Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

# *Theory of Machines and Automatic Control*

Winter 2019/2020

**Lecturer: Sebastian Korczak, PhD Eng.**

# Lecture 10

Classification of basic automatic systems with examples.

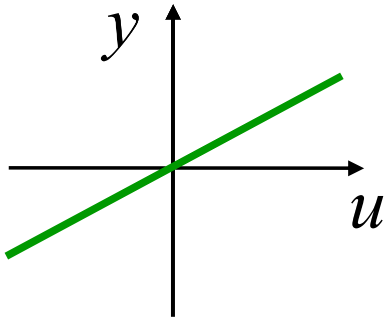
# Classification of basic automatic systems

| Element name | Transfer function |
|---|---|
| proportional | $k$ |
| first order (inertial) | $\frac{k}{Ts+1}$ |
| integrator | $\frac{k}{s}$ |
| differentiator | $ks$ |
| differentiator with inertia | $\frac{ks}{Ts+1}$ |
| delay | $e^{-\tau s}$ |
| second order (oscillator) | $\frac{k}{T_1^2 s^2 + T_2 s + 1}$ |

# Proportional element

1. Element equation: $y(t)=ku(t)$ $k\in\mathbb{R}$ $u(t)$ - input, $y(t)$ - output

2. Static characteristic (steady state): $y=ku$ for $\frac{dy}{dt}=0 \wedge \frac{du}{dt}=0$

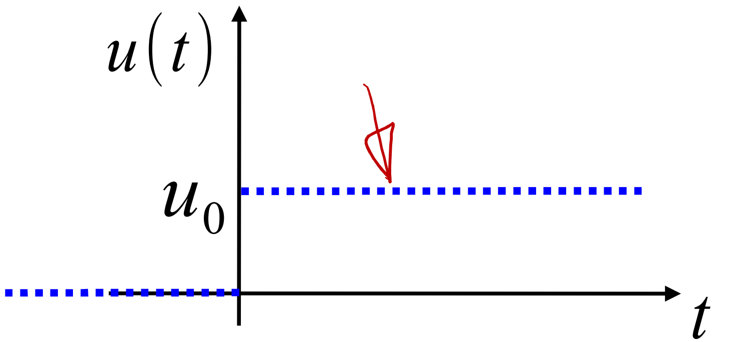


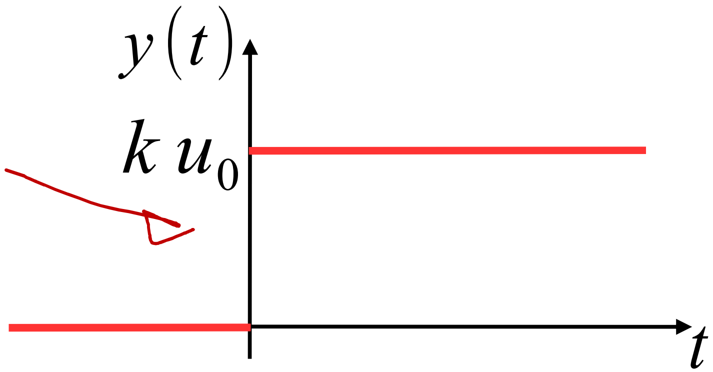

✓

for $k>0$

3. Transfer function: $\boxed{H(s)=k}$

4. Step response: $y(t)=k\,u_0\,1(t)$ for $u(t)=u_0\,1(t)$

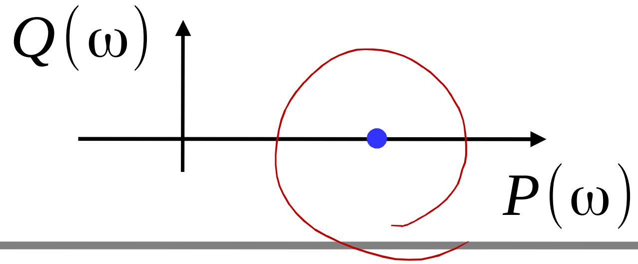


# Proportional element

5. Frequency response: $\boxed{H(j\omega)=k}$ $\qquad P(\omega)=k,\ Q(\omega)=0$

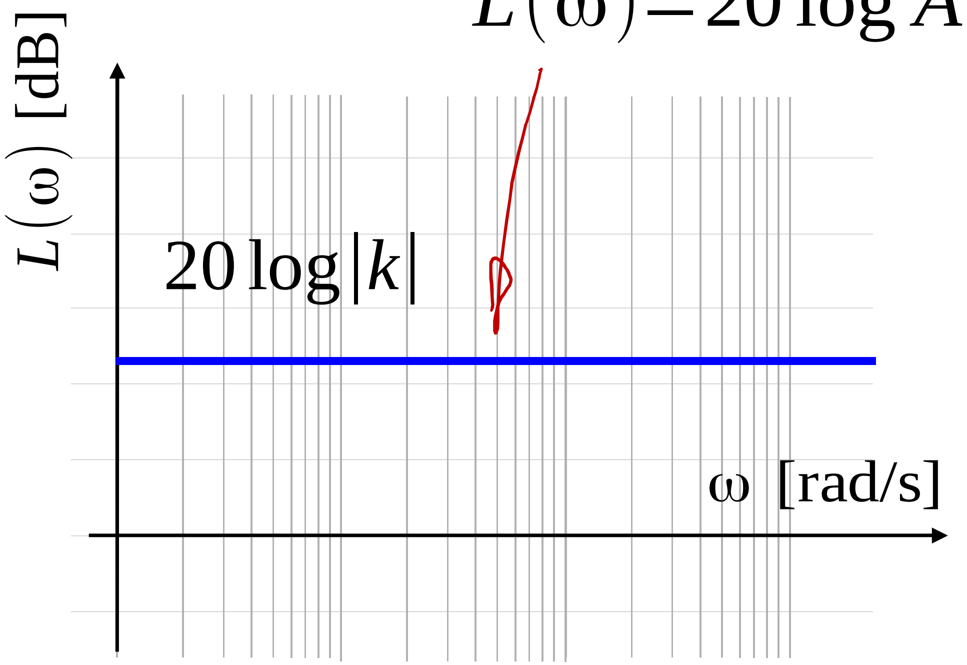
6. Nyquist plot:

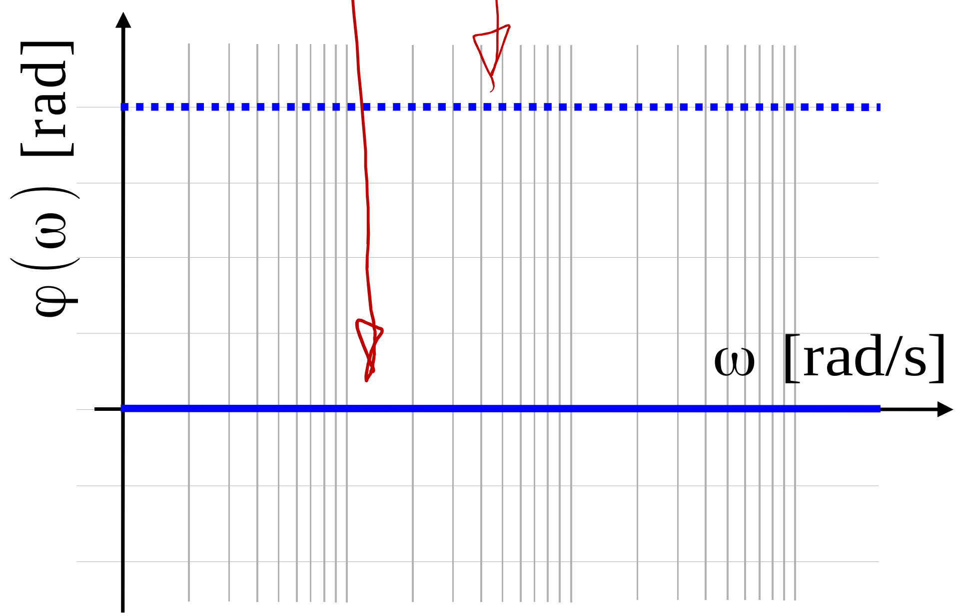



for $k>0$

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k|$ $\qquad \varphi(\omega)=\arctan\frac{Q}{P}=\begin{cases}0, \text{ dla } k\geq 0\\ \pi, \text{ dla } k<0\end{cases}$

$$L(\omega)=20\log A(\omega)$$

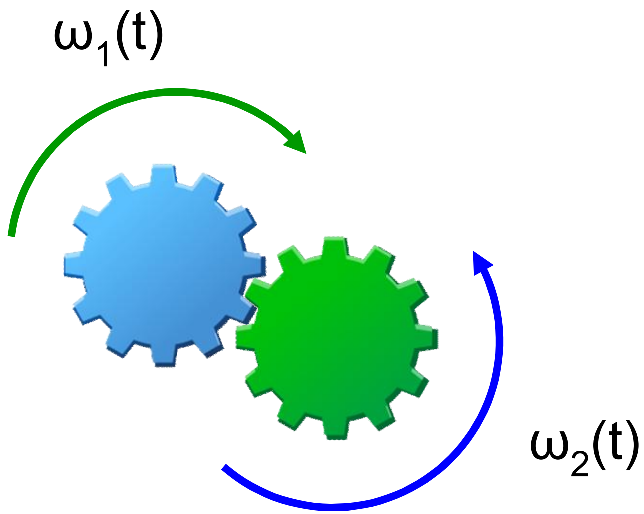


# Proportional element

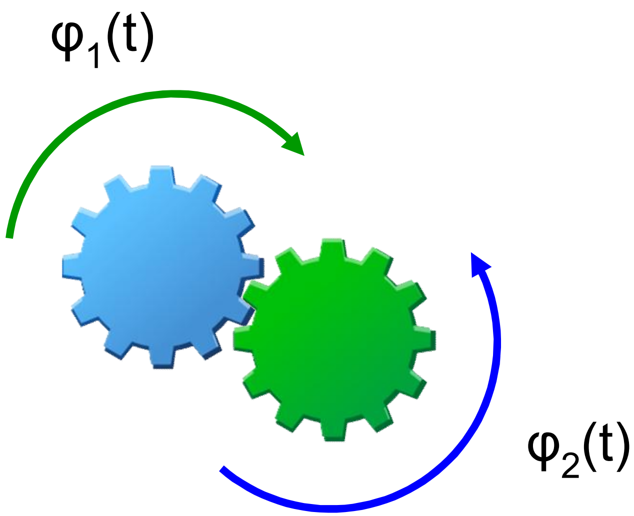
## Examples

1




GEARBOX:
input – angular velocity $\omega_1(t)$
output – angular velocity $\omega_2(t)$

2

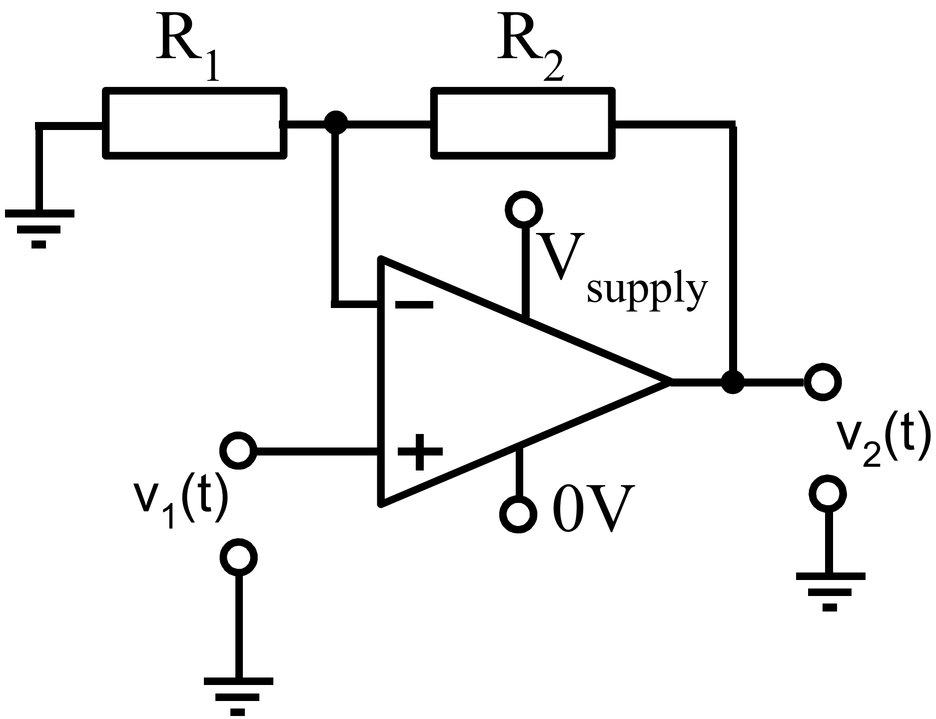



GEARBOX:
input – rotation angle $\varphi_1(t)$
output – rotation angle $\varphi_2(t)$

# Proportional element

## Examples

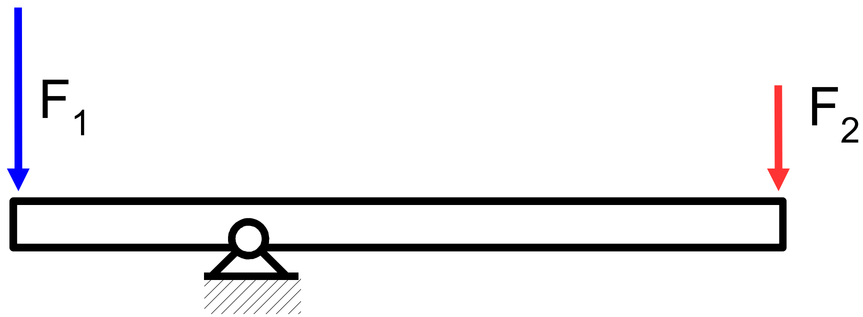
3




OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = v_1(t)\left(1 + \frac{R_2}{R_1}\right)$$

4

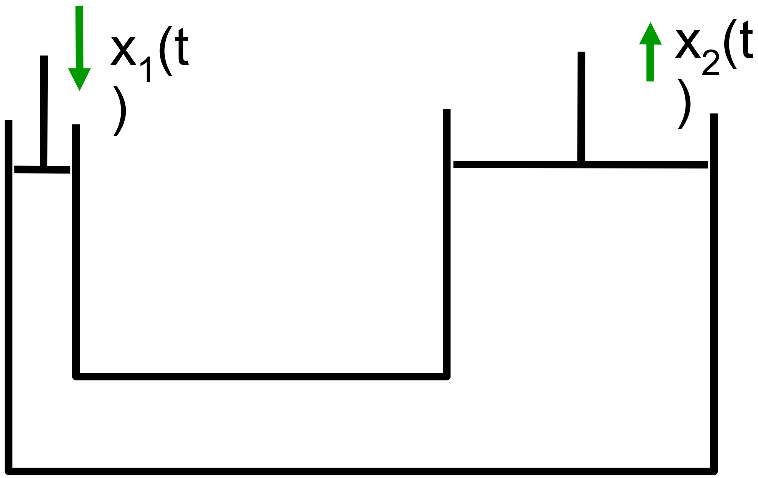



BEAM in steady state:
input – force $F_1$
output – force $F_2$

# Proportional element

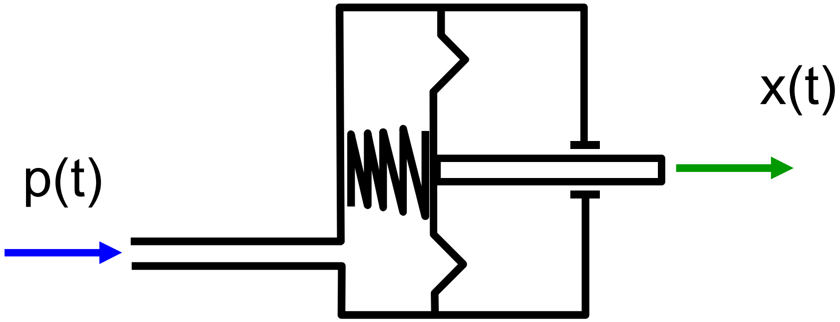
## Examples

5




HYDRAULIC LEVER:
input – displacement $x_1(t)$
output – displacement $x_2(t)$

6




PRESSURE ACTUATOR:
input – pressure $p_1(t)$
output – displacement $x(t)$

# First-order inertial element

1. Element equation: $T\frac{dy(t)}{dt}+y(t)=ku(t)$  $k \in \mathbb{R}$  $T \in \mathbb{R}_+$

$u(t)$ - input

$y(t)$ - output

# First-order inertial element

1. Element equation: $T\cancel{\frac{dy(t)}{dt}}^{=0}+y(t)=ku(t)$

$u(t)$ - input
$y(t)$ - output

---

2. Static characteristic (steady state):

$u=\text{const.}$

$\frac{dy}{dt}=0$

$y=k\cdot u$

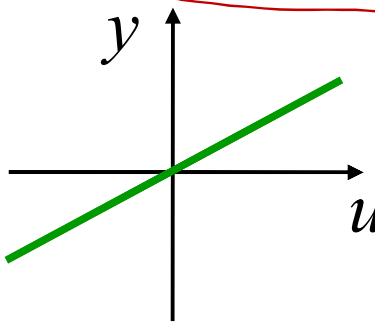
# First-order inertial element

1. Element equation: $T\dfrac{dy(t)}{dt}+y(t)=ku(t)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=ku$

for $\dfrac{dy}{dt}=0 \wedge \dfrac{du}{dt}=0$




for $k>0$

$\mathcal{L}$

3. Transfer function: $T s Y(s) + Y(s) = k \cdot U(s)$

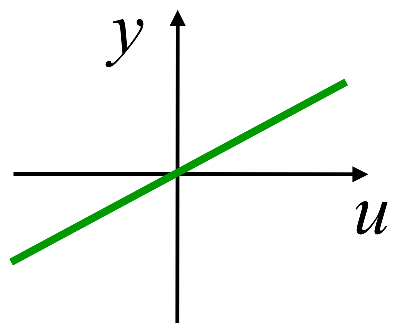
$$H(s)=\frac{Y(s)}{U(s)}=\frac{k}{Ts+1}$$

# First-order inertial element

1. Element equation: $T\frac{dy(t)}{dt}+y(t)=ku(t)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=ku$ for $\frac{dy}{dt}=0 \wedge \frac{du}{dt}=0$




for $k>0$

3. Transfer function: $H(s)=\frac{k}{Ts+1}$

# First-order inertial element

4. Step response: $u(t) = u_0 \mathbb{1}(t)$ $\mathcal{L}\{u(t)\} = u_0 \frac{1}{s}$

$$Y(s) = U(s)H(s) = u_0 \frac{1}{s} \frac{k}{Ts+1} = \frac{ku_0}{s(Ts+1)} =$$

$$= ku_0 \boxed{\frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = ku_0\left(1 - e^{-\frac{t}{T}}\right)$$

$$\boxed{\frac{ku_0}{s(Ts+1)} = \frac{ku_0}{Ts\left(s+\frac{1}{T}\right)} = ku_0 \frac{\frac{1}{T}}{s\left(s+\frac{1}{T}\right)}}$$

# First-order inertial element

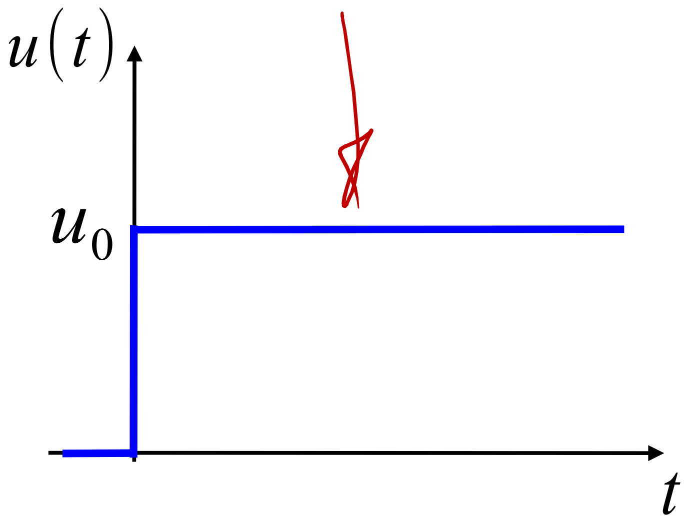
4. Step response:

