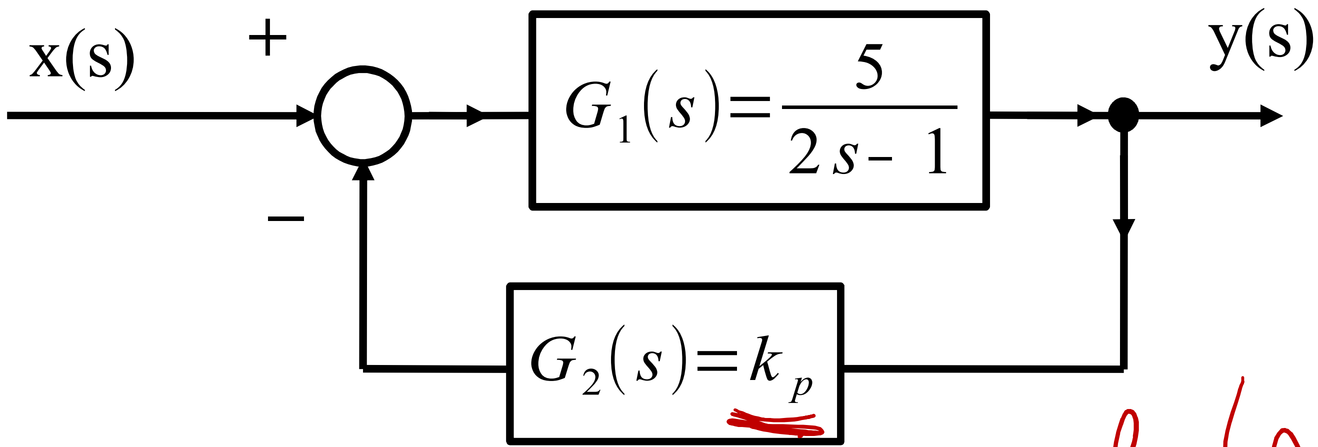
$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{1}{s^2+4s+6}=\frac{1}{(s-s_1)(s-s_2)}$$

$$s_1=-2-2\sqrt{2}j,\ \ s_2=-2+\sqrt{2}j$$

$$\Re(s_1)<0\ \wedge\ \Re(s_2)<0\ \Rightarrow$$ system is stable from general stability criterion

Example 2

Choose values of K_P to obtain system stability using the general stability criterion



$$H(s)=\frac{G_1}{1+G_1G_2}=\frac{\frac{5}{2s-1}}{1+\frac{5k_p}{2s-1}}=\frac{5}{2s-1+5k_p}=\frac{\frac{5}{2}}{s-\frac{1-5k_p}{2}}$$