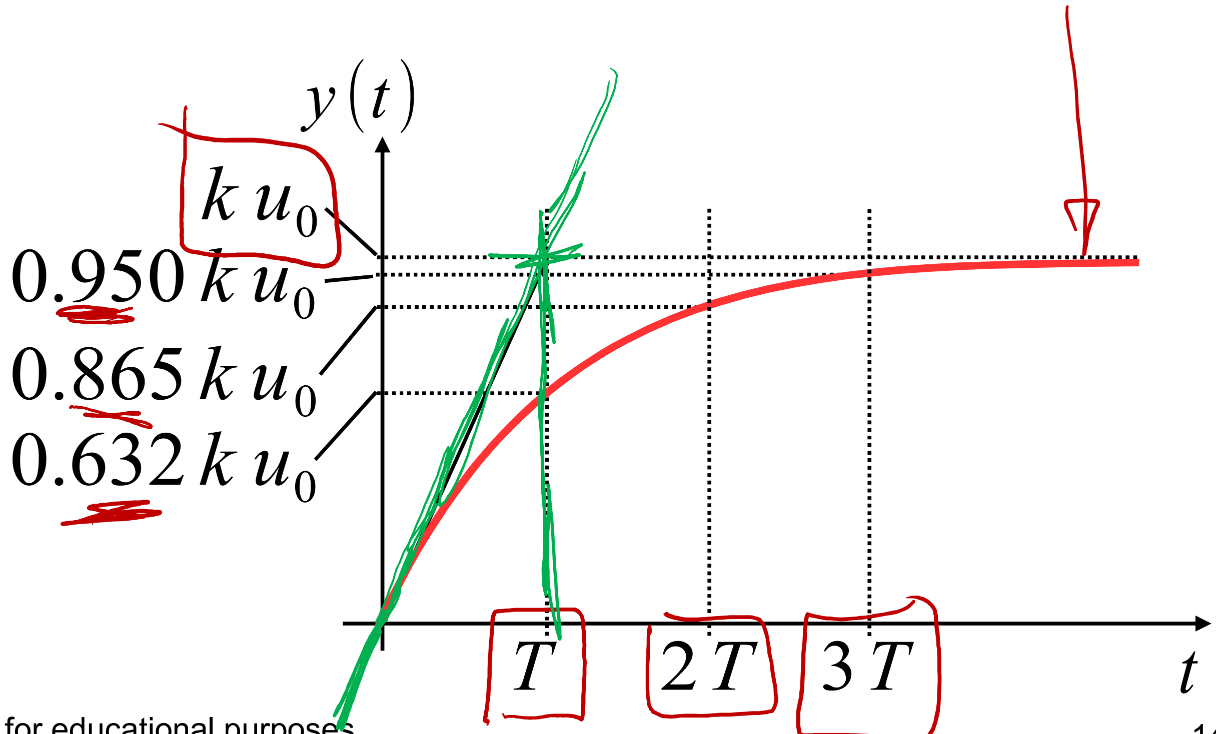
input: $u(t) = u_0 1(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Laplace of output: $Y(s) = H(s)U(s) = \frac{k u_0}{s(Ts+1)}$

output: $y(t) = L^{-1}\{Y(s)\} = k u_0 (1 - e^{-t/T})$



# First-order inertial element

5. Frequency response: $H(j\omega) = \dfrac{k}{Tj\omega + 1} = P(\omega) + jQ(\omega)$

$$\frac{k}{1+jT\omega} \cdot \frac{1-jT\omega}{1-jT\omega} = \frac{k - jkT\omega}{1^2 - j^2\omega^2T^2}$$

$$+$$

$$P(\omega) = \frac{k}{1+\omega^2T^2} \quad ; \quad Q(\omega) = \frac{-kT\omega}{1+T^2\omega^2}$$

# First-order inertial element

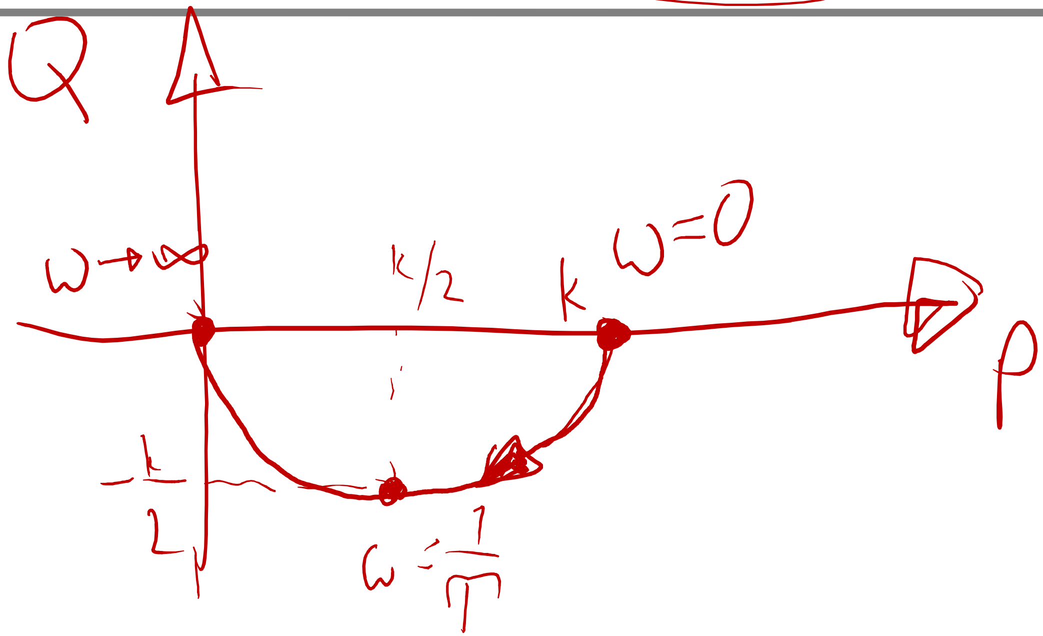
5. Frequency response: $H(j\omega)=\dfrac{k}{Tj\omega+1}$

$$P(\omega)=\frac{k}{T^2\omega^2+1},\quad Q(\omega)=\frac{-kT\omega}{T^2\omega^2+1}$$

6. Nyquist plot:

$P\left(\omega=\frac{1}{T}\right)=\frac{k}{2}$

$Q\left(\omega=\frac{1}{T}\right)=-\frac{k}{2}$




$k>0$

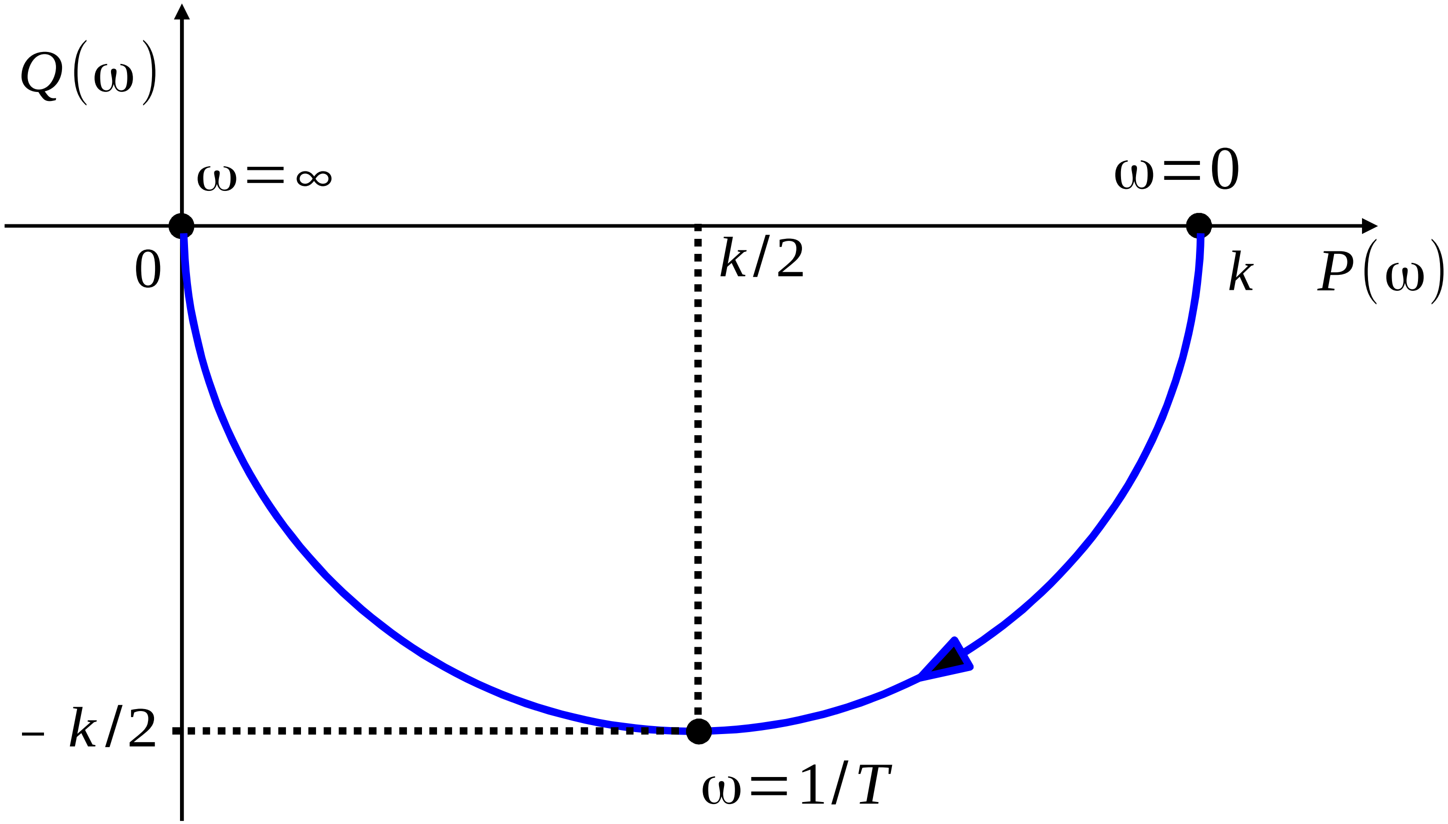
# First-order inertial element

5. Frequency response: $H(j\omega) = \dfrac{k}{Tj\omega + 1}$

$$P(\omega) = \frac{k}{T^2\omega^2 + 1}, \quad Q(\omega) = \frac{-kT\omega}{T^2\omega^2 + 1}$$

6. Nyquist plot:

for $k > 0$



# First-order inertial element

7. Bode plot:

$$P(\omega)=\frac{k}{T^2\omega^2+1},\quad Q(\omega)=\frac{-kT\omega}{T^2\omega^2+1}$$

$$A(\omega)=\sqrt{P^2+Q^2}=\sqrt{\frac{k^2+k^2T^2\omega^2}{(T^2\omega^2+1)^2}}=\sqrt{\frac{k^2\cancel{(1+T^2\omega^2)}}{(T^2\omega^2+1)^{\cancel{2}}}}$$

$$=\frac{|k|}{\sqrt{T^2\omega^2+1}}$$

$$L(\omega)[dB]=20\log_{10}A(\omega)=20\log|k|-20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega)[rad]=\operatorname{atan}\frac{Q}{P}=\operatorname{atan}(-T\omega)=-\operatorname{atan}(T\omega)$$

# First-order inertial element

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k| / \sqrt{T^2\omega^2 + 1}$

$$L(\omega) = 20\log A(\omega) = 20\log|k| - 20\log\sqrt{T^2\omega^2+1}$$

$\omega = \frac{1}{T}$

$$\varphi(\omega) = \arctan\frac{Q}{P} = \arctan(-T\omega)$$

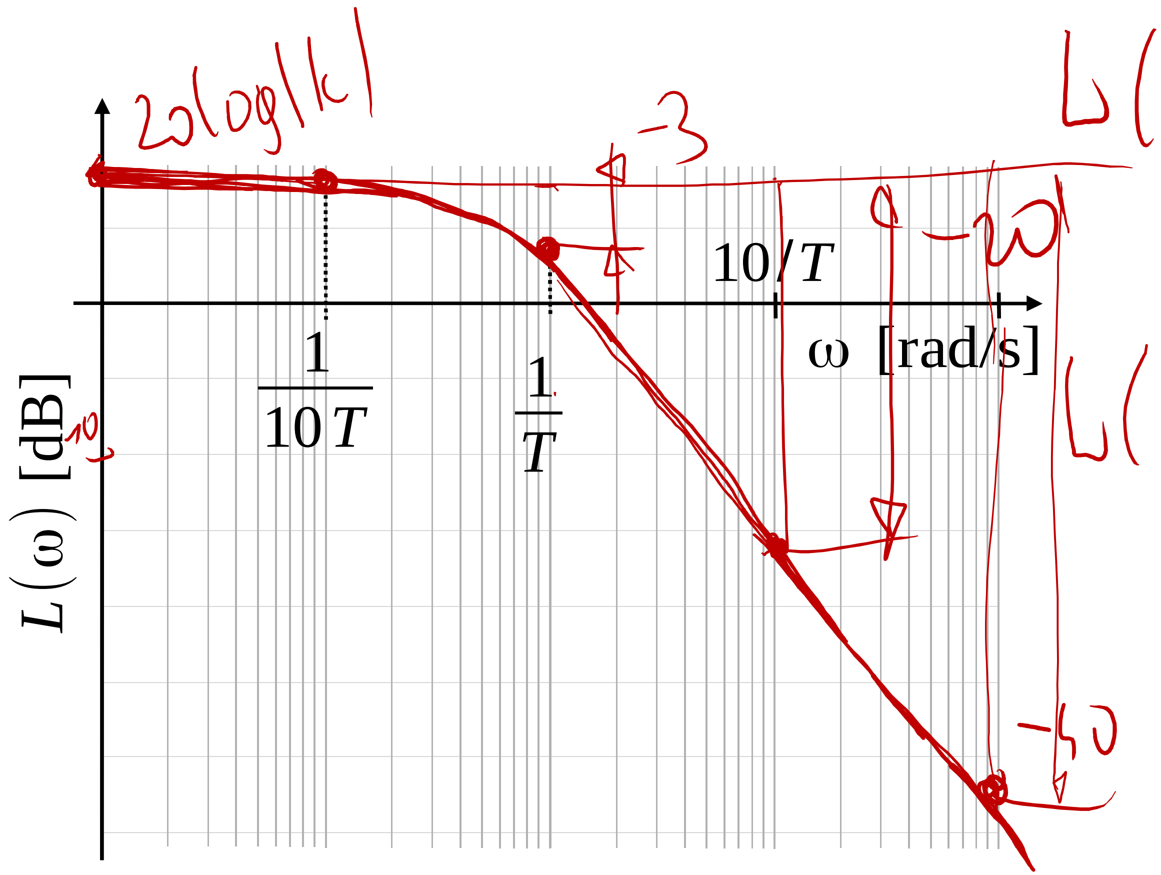
# First-order inertial element

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k|/\sqrt{T^2\omega^2+1}$

$$L(\omega)=20\log A(\omega)=20\log|k|-20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-T\omega)$$

~~for $k>0$~~




$$L\left(\frac{1}{T}\right)=20\log|k|-20\log\sqrt{2}$$
$$=20\log|k|-3$$

$$L\left(\frac{10}{T}\right)=20\log|k|-20\log\sqrt{101}$$
$$=20\log|k|-20$$

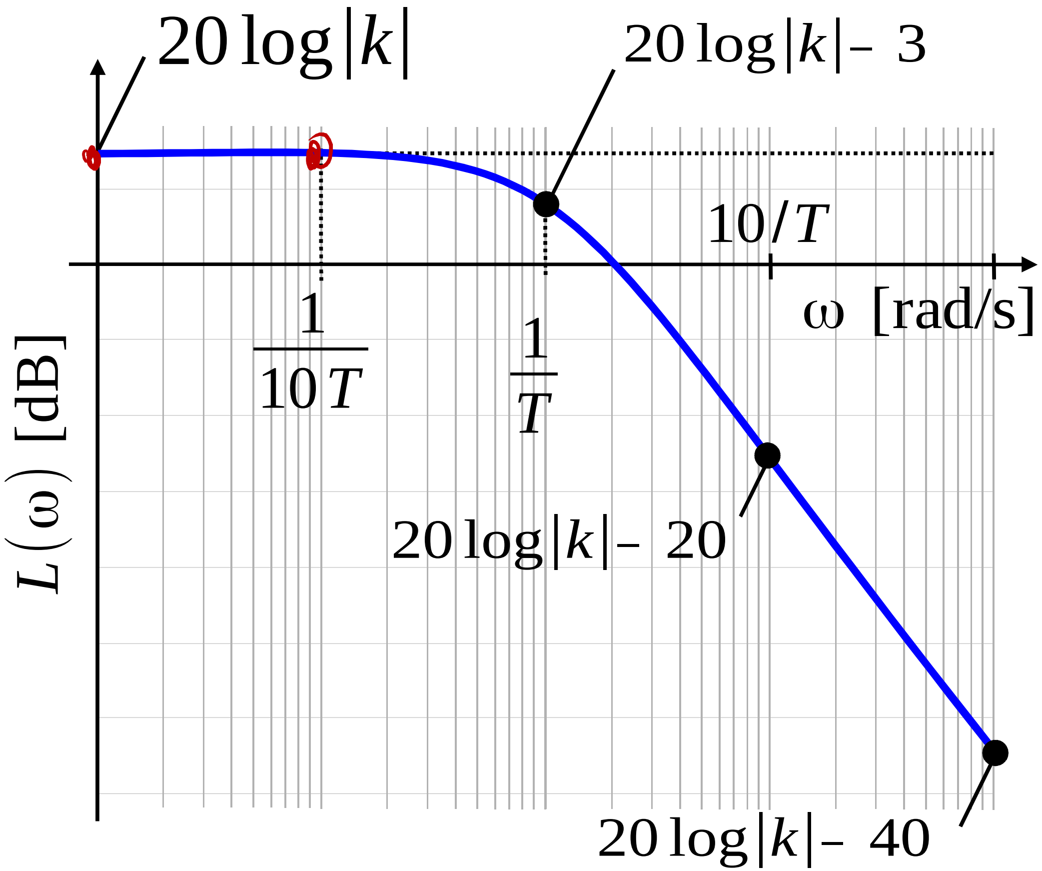
$$L\left(\frac{1}{10T}\right)=20\log|k|-\underbrace{20\log\sqrt{\frac{101}{100}}}_{\approx 0}$$
$$=20\log|k|$$

# First-order inertial element

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k| / \sqrt{T^2\omega^2 + 1}$

$$L(\omega) = 20\log A(\omega) = 20\log|k| - 20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega) = \arctan\frac{Q}{P} = \arctan(-T\omega) \qquad \text{for } k>0$$



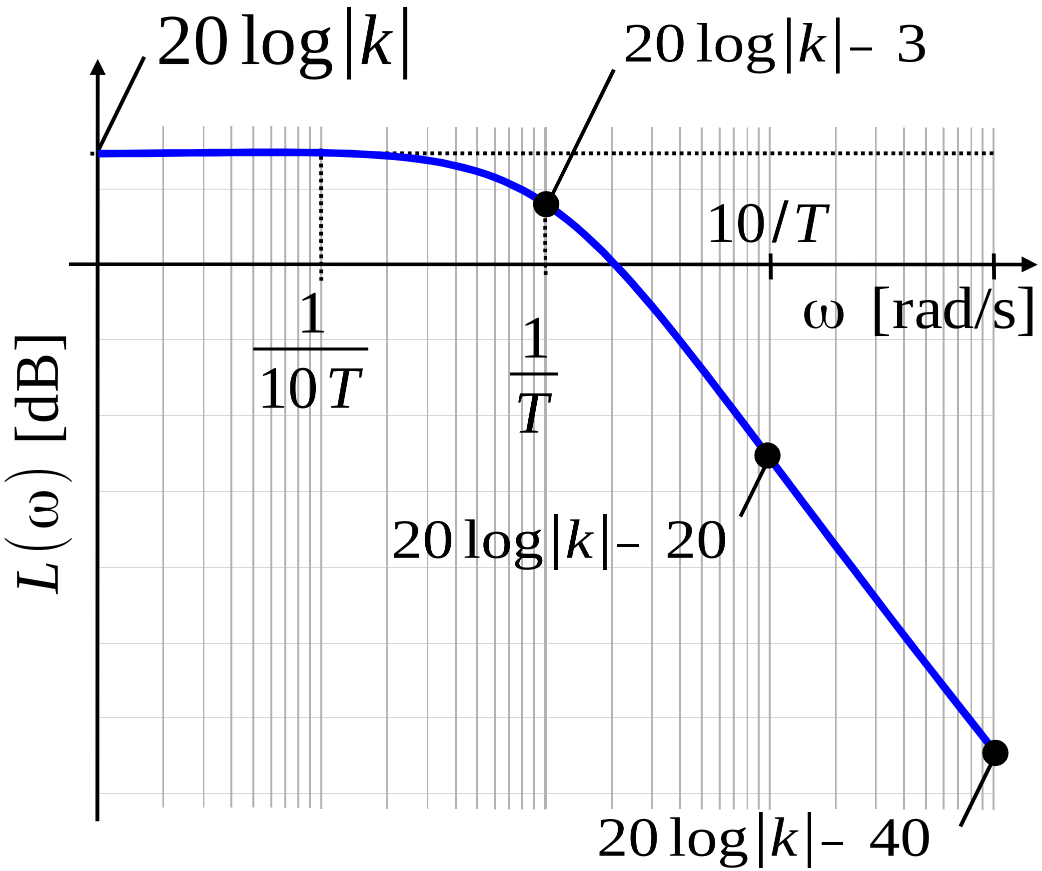
# First-order inertial element

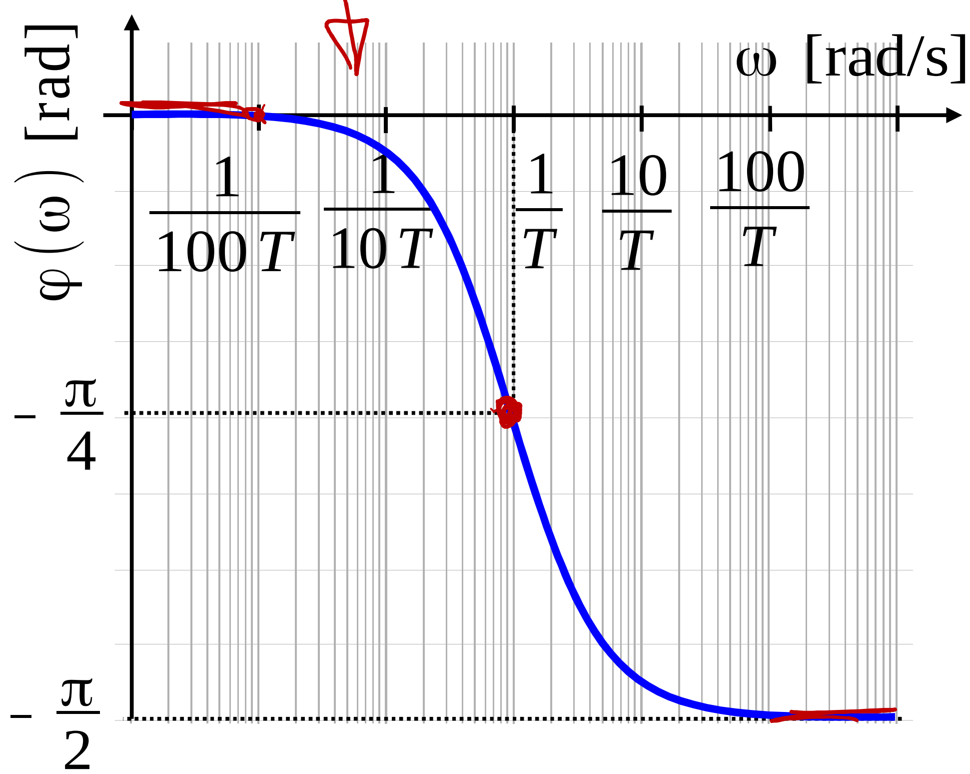
7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k|/\sqrt{T^2\omega^2+1}$

$$L(\omega)=20\log A(\omega)=20\log|k|-20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-T\omega)$$

for $k>0$

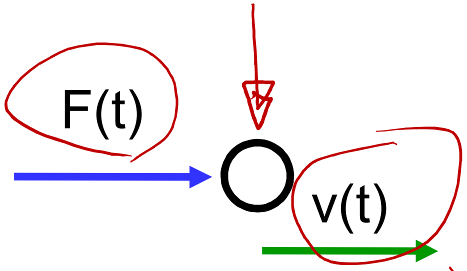


# First-order inertial element

Examples

1




$$m\frac{dv(t)}{dt} + c\,v(t) = F(t)$$

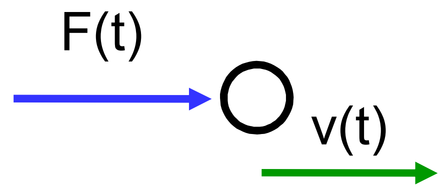
LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
input – force F(t)
output – velocity v(t)

example: car is driving on a flat surface with air resistance proportional To its velocity, described using machine equation of motion, with assumption of constant reduced mass.

# First-order inertial element

## Examples

1




LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
input – force F(t)
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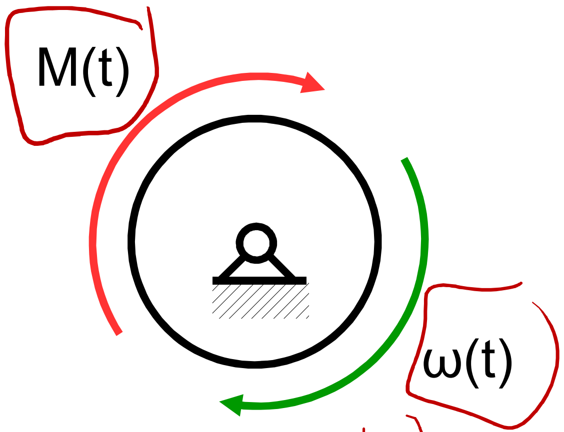
2



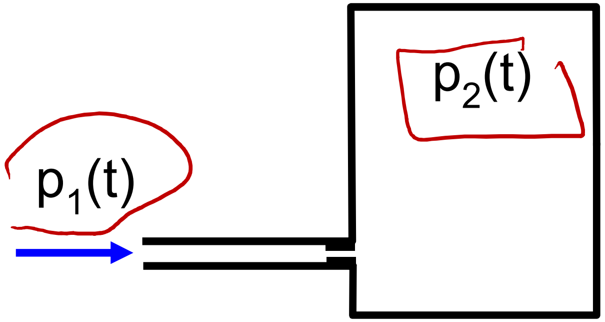

ANGULAR MOTION OF A RIGID BODY WITH LINEAR DAMPING:
input – torque M(t)
output – angular velocity ω(t)

$$I_0 \frac{d\omega}{dt} + c\omega = M(t)$$

# First-order inertial element

## Examples

3

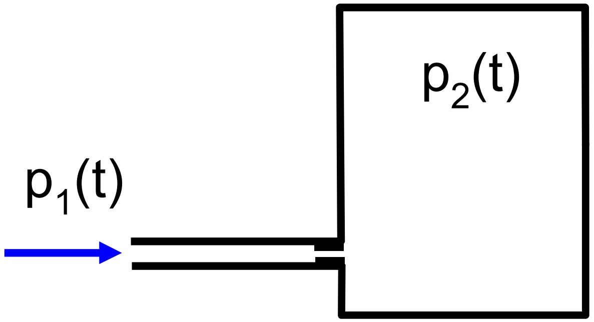



AIR CONTAINER:
input – pressure $p_1(t)$
output – pressure $p_2(t)$

# First-order inertial element

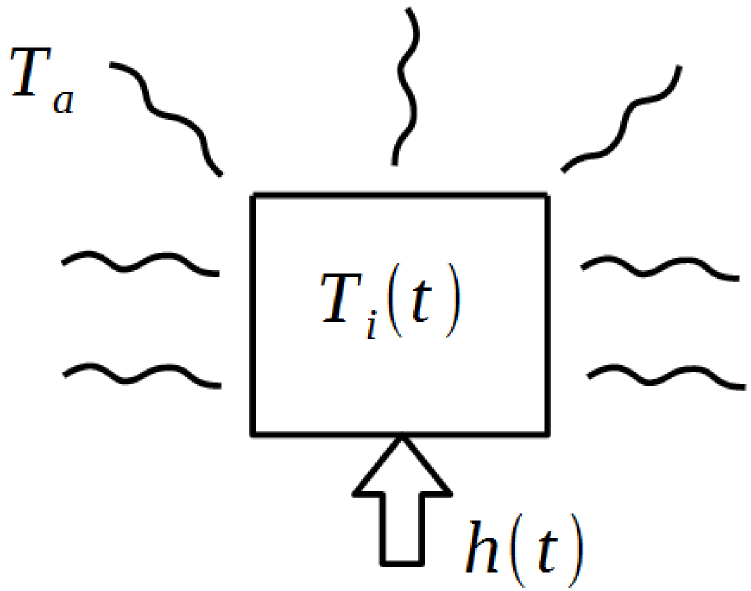
Examples

3

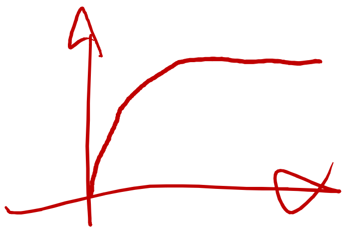



AIR CONTAINER:
input – pressure $p_1(t)$
output – pressure $p_2(t)$

4




HEATED OBJECT WITH SMALL INERTIA:
input – heater power $h(t)$
output – object temperature $T_i(t)$



# Integrator

$y(t) = k \int_0^t u(\tau) d\tau$

1. Element equation: $\frac{dy(t)}{dt} = k u(t)$

$u(t)$ - input
$y(t)$ - output

# Integrator

1. Element equation: $\frac{dy(t)}{dt} = k\,u(t)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state):

u = const.
y = const.

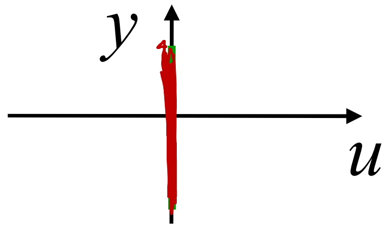
u = 0 , y - any

# Integrator

1. Element equation: $\frac{dy(t)}{dt} = k\,u(t)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $u = 0$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$




3. Transfer function: $s\,Y(s) = k\,U(s)$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k}{s}$$

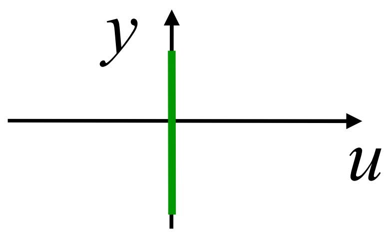
(sometimes $\frac{1}{Ts}$)

# Integrator

1. Element equation: $\frac{dy(t)}{dt} = k\,u(t)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $u = 0$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$




3. Transfer function: $H(s) = \frac{k}{s}$

# Integrator

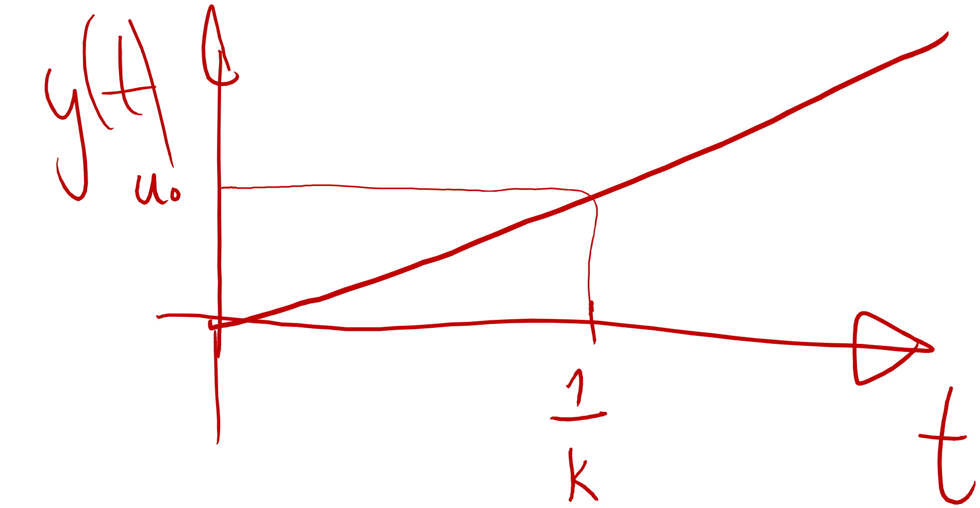
4. Step response:

input: $u(t)=u_0 1(t)$

Laplace of input: $U(s)=u_0 \frac{1}{s}$

$$Y(s) = H(s)\cdot U(s) = \frac{k}{s}\cdot u_0 \frac{1}{s} = \frac{k u_0}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{k u_0}{s^2}\right\} = k u_0 \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = k u_0 \cdot t$$



# Integrator

## 4. Step response:

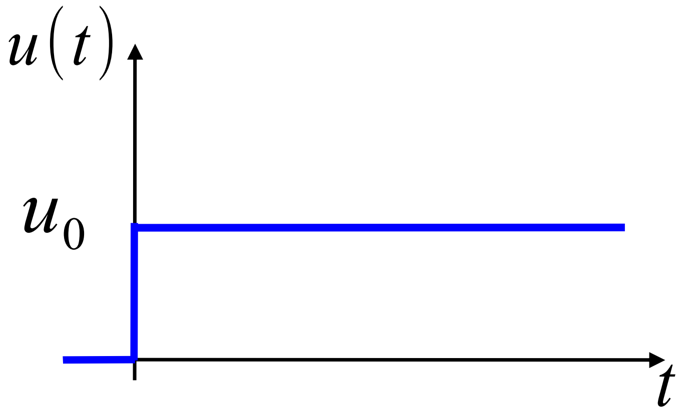
input: $u(t) = u_0 1(t)$

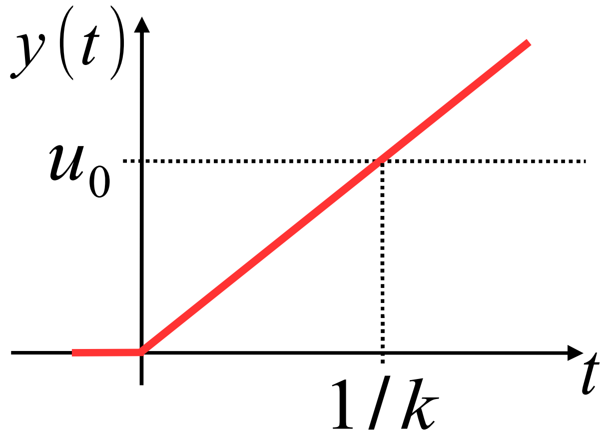
Laplace of input: $U(s) = u_0 \frac{1}{s}$

Laplace of output: $Y(s) = H(s)U(s) = \frac{k u_0}{s^2}$

output: $y(t) = L^{-1}\{Y(s)\} = k u_0 t$

for $k > 0$



# Integrator

5. Frequency response: $H(j\omega) = \frac{k}{j\omega}$

$$H(j\omega) = \frac{k}{j\omega} \cdot \frac{j}{j} = \frac{jk}{-\omega} = P(\omega) + jQ(\omega)$$

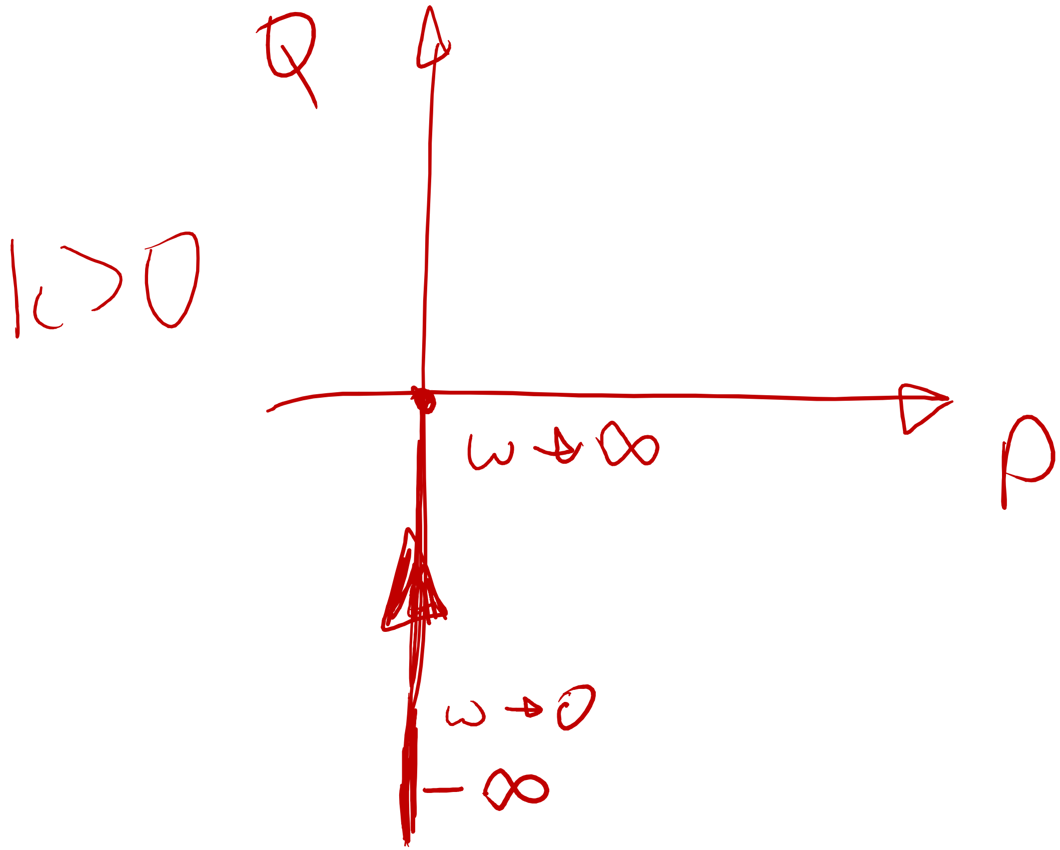
$$P(\omega) = 0 \; ; \qquad Q(\omega) = -\frac{k}{\omega}$$

# Integrator

5. Frequency response: $H(j\omega) = \dfrac{k}{j\omega}$

$$P(\omega) = 0, \quad Q(\omega) = -\frac{k}{\omega}$$

6. Nyquist plot:



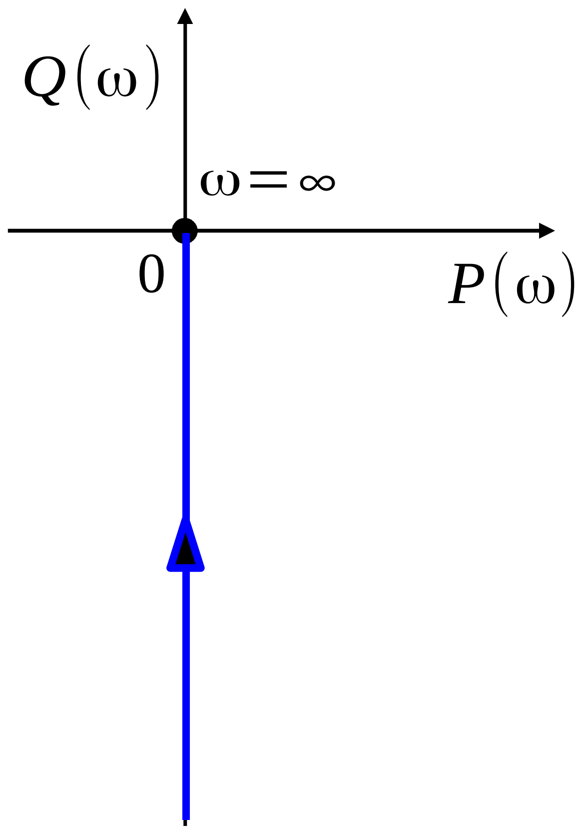
# Integrator

5. Frequency response: $H(j\omega) = \frac{k}{j\omega}$

$$P(\omega) = 0, \;\; Q(\omega) = -\frac{k}{\omega}$$

6. Nyquist plot:

for $k > 0$



# Integrator

7. Bode plot:

$$P(\omega)=0, \quad Q(\omega)=-\frac{k}{\omega}$$

$$A(\omega)=\sqrt{P^2+Q^2}=\left|\frac{k}{\omega}\right|$$

$$L(\omega)[dB]=20\log\left|\frac{k}{\omega}\right|$$

$$\varphi(\omega)[rad]=\mathrm{atan}\frac{Q}{P}=\mathrm{atan}(-\infty)=-\frac{\pi}{2}$$

# Integrator

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = \left|\frac{k}{\omega}\right|$