$p_1 = \frac{1-5k_p}{2}$

$$Re(p_1)=\frac{1-5k_p}{2}$$

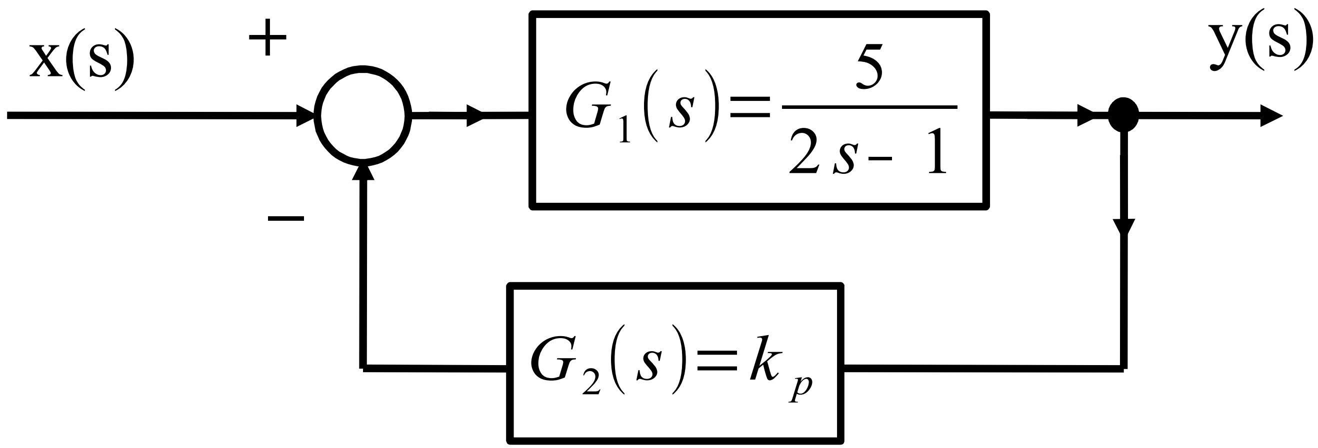
$$\frac{1-5k_p}{2}<0$$

$$-5k_p<-1$$

$$k_p>\frac{1}{5}$$

Example 2

Choose values of K_P to obtain system stability using the general stability criterion

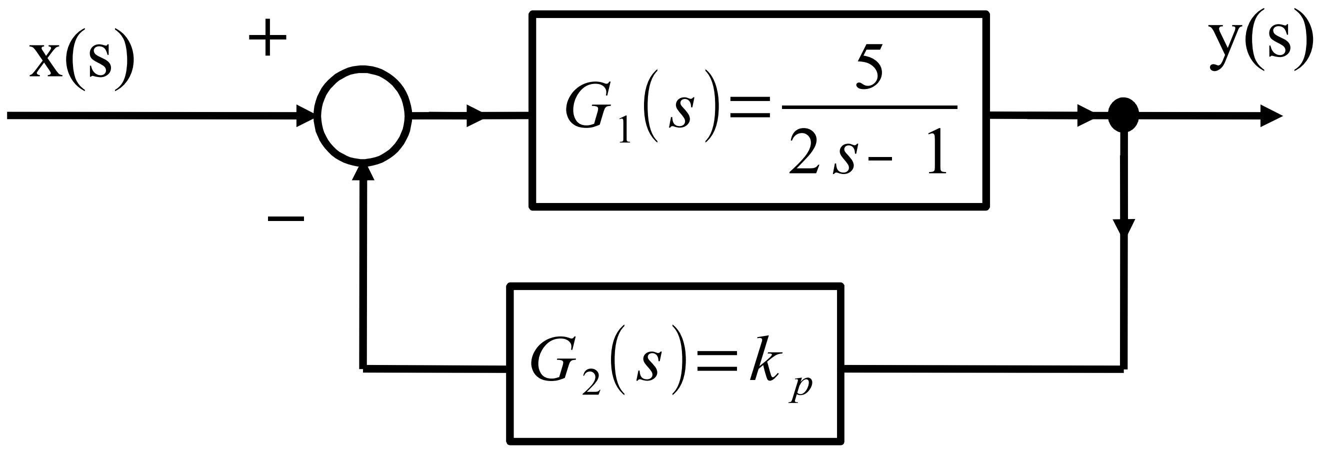


$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

Example 2

Choose values of K_P to obtain system stability using the general stability criterion



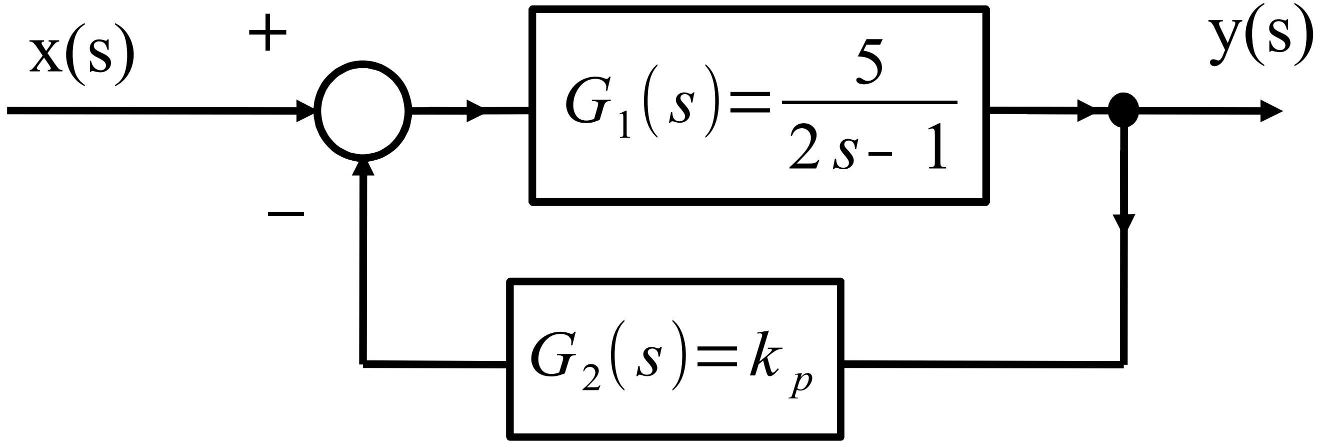
$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