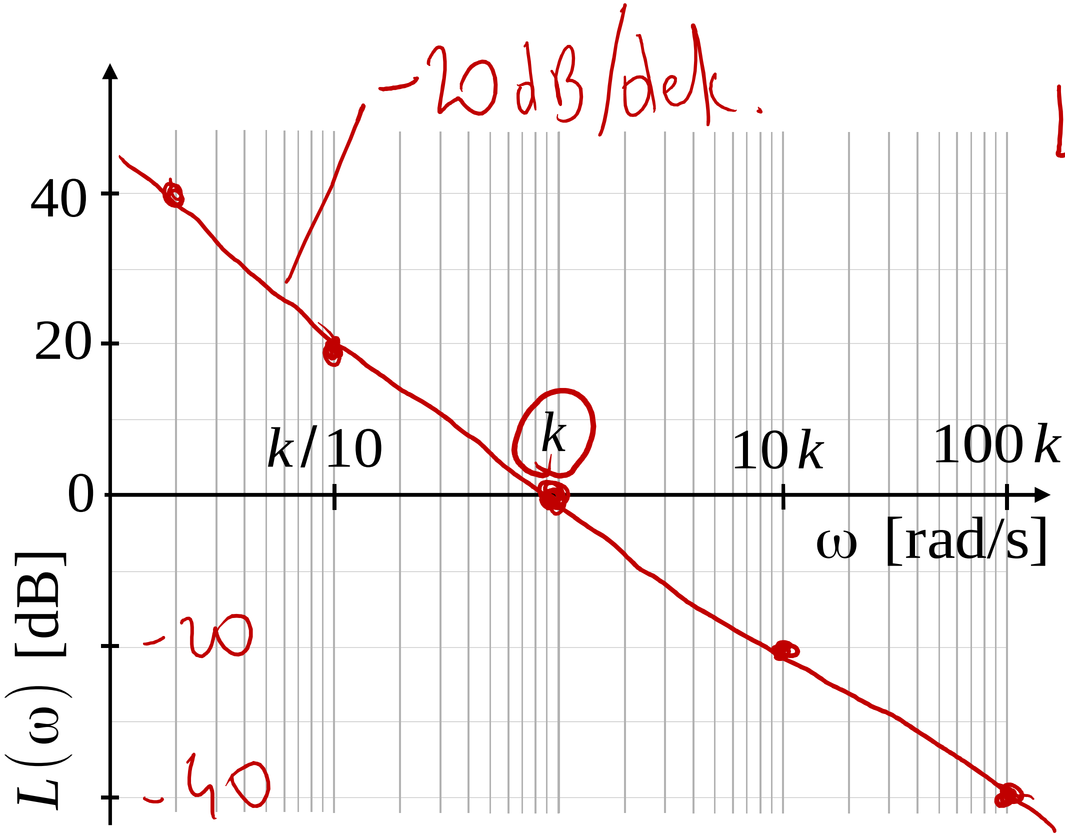
$$L(\omega) = 20\log A(\omega) = 20\log\left|\frac{k}{\omega}\right| \qquad \varphi(\omega) = \arctan\frac{Q}{P} = \arctan(-\infty)$$

# Integrator

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=\left|\frac{k}{\omega}\right|$ for $k>0$

$$L(\omega)=20\log A(\omega)=20\log\left|\frac{k}{\omega}\right|$$

$$\varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-\infty)$$




$$L(\omega=k)=20\log\left|\frac{k}{k}\right|=0$$

$$L(\omega=10k)=20\log\left|\frac{k}{10k}\right|=-20$$

$$L(\omega=100k)=20\log\left|\frac{1}{100}\right|=-40$$

$$L\left(\omega=\frac{k}{10}\right)=20\log|10|=20$$

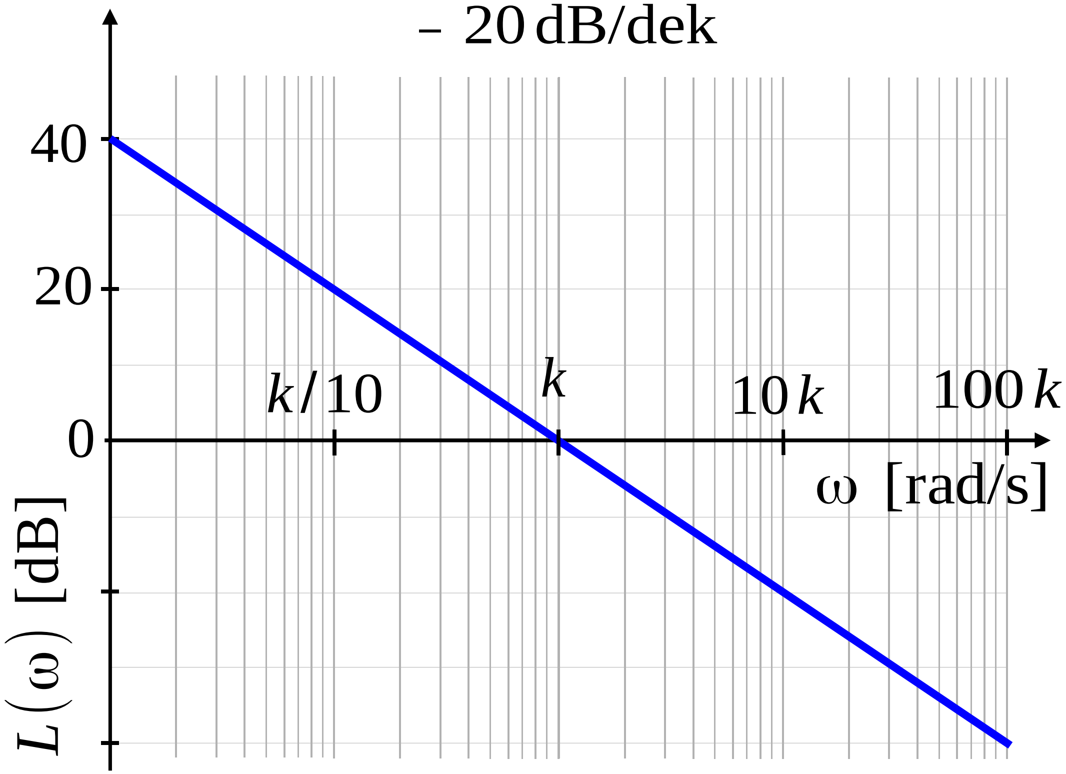
$$L\left(\omega=\frac{k}{100}\right)=20\log|100|=40$$

# Integrator

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=\left|\frac{k}{\omega}\right|$ for $k>0$

$$L(\omega)=20\log A(\omega)=20\log\left|\frac{k}{\omega}\right| \qquad \varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-\infty)$$

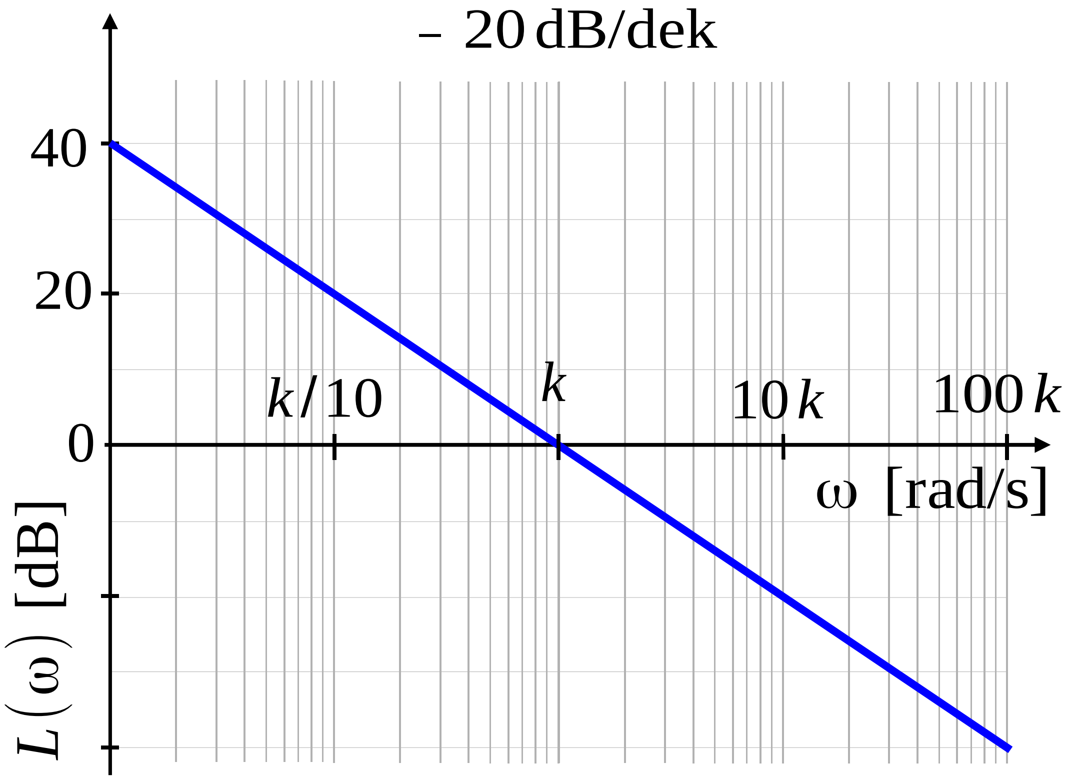
$=-\frac{\pi}{2}$

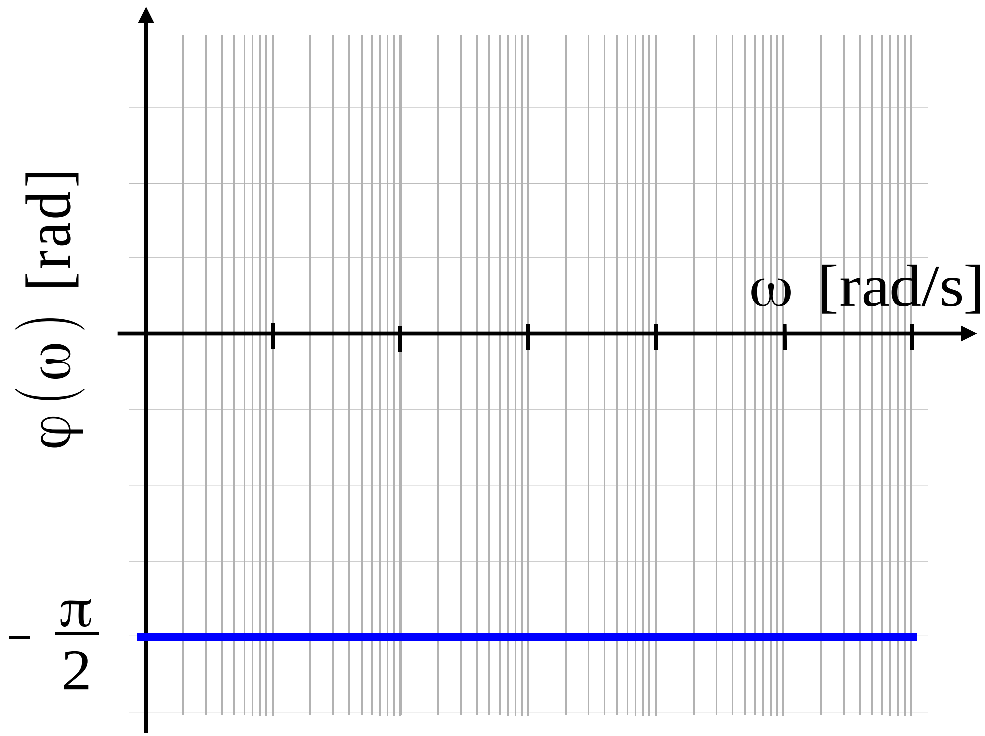


# Integrator

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|\frac{k}{\omega}|$ for $k>0$

$$L(\omega)=20\log A(\omega)=20\log|\frac{k}{\omega}| \qquad \varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-\infty)$$



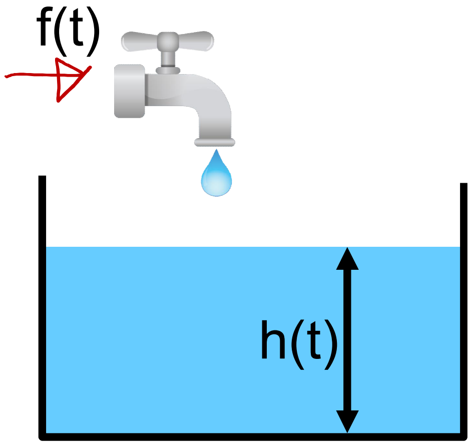



# Integrator

## Examples

1

[m³/h]

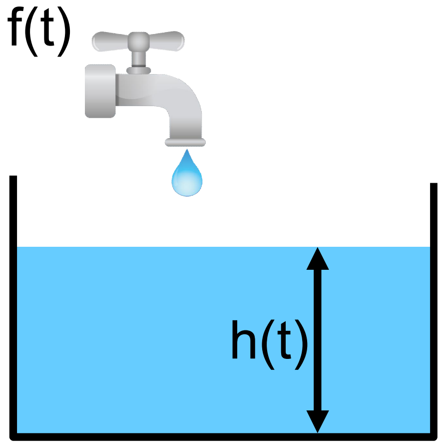



PRISM LIQUID TANK:
input – liquid inflow f(t)
output – liquid level h(t)

# Integrator

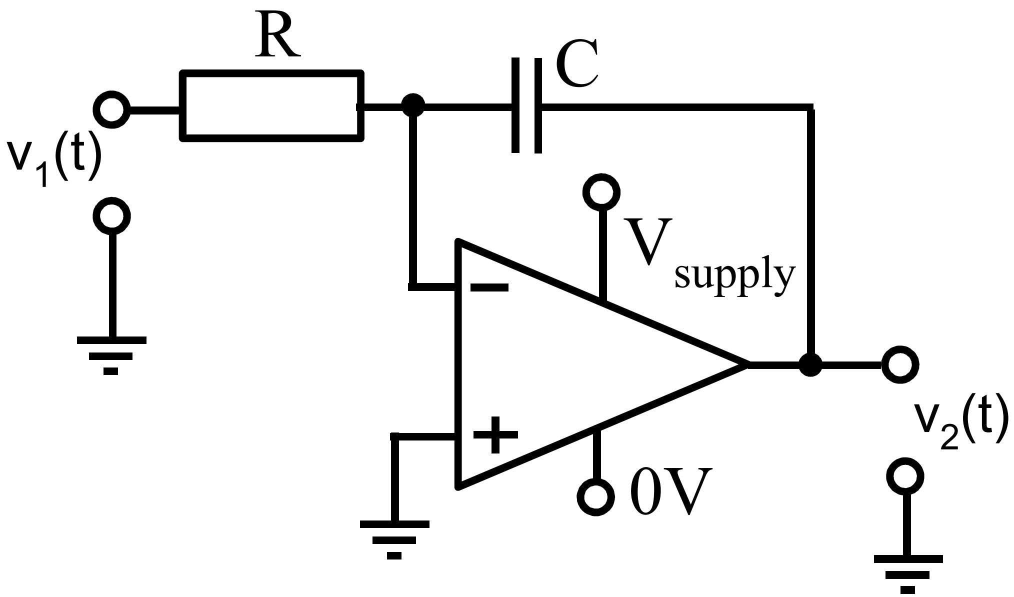
## Examples

1




PRISM LIQUID TANK:
input – liquid inflow f(t)
output – liquid level h(t)

2




OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

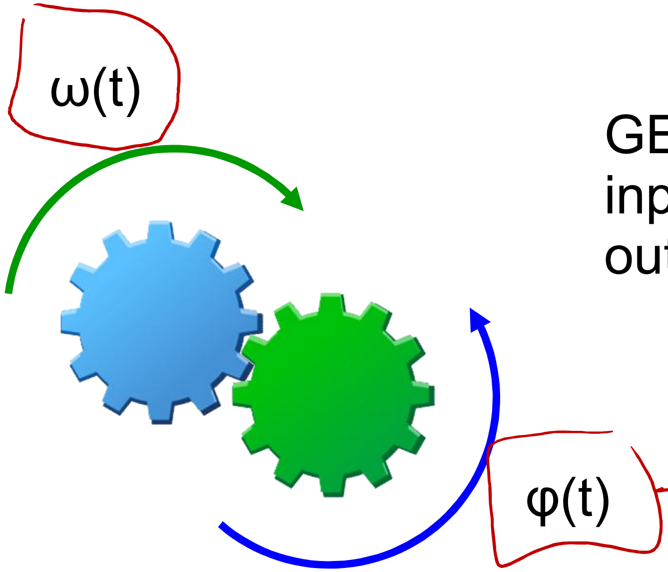
$$v_2(t) = \frac{1}{RC} \int_0^t v_1(t)\,dt$$

# Integrator

$$\omega(t) = \frac{d\varphi(t)}{dt}$$

Examples

3



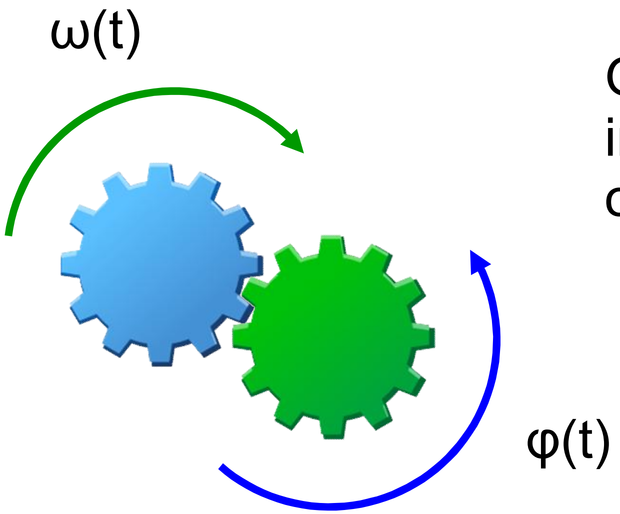

$$= \int \omega(t)\, dt$$

GEARBOX:
input – angular velocity ω(t)
output – rotation angle φ(t)

# Integrator

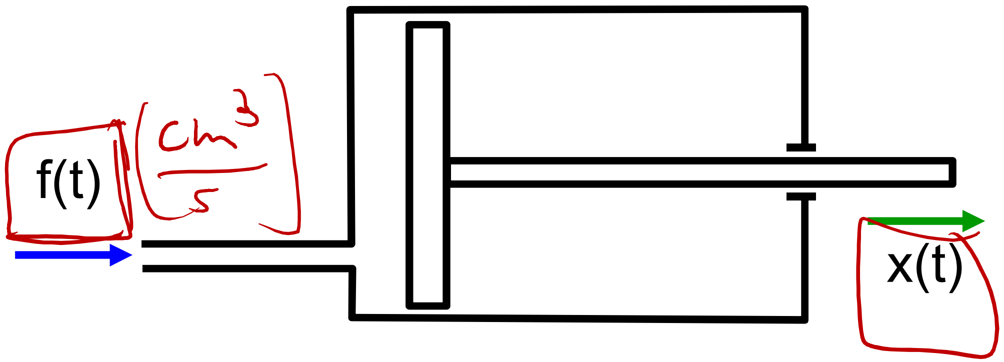
Examples

3




GEARBOX:
input – angular velocity ω(t)
output – rotation angle φ(t)

4




HYDRAULIC CYLINDER:
input – volume inflow f(t)
output – displacement x(t)

# Differentiator

1. Element equation: $y(t)=k\frac{du(t)}{dt}$

$k \in \mathbb{R}$

$u(t)$ - input
$y(t)$ - output

# Differentiator

1. Element equation: $y(t) = k\dfrac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state):

$u = const.$

$\dfrac{du}{dt} = 0$

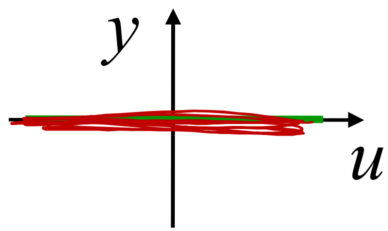
$y(t) = 0$

# Differentiator

1. Element equation: $y(t) = k\dfrac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y = 0$ for $\dfrac{dy}{dt} = 0 \wedge \dfrac{du}{dt} = 0$




3. Transfer function: $Y(s) = k\,s\,U(s)$

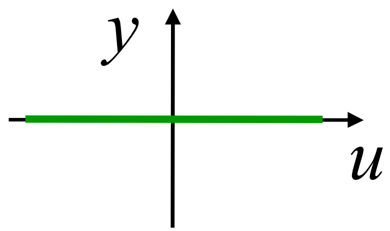
$$H(s) = \frac{Y(s)}{U(s)} = k \cdot s$$

# Differentiator

1. Element equation: $y(t) = k\,\frac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y = 0$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$