System is stable, if $\Re(p_1)<0$

Example 2

Choose values of K_P to obtain system stability using the general stability criterion



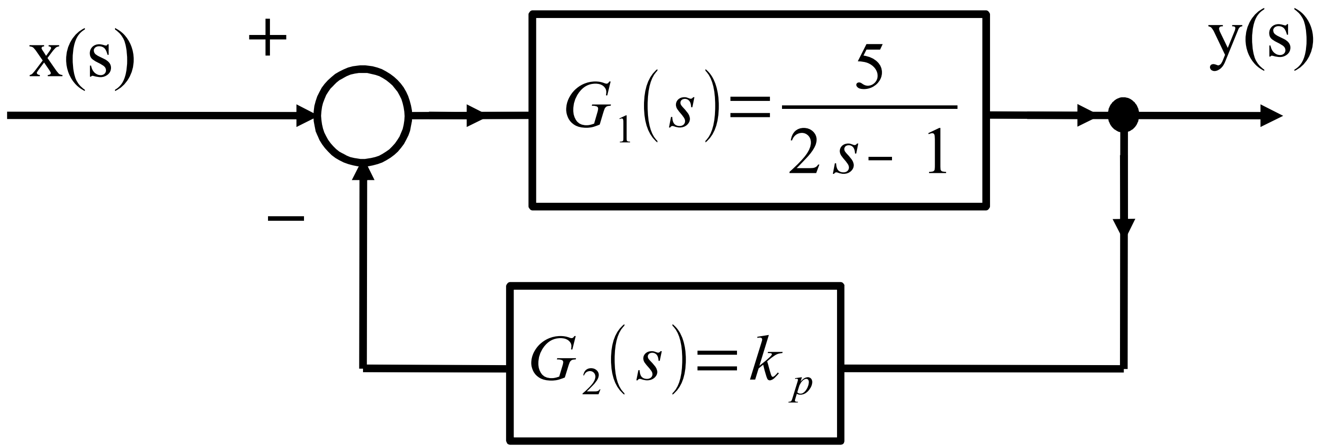
$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

System is stable, if $\Re\left(p_1\right)<0 \Rightarrow k_p>\frac{1}{5}$

Example 2

Choose values of K_P to obtain system stability using the general stability criterion