3. Transfer function: $H(s) = k\,s$

# Differentiator

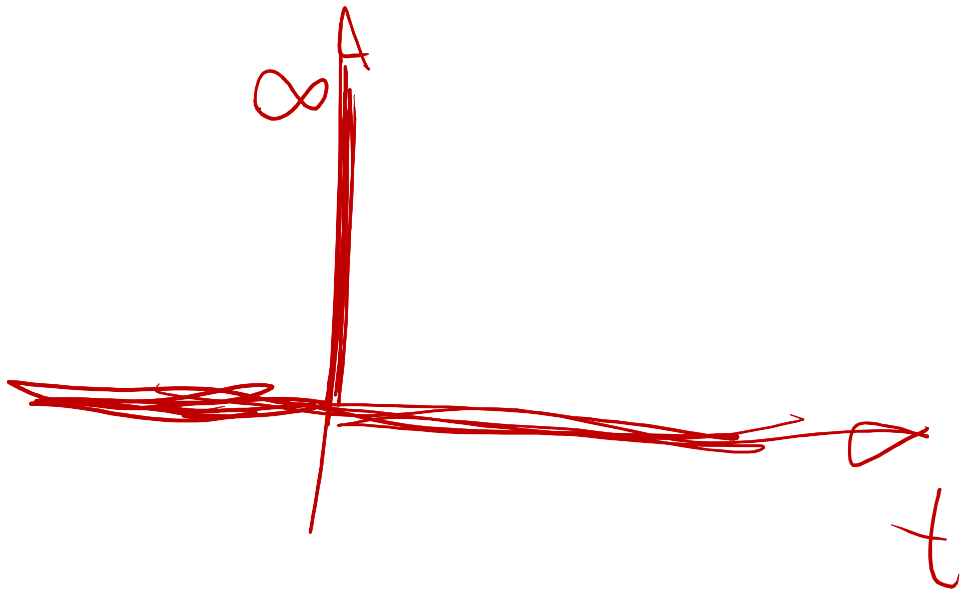
4. Step response:

input: $u(t)=u_0 1(t)$

Laplace of input: $U(s)=u_0 \frac{1}{s}$

$$Y(s)=H(s)\cdot U(s)=k\cdot s\cdot u_0\frac{1}{s}=ku_0\cdot 1$$

$$y(t)=\mathcal{L}^{-1}\{ku_0\}=ku_0\mathcal{L}^{-1}\{1\}=ku_0\,\delta(t)$$



(iDEAL) 

# Differentiator

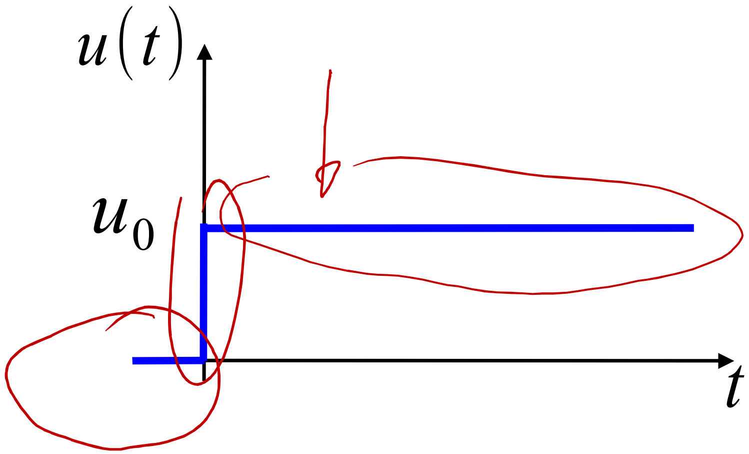
## 4. Step response:

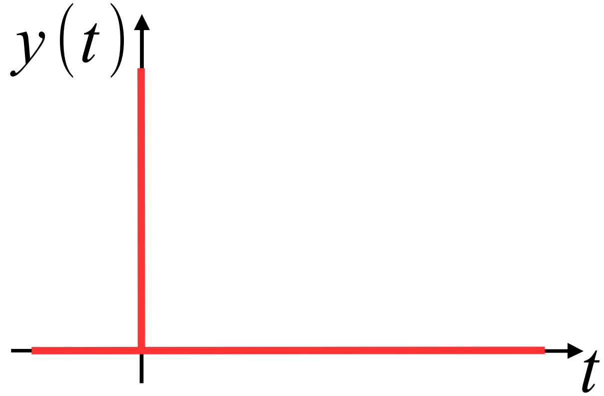
input: $u(t)=u_0 1(t)$

Laplace of input: $U(s)=u_0 \frac{1}{s}$

Laplace of output: $Y(s)=H(s)U(s)=k u_0$

output: $y(t)=L^{-1}\{Y(s)\}=k u_0 \delta(t)$



# Differentiator

5. Frequency response: $H(j\omega) = k \cdot j\omega$

$P(\omega) = 0; \quad Q(\omega) = k\omega$

# Differentiator

5. Frequency response: $H(j\omega)=jk\omega$

$$P(\omega)=0\,,\;\;Q(\omega)=k\omega$$

6. Nyquist plot:

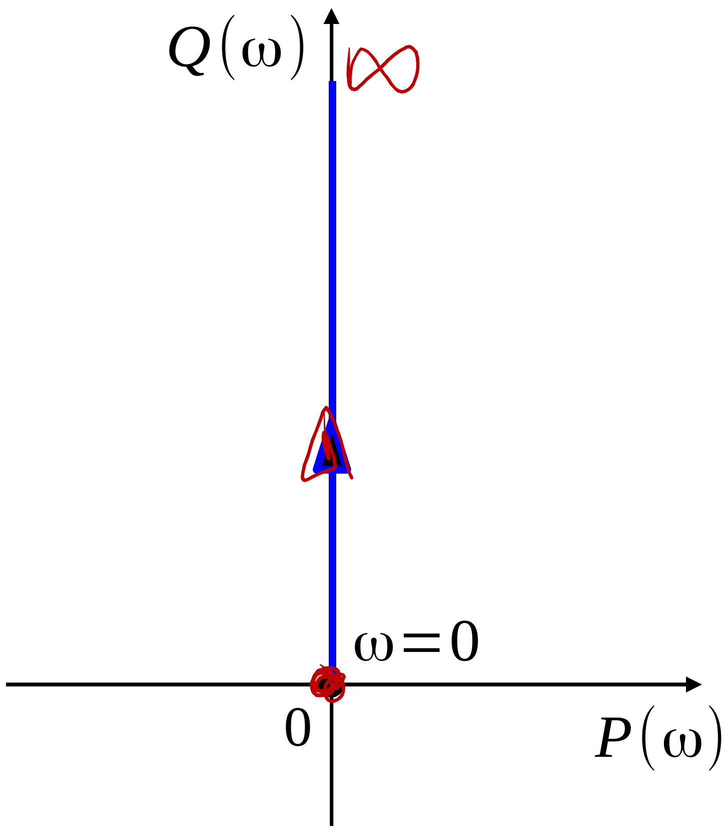
# Differentiator

5. Frequency response: $H(j\omega)=jk\omega$

$$P(\omega)=0,\ \ Q(\omega)=k\omega$$

6. Nyquist plot:

for $k>0$



# Differentiator

7. Bode plot:

$$P(\omega)=0, \quad Q(\omega)=k\omega$$

$A(\omega)=\sqrt{P^2+Q^2}=|k\omega|$

$L(\omega)=20\log|k\omega|$

$\varphi(\omega)=\operatorname{atan}\frac{Q}{P}=\operatorname{atan}(\infty)=\frac{\pi}{2}$

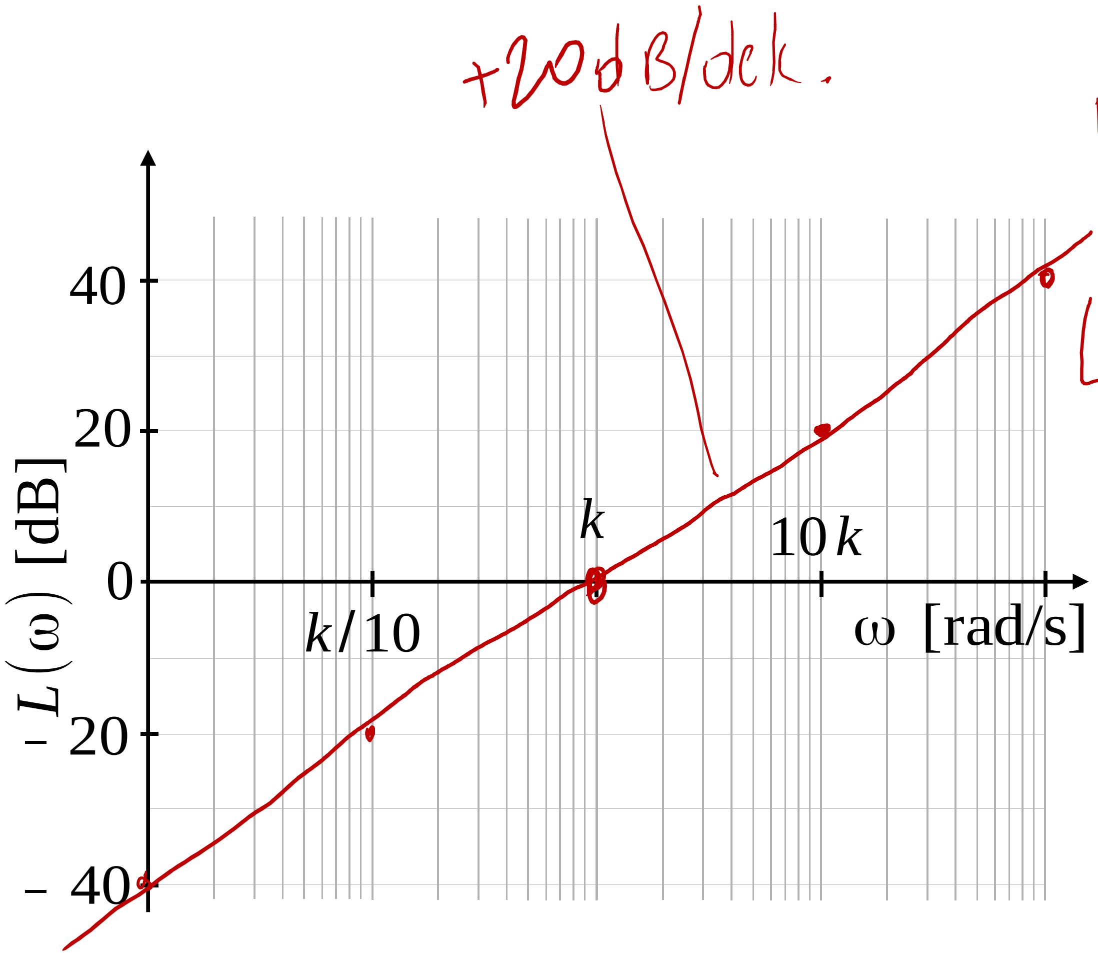
ASSUME:

$k>0$

# Differentiator

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k\omega|$

$$L(\omega)=20\log A(\omega)=20\log|k\omega| \qquad \varphi(\omega)=\arctan\frac{Q}{P}=\arctan(\infty)$$




$L\left(\omega=\frac{1}{k}\right)=20\log 1=0$

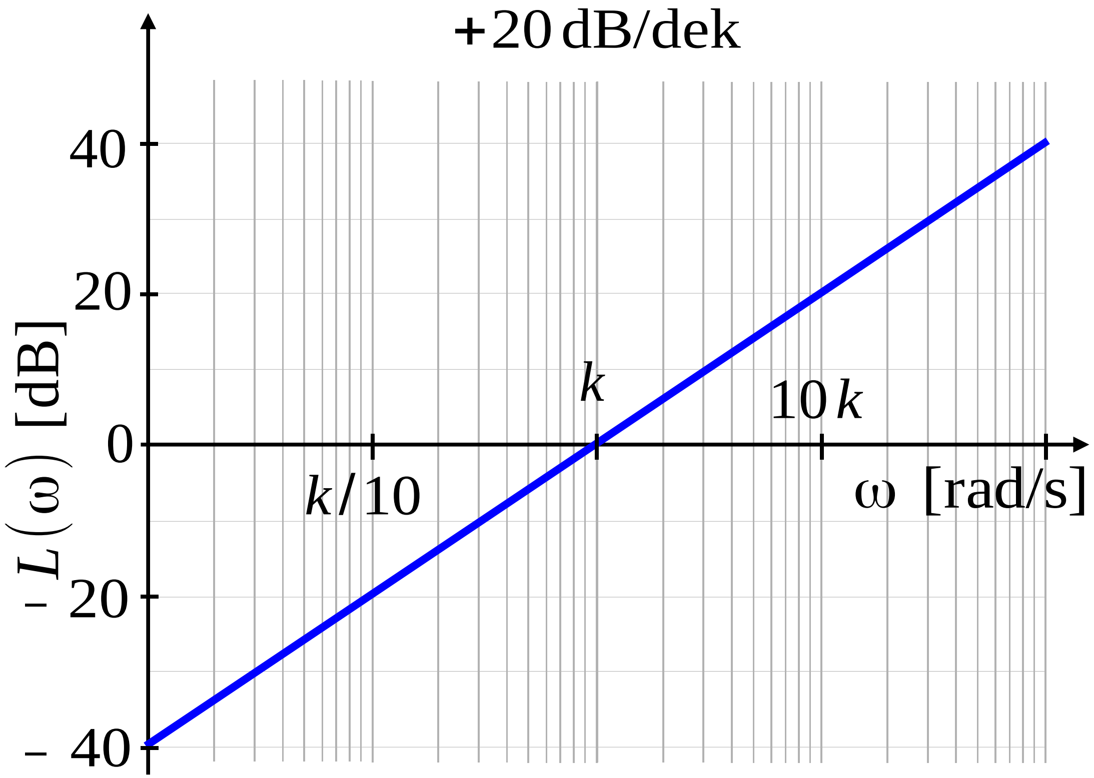
$L\left(\omega=\frac{10}{k}\right)=20\log 10=20$

$L\left(\omega=\frac{1}{10k}\right)=20\log\frac{1}{10}=-20$

# Differentiator

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k\omega|$

$$L(\omega)=20\log A(\omega)=20\log|k\omega| \qquad \varphi(\omega)=\arctan\frac{Q}{P}=\arctan(\infty)$$

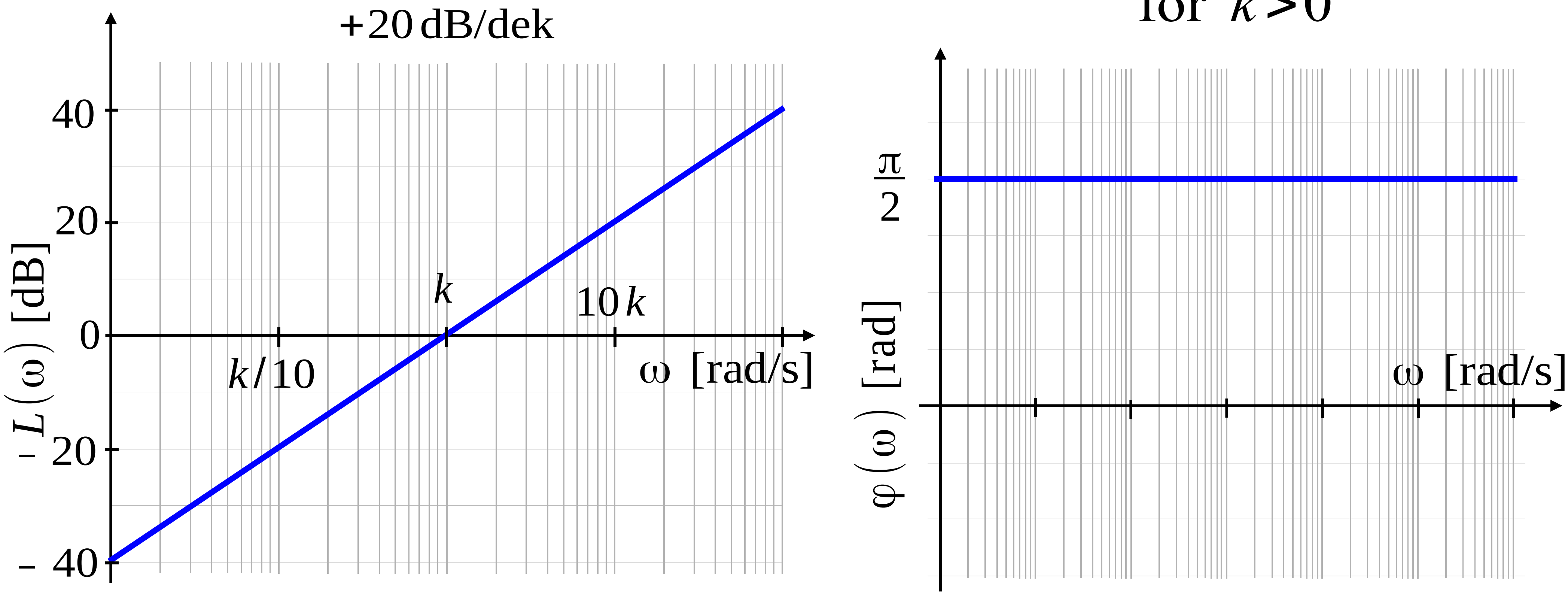


# Differentiator

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k\omega|$

$$L(\omega) = 20\log A(\omega) = 20\log|k\omega| \qquad \varphi(\omega) = \arctan\frac{Q}{P} = \arctan(\infty)$$

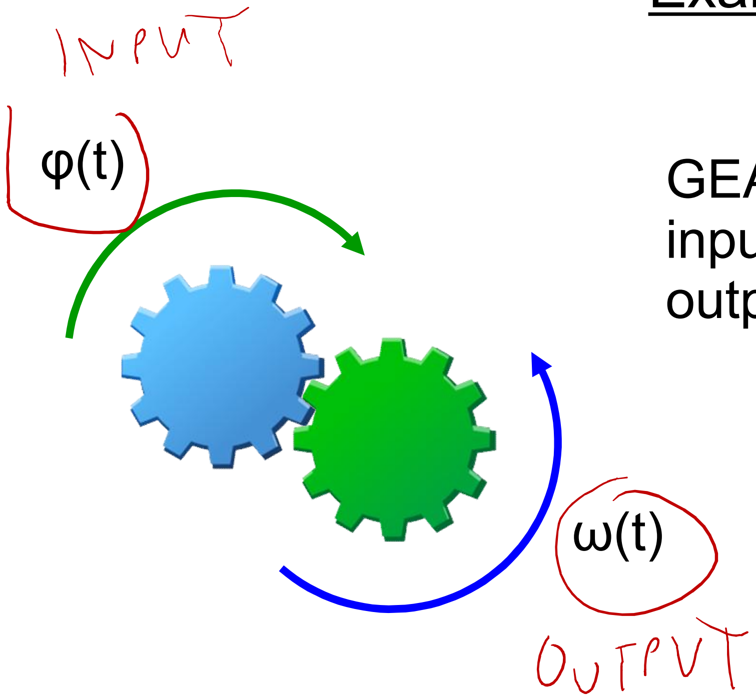
for $k > 0$



# Differentiator IDEAL

## Examples

1

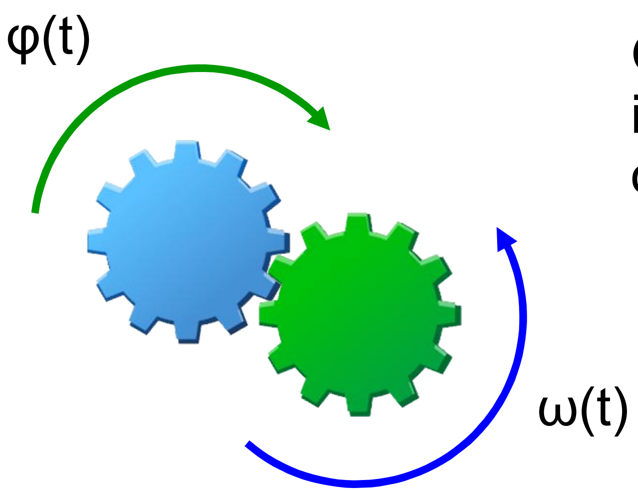



GEARBOX:
input – rotation angle φ(t)
output – angular velocity ω(t)