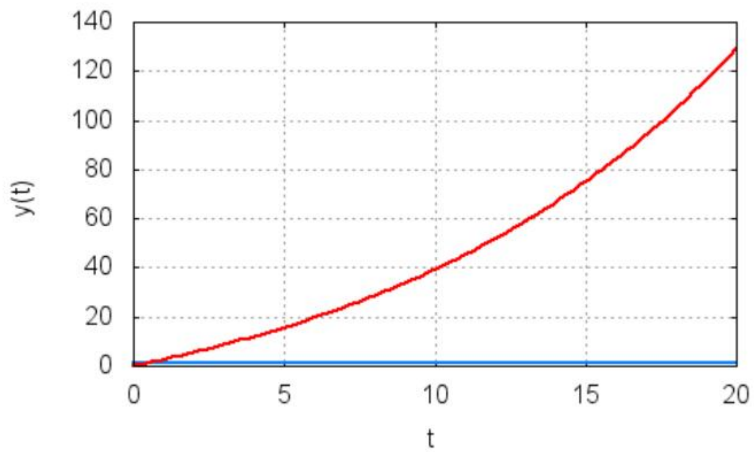


$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

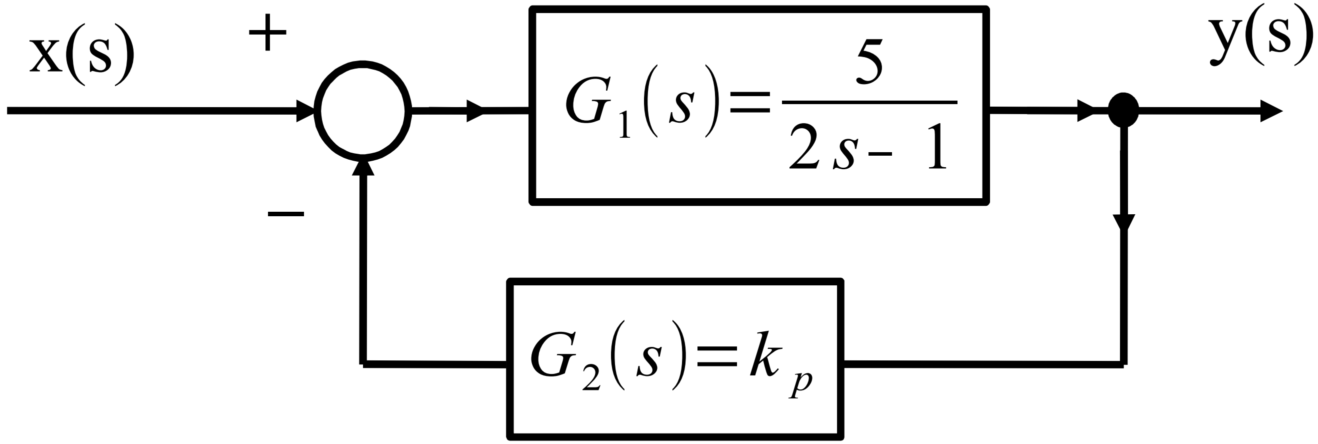
System is stable, if $\Re(p_1)<0 \quad \Rightarrow \quad k_p>\frac{1}{5}$

$k_p=\frac{1}{6}$ (unstable)



Example 2

Choose values of K_P to obtain system stability using the general stability criterion