# Differentiator

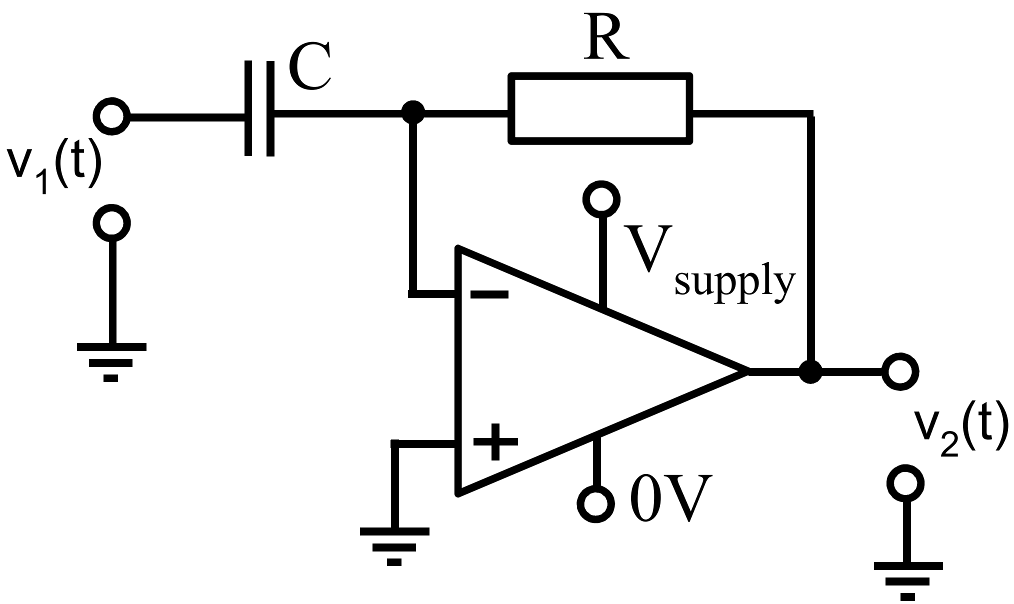
## Examples

1




GEARBOX:
input – rotation angle φ(t)
output – angular velocity ω(t)

2




OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = -\ RC\frac{dv_1(t)}{dt}$$

# Real differentiator (derivative+1st order)

$k \in \mathbb{R}$

1. Element equation: $T\dfrac{dy(t)}{dt}+y(t)=k\dfrac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

$T \in \mathbb{R}_+$

# Real differentiator (derivative+1st order)

1. Element equation: $T\underbrace{\frac{dy(t)}{dt}}^{=0}+y(t)=k\frac{du(t)}{dt}^{=0}$  $u(t)$ - input, $y(t)$ - output

2. Static characteristic (steady state):

$u=const$
$y=const$

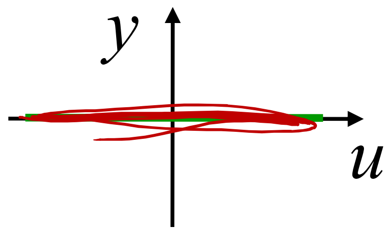
$y(t)=0$

# Real differentiator (derivative+1st order)

1. Element equation: $T\frac{dy(t)}{dt}+y(t)=k\frac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=0$ for $\frac{dy}{dt}=0 \wedge \frac{du}{dt}=0$




3. Transfer function: $Ts\,Y(s)+Y(s)=k\,s\,U(s)$

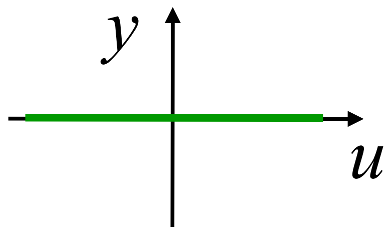
$$H(s)=\frac{Y(s)}{U(s)}=\frac{ks}{Ts+1}$$

# Real differentiator (derivative+1st order)

1. Element equation: $T\dfrac{dy(t)}{dt}+y(t)=k\dfrac{du(t)}{dt}$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=0$ for $\dfrac{dy}{dt}=0\wedge\dfrac{du}{dt}=0$




3. Transfer function: $H(s)=\dfrac{k\,s}{Ts+1}$

# Real differentiator (derivative+1st order)

4. Step response:

input: $u(t) = u_0 1(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

# Real differentiator (derivative+1st order)

4. Step response:

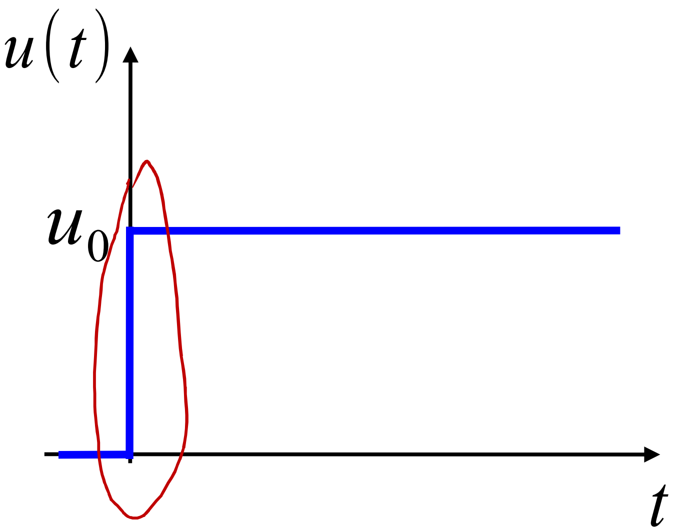
input: $u(t)=u_0 1(t)$

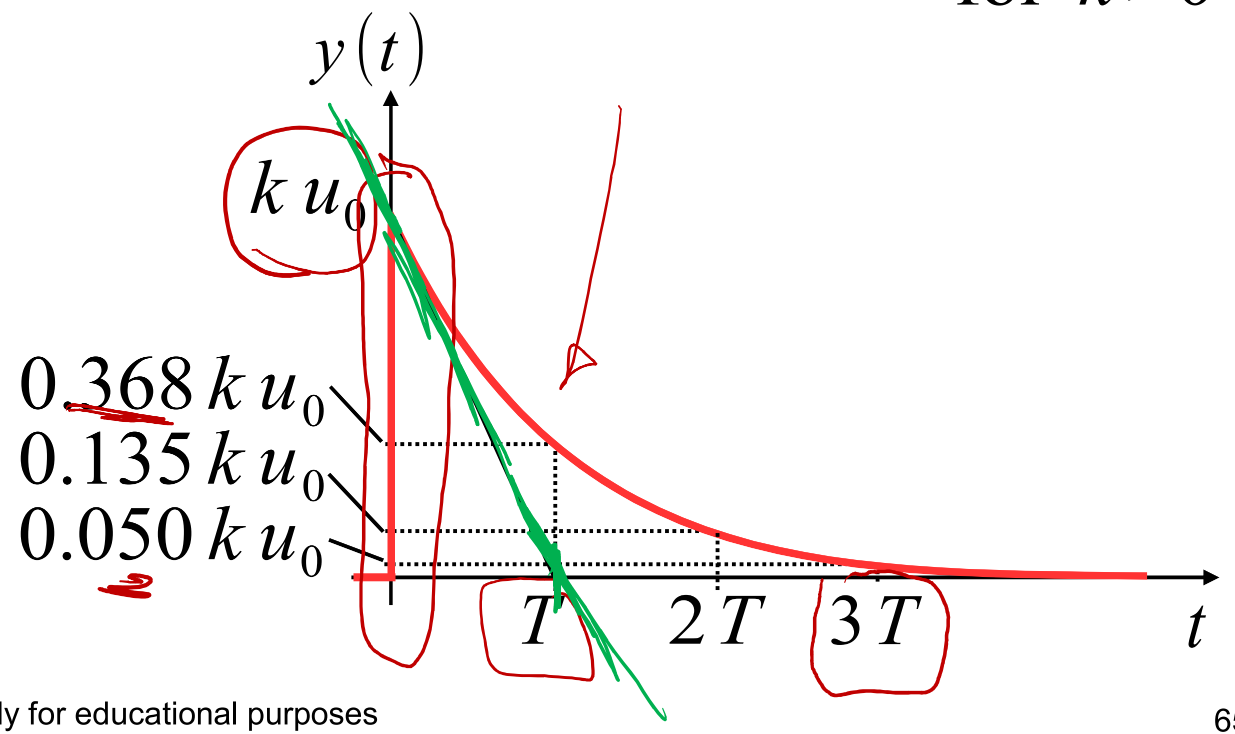
Laplace of input: $U(s)=u_0\frac{1}{s}$

Laplace of output: $Y(s)=H(s)U(s)=\frac{k u_0}{Ts+1}$

output: $y(t)=L^{-1}\{Y(s)\}=\boxed{k u_0 e^{-t/T}}$

for $k>0$



# Real differentiator (derivative+1st order)

5. Frequency response: $H(j\omega) = \frac{k \cdot j\omega}{Tj\omega + 1}$

$$\frac{jk\omega}{1+jT\omega} \cdot \frac{1-jT\omega}{1-jT\omega} = \frac{jk\omega + kT\omega^2}{1^2 + T^2\omega^2}$$

$$P(\omega) = \frac{kT\omega^2}{1+T^2\omega^2} \; ; \quad Q(\omega) = \frac{k\omega}{1+T^2\omega^2}$$

# Real differentiator (derivative+1st order)

5. Frequency response: $H(j\omega)=\dfrac{k\,j\omega}{Tj\omega+1}$

$$P(\omega)=\frac{k\,T\,\omega^2}{T^2\omega^2+1},\quad Q(\omega)=\frac{k\,\omega}{T^2\omega^2+1}$$

6. Nyquist plot:

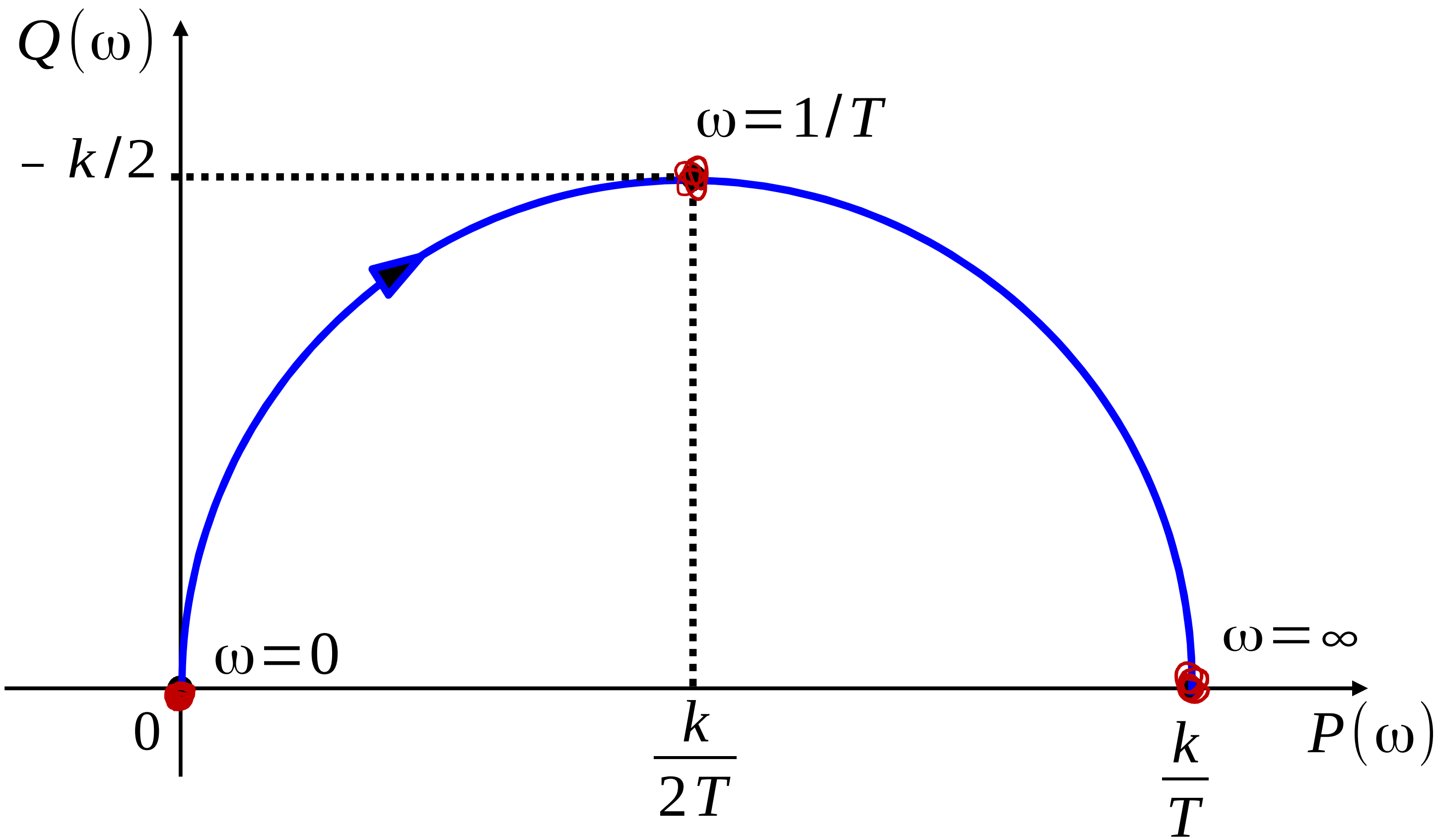
# Real differentiator (derivative+1st order)

5. Frequency response: $H(j\omega)=\dfrac{k\,j\omega}{Tj\omega+1}$

$$P(\omega)=\frac{kT\omega^2}{T^2\omega^2+1},\quad Q(\omega)=\frac{k\omega}{T^2\omega^2+1}$$

6. Nyquist plot:

for $k>0$



# Real differentiator (derivative+1st order)

7. Bode plot:

$$P(\omega)=\frac{kT\omega^2}{T^2\omega^2+1},\quad Q(\omega)=\frac{k\omega}{T^2\omega^2+1}$$

$A(\omega)$

$L(\omega)$

$\varphi(\omega)$

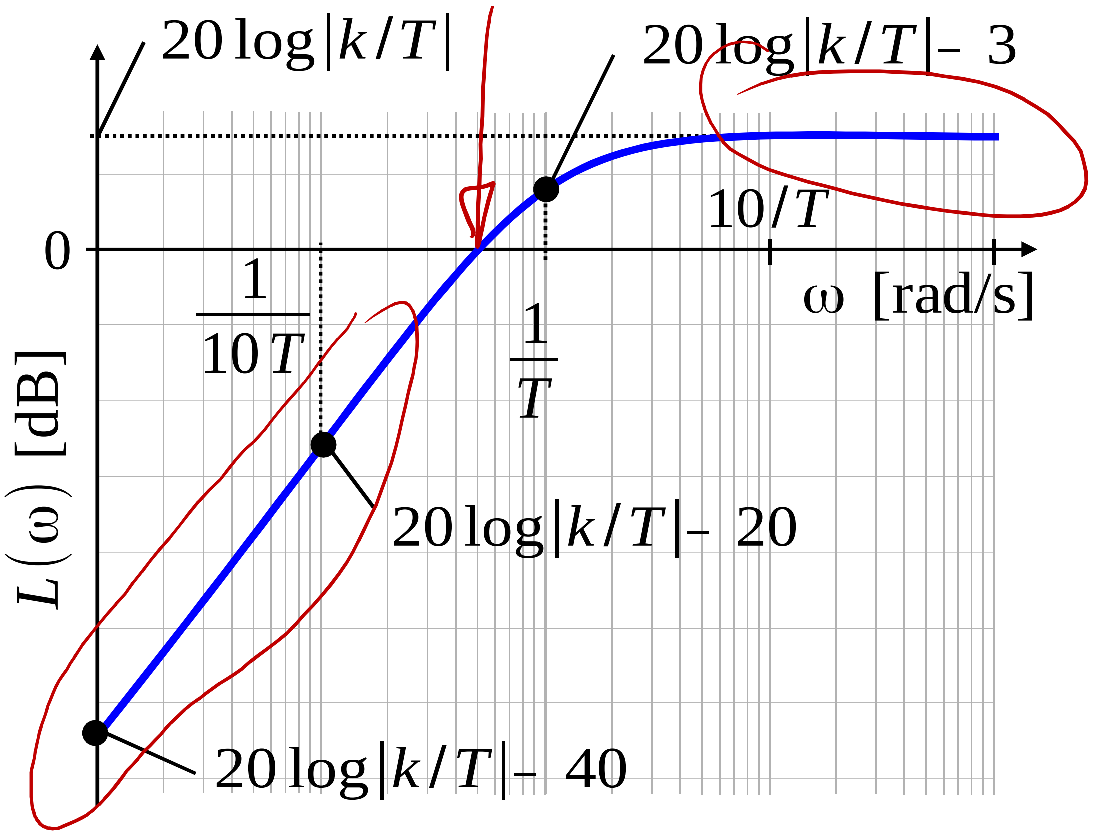
# Real differentiator (derivative+1st order)

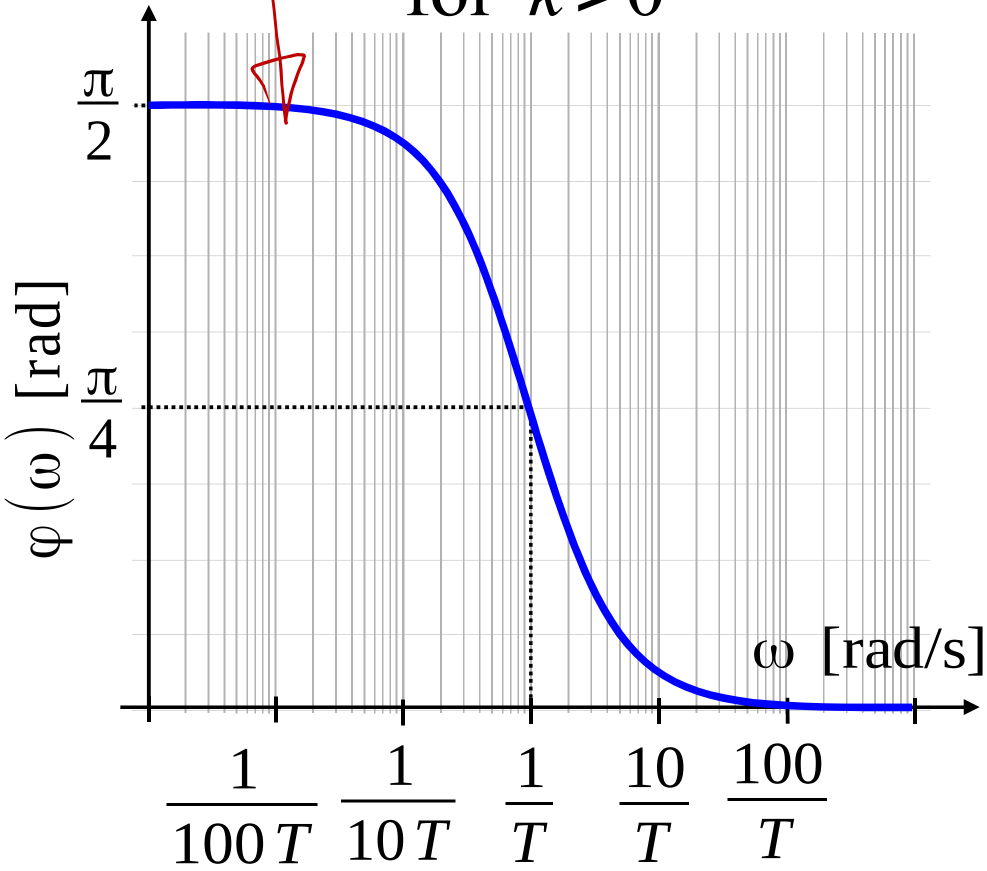
7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=|k\omega|/\sqrt{T^2\omega^2+1}$

$$L(\omega)=20\log A(\omega)=20\log|k\omega|-20\log\sqrt{T^2\omega^2+1}$$

$$\varphi(\omega)=\arctan\frac{Q}{P}=\arctan\left(\frac{1}{T\omega}\right)$$

for $k>0$

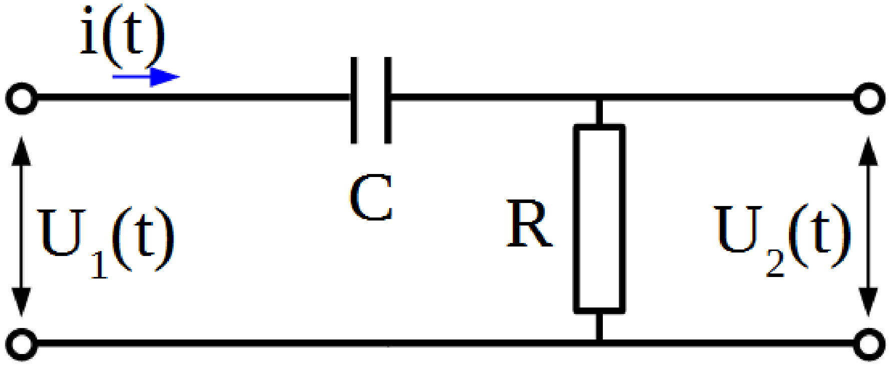


# Real differentiator (derivative+1st order)

Examples

1




RC CIRCUIT:
input – voltage $u_1(t)$
output – voltage $u_2(t)$

# Delay

1. Element equation: $y(t)=u(t-\tau)$ $\tau \in \mathbb{R}_+$

$u(t)$ - input
$y(t)$ - output

# Delay

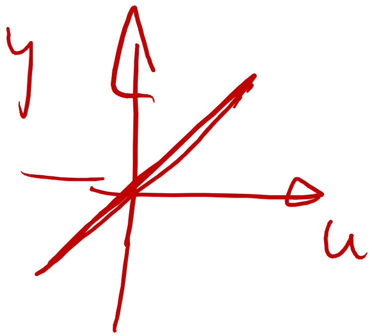
1. Element equation: $y(t)=u(t-\tau)$

$u(t)$ - input

$y(t)$ - output

2. Static characteristic (steady state):

$u = const.$ $y = const.$



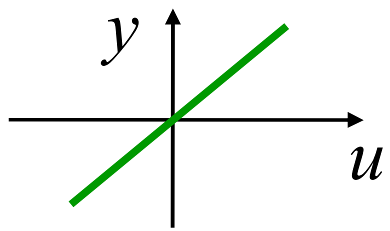

$u = y$

# Delay

1. Element equation: $y(t)=u(t-\tau)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=u$ for $\frac{dy}{dt}=0 \wedge \frac{du}{dt}=0$