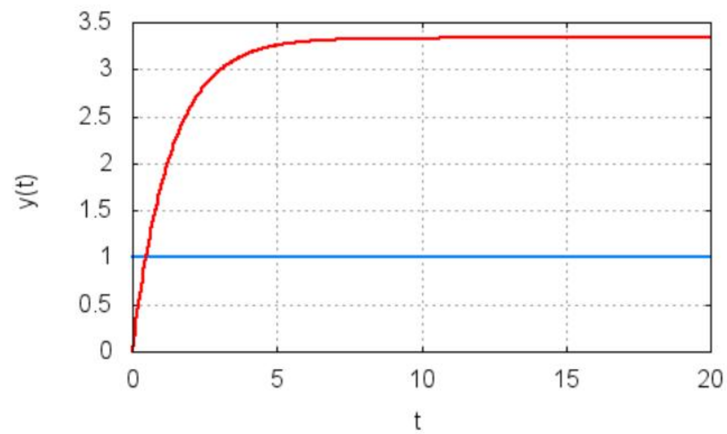


$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

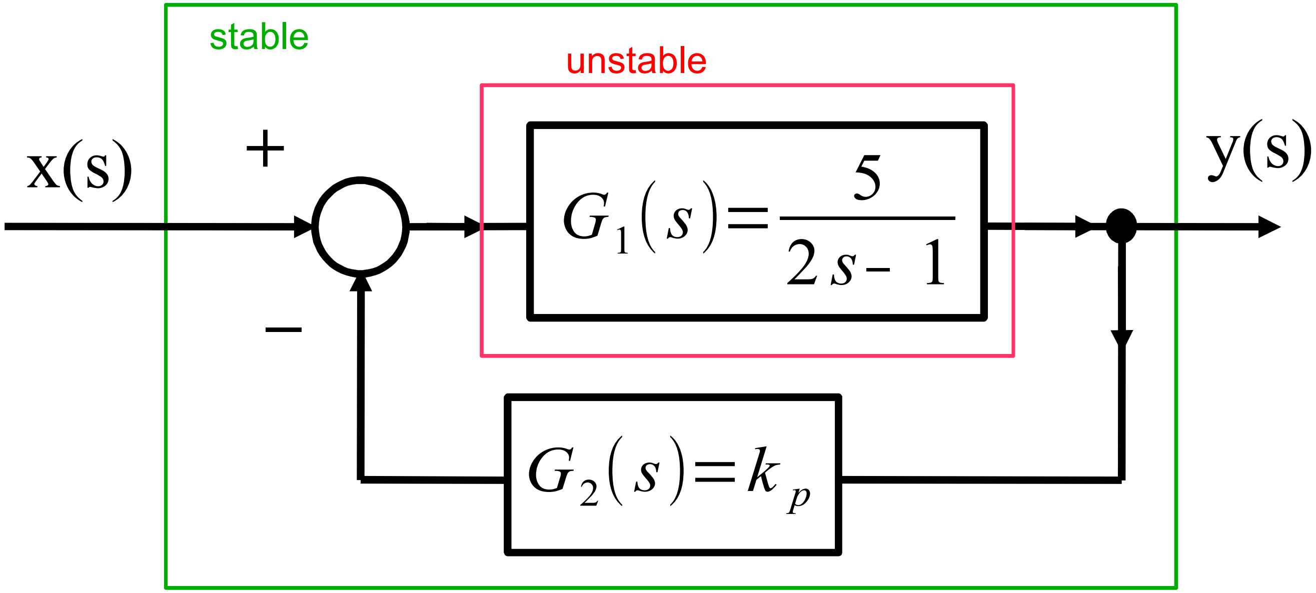
System is stable, if $\Re(p_1)<0 \quad \Rightarrow \quad k_p>\frac{1}{5}$

$k_p=\frac{1}{2}$ (stable)



Example 2

Choose values of K_P to obtain system stability using the general stability criterion

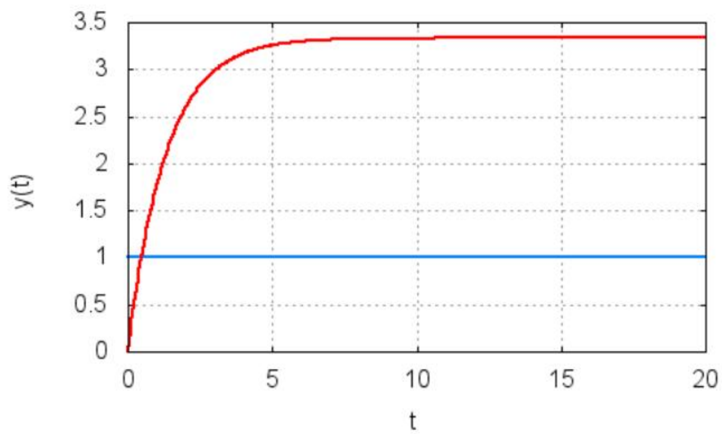


$$G(s)=\frac{Y(s)}{X(s)}=\frac{G_1}{1+G_1G_2}=\frac{5}{2}\frac{1}{s-\left(\frac{1}{2}-\frac{5}{2}k_p\right)}$$

$$p_1=\left(\frac{1}{2}-\frac{5}{2}k_p\right)$$

System is stable, if $\Re(p_1)<0 \quad \Rightarrow \quad k_p>\frac{1}{5}$

$k_p=\frac{1}{2}$ (stable)



List of important exam topics

- Mechanism mobility calculation
- Procedure analytical method
- Equation of machine motion
- Nonuniformity and flyweel
- Transfer function of LTI SISO
- Step response & Bode Plot of a given system
- Block diagram algebra
- PID controller & tuning
- General stability criterion
- Hurwitz criterion
- particular Nyquist criterion