3. Transfer function: $Y(s)=U(s)e^{-\tau s}$

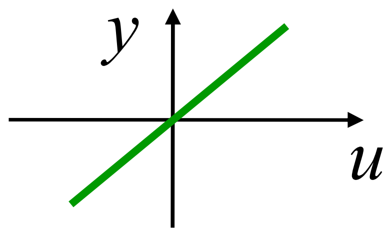
$$H(s)=\frac{Y(s)}{U(s)}=e^{-\tau s}$$

# Delay

1. Element equation: $y(t)=u(t-\tau)$

$u(t)$ - input
$y(t)$ - output

2. Static characteristic (steady state): $y=u$ for $\frac{dy}{dt}=0 \wedge \frac{du}{dt}=0$




3. Transfer function: $H(s)=e^{-\tau s}$

# Delay

4. Step response: input: $u(t)=u_0 1(t)$

Laplace of input: $U(s)=u_0 \frac{1}{s}$

$$Y(s)=H(s)\cdot U(s)=e^{-\tau s}\cdot u_0\frac{1}{s}=u_0\frac{1}{s}e^{-\tau s}$$

$$y(t)=\mathcal{L}^{-1}\left\{u_0\frac{1}{s}e^{-\tau s}\right\}=u_0\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}*\mathcal{L}^{-1}\left\{e^{-\tau s}\right\}_s$$

$$=u_0\,\mathbb{1}(t)*\mathbb{1}(t-\tau)=u_0\,\mathbb{1}(t-\tau)$$

# Delay

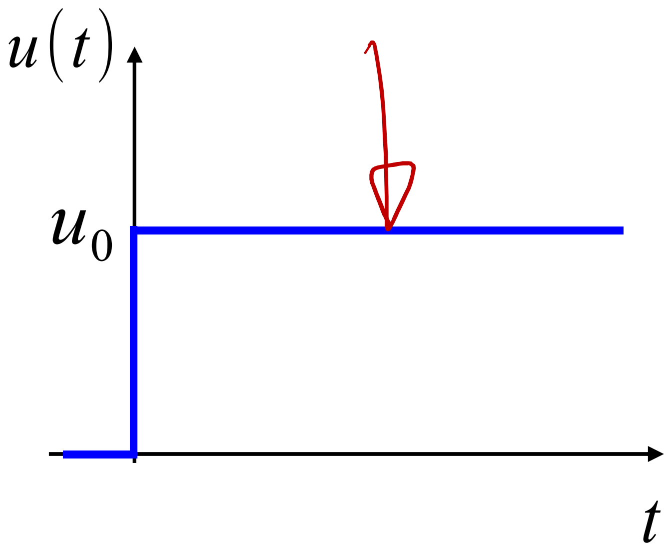
4. Step response:

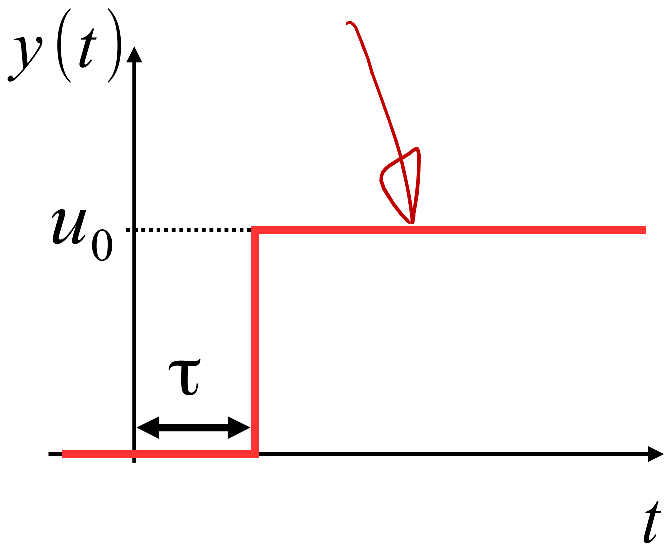
input: $u(t) = u_0 1(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Laplace of output: $Y(s) = H(s)U(s) = \frac{u_0}{s} e^{-\tau s}$

output: $y(t) = L^{-1}\{Y(s)\} = u_0 1(t - \tau)$



# Delay

5. Frequency response: $H(s) = e^{-\tau s}$

$e^{-ix} = \cos x - i \sin x$

$H(j\omega) = e^{-\tau j\omega} = \cos(\tau\omega) - j\sin(\tau\omega)$

$P(\omega) = \cos(\tau\omega)$ $\qquad Q(\omega) = -\sin(\tau\omega)$

# Delay

5. Frequency response: $H(j\omega)=e^{-\tau j\omega}$

$$\boxed{e^{-jx}=\cos x-j\sin x}$$

$$P(\omega)=\cos(\tau\omega),\quad Q(\omega)=-\sin(\tau\omega)$$

6. Nyquist plot:

# Delay

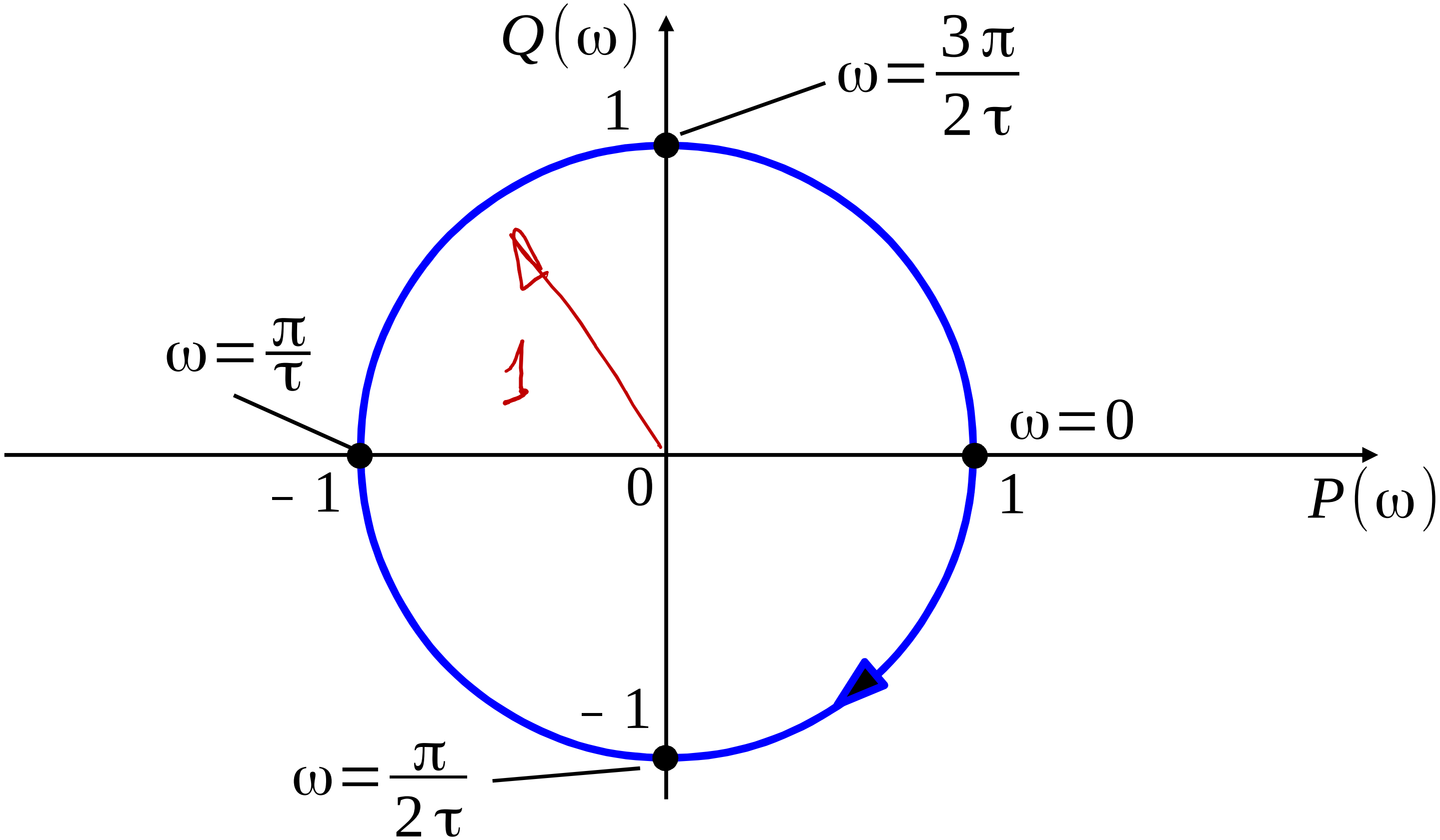
5. Frequency response: $H(j\omega)=e^{-\tau j\omega}$

$$\boxed{e^{-jx}=\cos x - j\sin x}$$

$$P(\omega)=\cos(\tau\omega),\quad Q(\omega)=-\sin(\tau\omega)$$

6. Nyquist plot:

for $k>0$



# Delay

7. Bode plot: $P(\omega)=\cos(\tau\omega),\ \ Q(\omega)=-\sin(\tau\omega)$

$$A(\omega)=\sqrt{P^2+Q^2}=\sqrt{\cos^2(\tau\omega)+\sin^2(\tau\omega)}=1$$

$$\varphi(\omega)=\operatorname{atan}(Q/P)=\operatorname{atan}\frac{-\sin\tau\omega}{\cos\tau\omega}=\operatorname{atan}(-\tan\tau\omega)=$$

$$=-\tau\omega$$

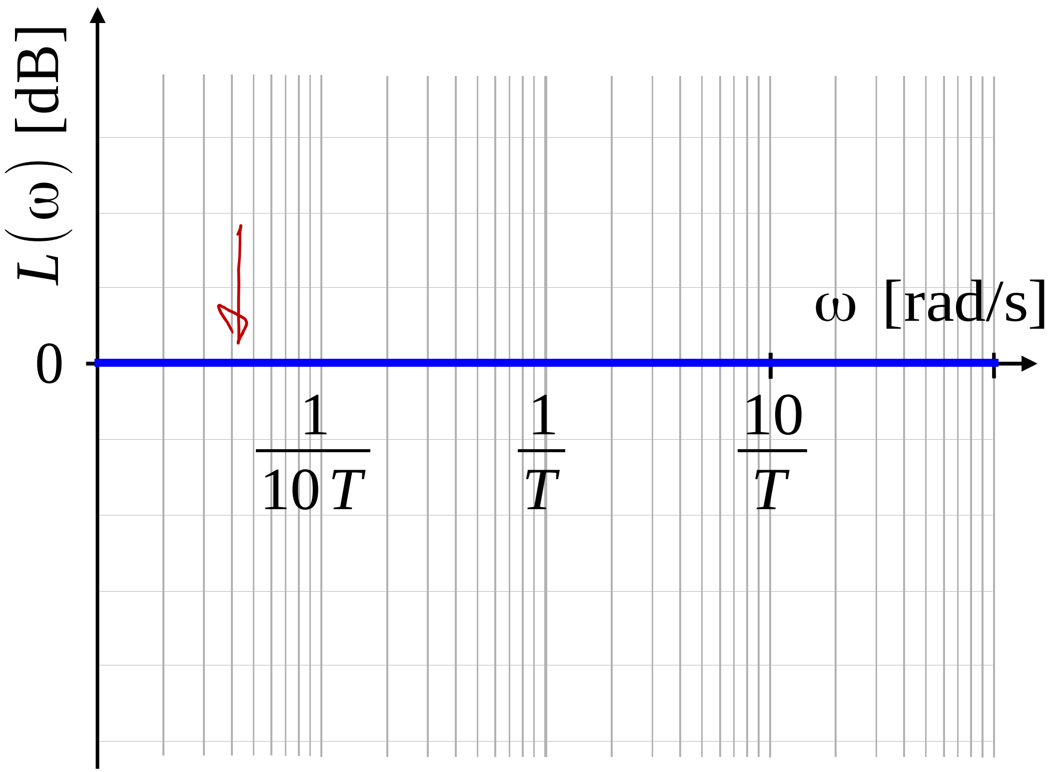
$$L(\omega)=20\log 1=0$$

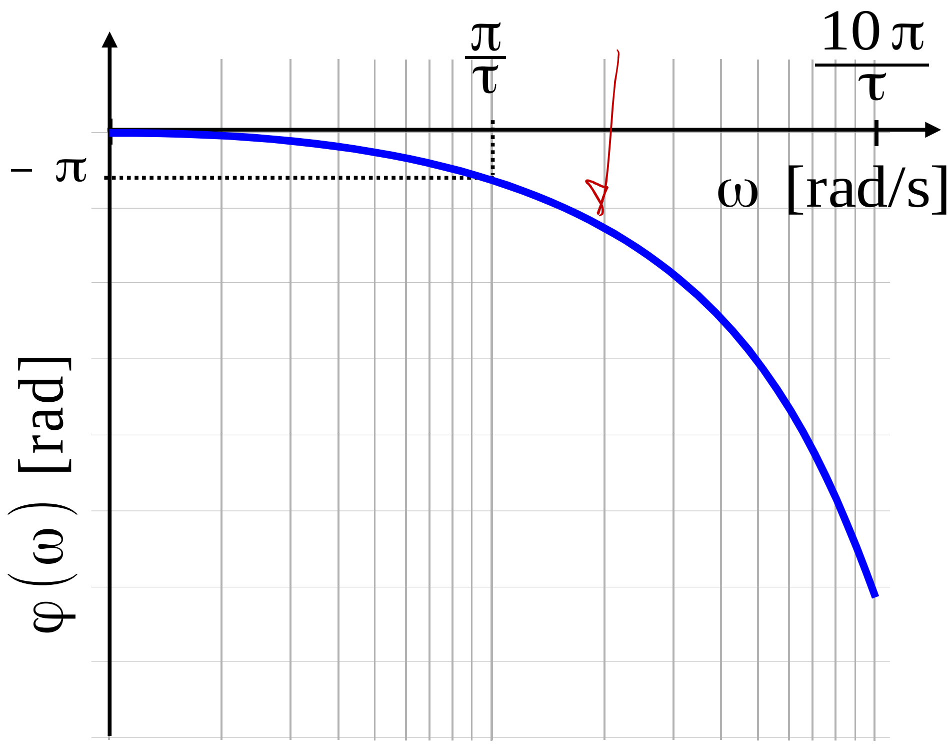
# Delay

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}=1$

$$L(\omega)=20\log A(\omega)=20\log 1=0$$

$$\varphi(\omega)=\arctan\frac{Q}{P}=\arctan(-\tan(\tau\omega))=-\tau\omega$$

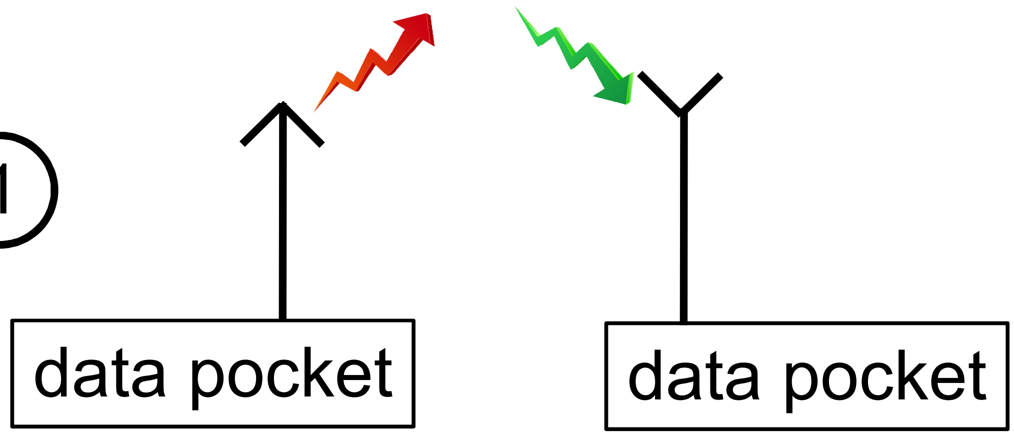


# Delay

Examples

1




WIRELESS TRANSMISSION:
input – sent data
output – received data

# Second-order inertial element

1. Element equation: $$T_1^2 \frac{d^2 y(t)}{dt^2} + T_2 \frac{dy(t)}{dt} + y(t) = k\,u(t)$$

$T_1, T_2 \in \mathbb{R}_+$

$k \in \mathbb{R}$

# Second-order inertial element

1. Element equation: $$T_1^2 \overset{0}{\frac{d^2 y(t)}{dt^2}} + T_2 \overset{0}{\frac{dy(t)}{dt}} + \overset{const}{y(t)} = k\, \overset{const}{u(t)}$$

2. Static characteristic (steady state): $y = k \cdot u$

# Second-order inertial element

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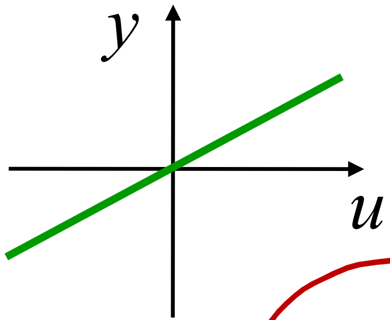


3. Transfer function:

$$\boxed{H(s) = \frac{Y(s)}{U(s)} = \frac{k}{T_1^2 s^2 + T_2 s + 1}}$$

$$T_1^2 s^2 Y(s) + T_2 s\, Y(s) + Y(s) = k \cdot U(s)$$

$$Y(s)\left(T_1^2 s^2 + T_2 s + 1\right) = k\, U(s)$$

# Second-order inertial element

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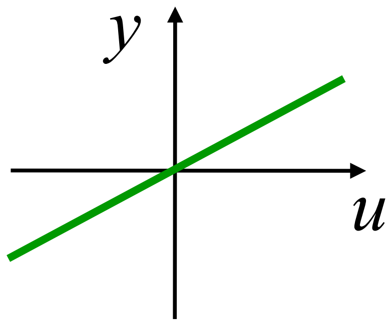


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3. Transfer function: $H(s) = \frac{k}{T_1^2 s^2 + T_2 s + 1}$

---

# Second-order inertial element

4. Step response:

# Second-order inertial element

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input: $u(t) = u_0 1(t)$

Laplace of input: $U(s) = u_0 \dfrac{1}{s}$

Laplace of output: $Y(s) = H(s)U(s) = \boxed{\dfrac{k u_0}{s\left(T_1^2 s^2 + T_2 s + 1\right)}}$

# Second-order inertial element

4. Step response:

input: $u(t)=u_0 1(t)$

Laplace of input: $U(s)=u_0 \frac{1}{s}$

Laplace of output: $Y(s)=H(s)U(s)=\frac{k u_0}{s\left(T_1^2 s^2+T_2 s+1\right)}$

output: $y(t)=L^{-1}\{Y(s)\}=$

$$=\begin{cases} k u_0 \omega_0^2\left(1-e^{-ht}\left(\cos\omega t+\frac{h}{\omega}\sin\omega t\right)\right), & \text{for } h\le\omega_0 \\ k u_0 \omega_0^2\left(1+e^{-ht}\left(\left(\frac{h+w}{2w}-1\right)e^{-wt}-\frac{h+w}{2w}e^{wt}\right)\right), & \text{for } h\ge\omega_0 \end{cases}$$

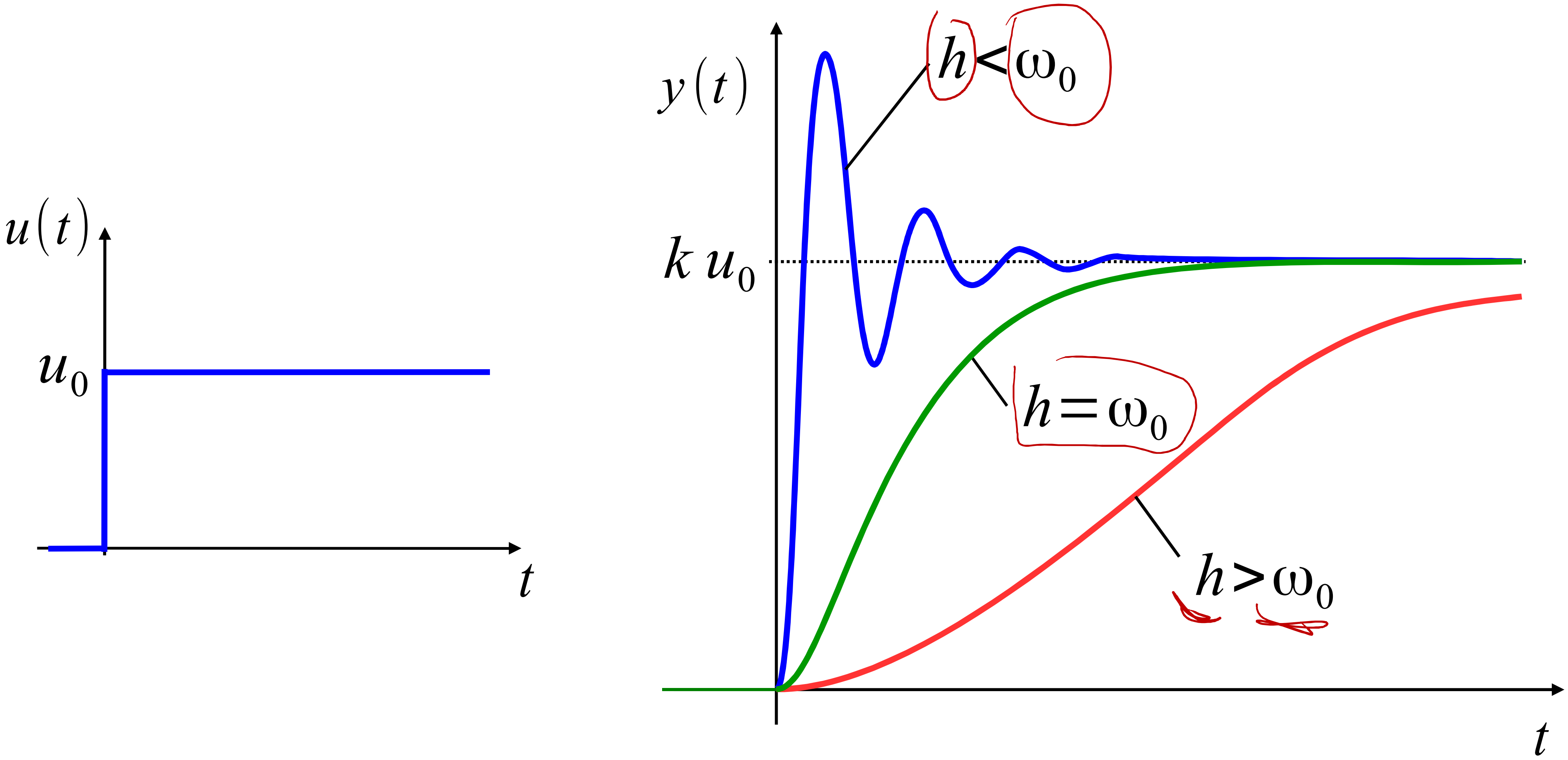
where: $h=\frac{T_2}{2T_1^2}$, $\omega_0=\frac{1}{T_1}$, $\omega=\sqrt{\omega_0^2-h^2}$, $w=\sqrt{h^2-\omega_0^2}$

damping

natural frequency

# Second-order inertial element

4. Step response:



# Second-order inertial element

5. Frequency response:

# Second-order inertial element

5. Frequency response: $H(j\omega)=\dfrac{k}{-T_1^2\omega^2+T_2 j\omega+1}$

$$P(\omega)=\frac{k\left(1-T_1^2\omega^2\right)}{\left(1-T_1^2\omega^2\right)^2+T_2^2\omega^2},\quad Q(\omega)=\frac{-kT_2\omega}{\left(1-T_1^2\omega^2\right)^2+T_2^2\omega^2}$$

# Second-order inertial element

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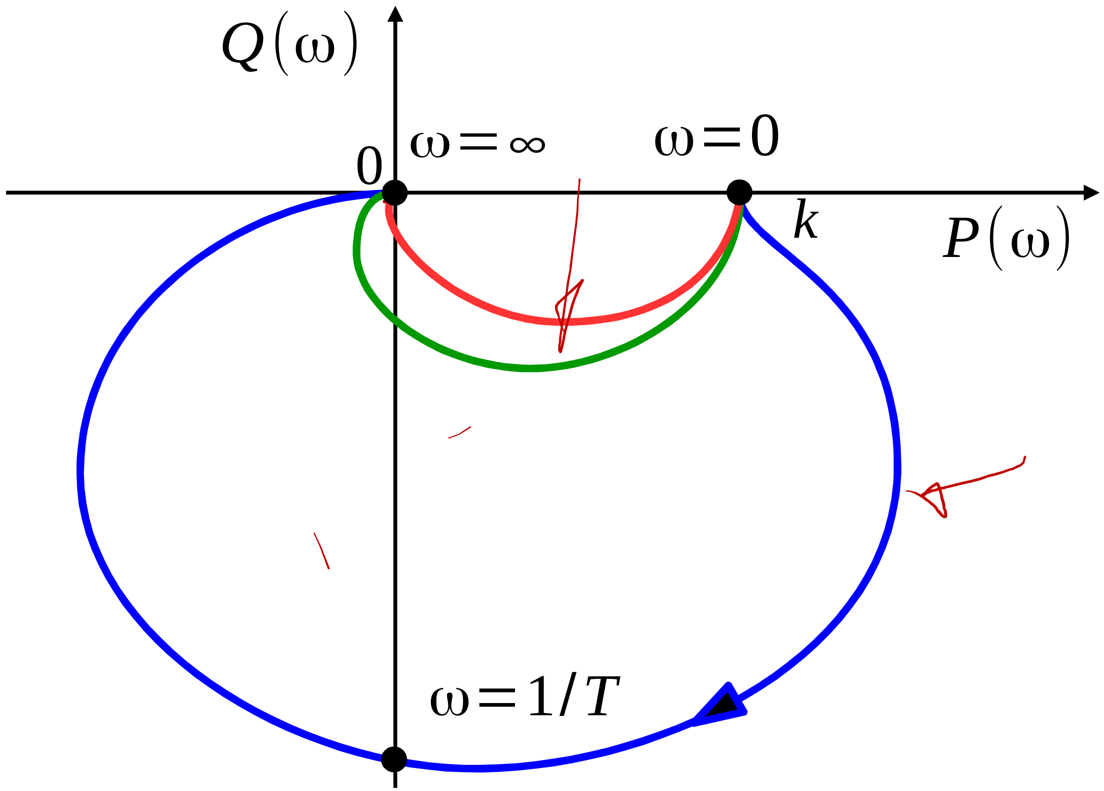
6. Nyquist plot:

for $k>0$




for $h<\omega_0$

for $h=\omega_0$

for $h>\omega_0$

# Second-order inertial element

7. Bode plot: $A(\omega)=\sqrt{P^2+Q^2}$

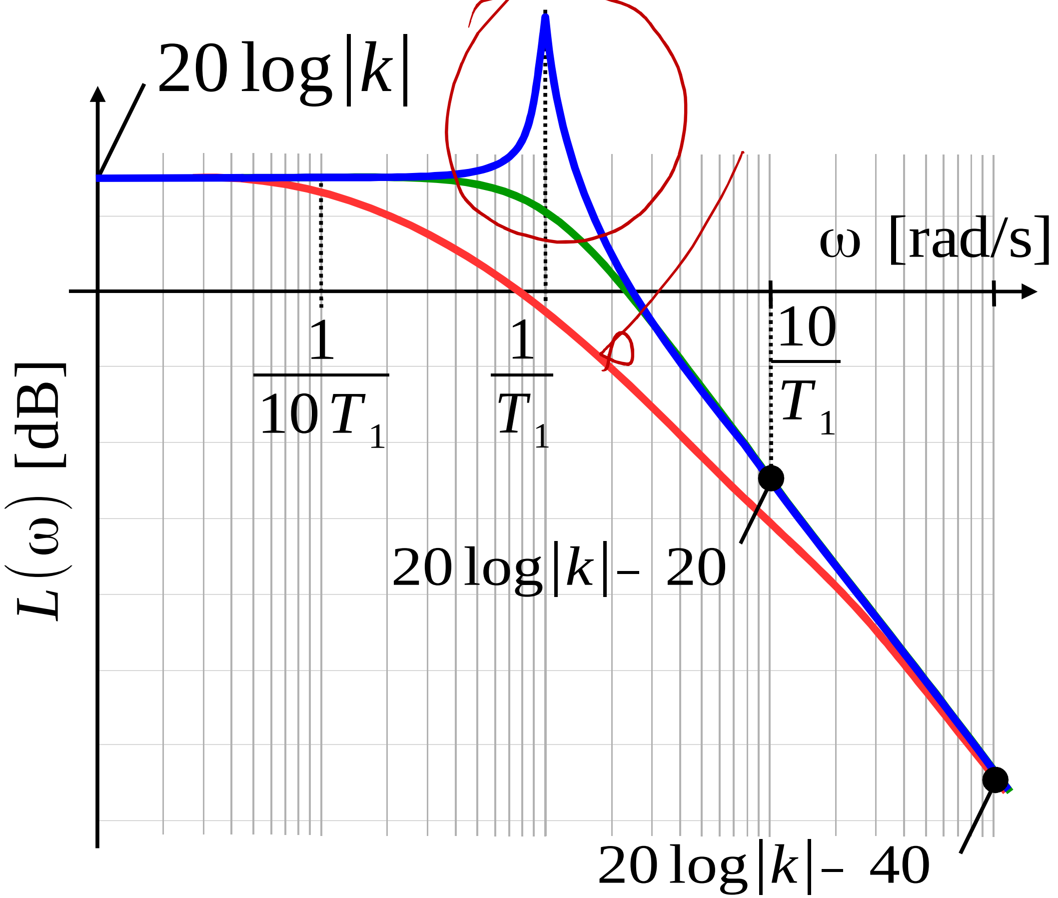
$L(\omega)=20\log A(\omega)$

$\varphi(\omega)=\arctan\dfrac{Q}{P}$

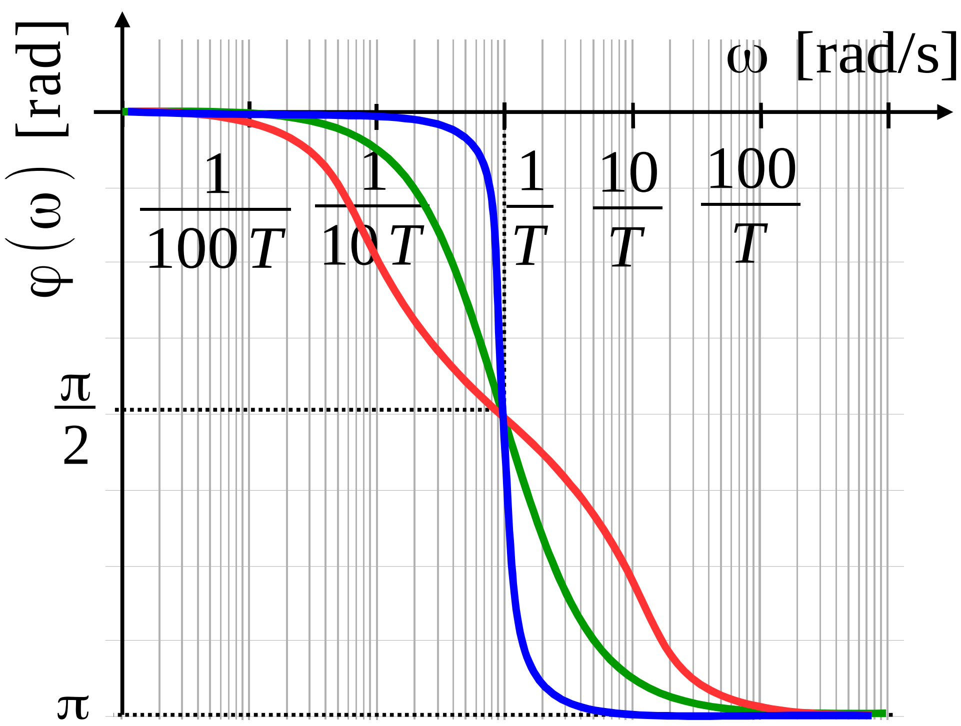
for $h<\omega_0$

for $h=\omega_0$

for $h>\omega_0$



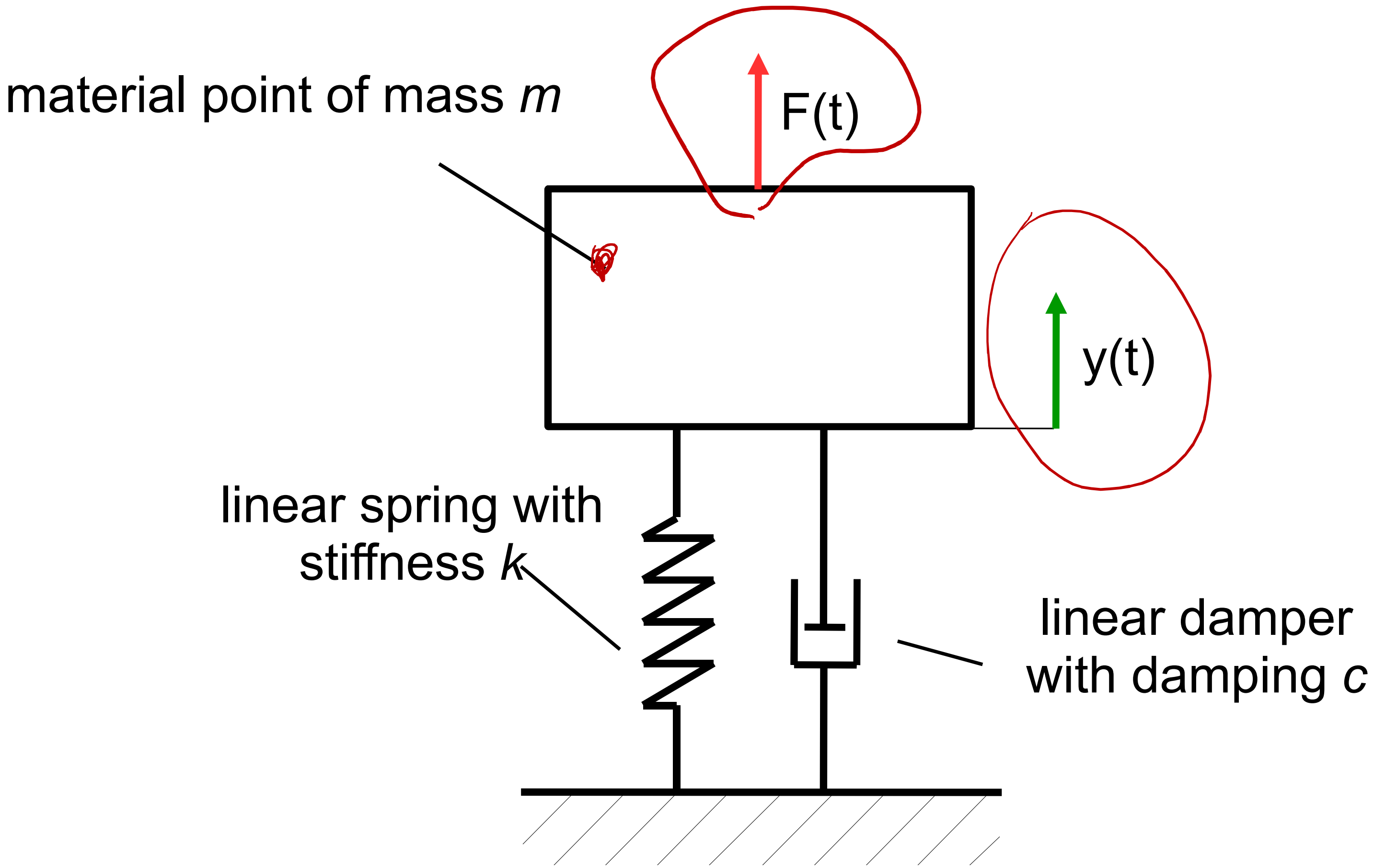

for $k>0$



# Second-order inertial element

Examples

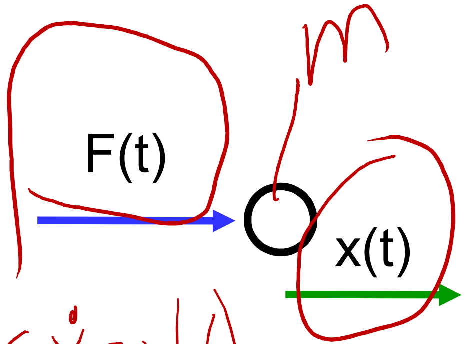
1 VIBRATING SYSTEM:
input – force F(t)
output – displacement y(t)



# Second-order inertial element

Examples

2




$m\ddot{x} + c\dot{x} = x(t)$

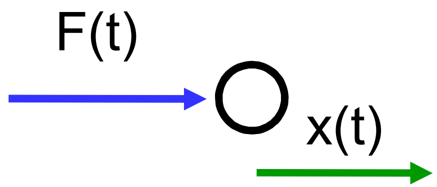
LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
input – force F(t)
output – displacement x(t)

example: car driving on a flat surface with air resistance proportional to velocity, described using machine equation of motion, with assumption of constant reduced mass.

# Second-order inertial element

## Examples

2




LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
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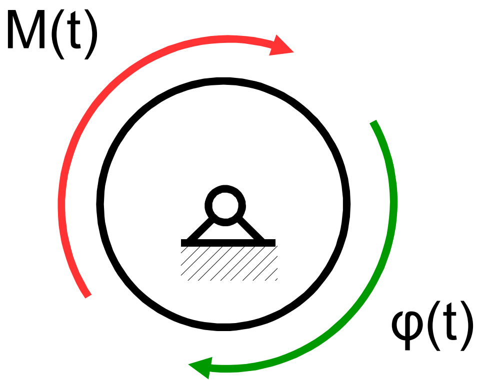
example: car driving on a flat surface with air resistance proportional to velocity, described using machine equation of motion, with assumption of constant reduced mass.

3

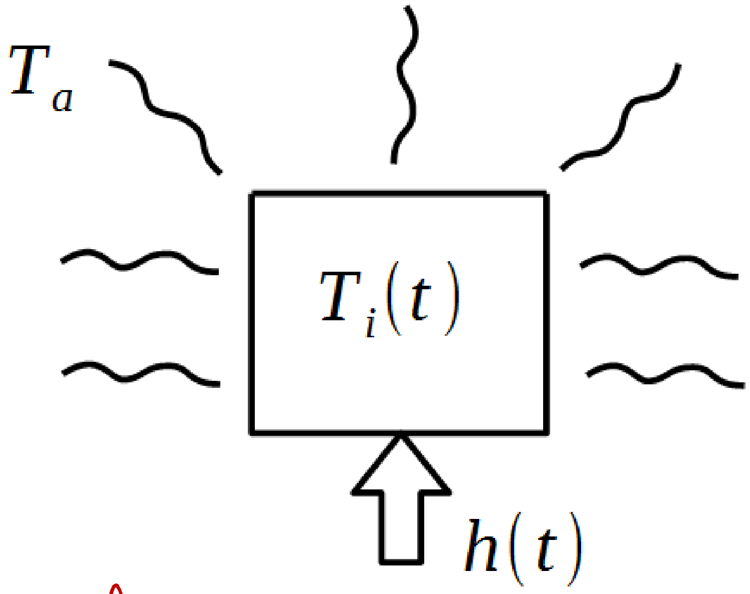



ANGULAR MOTION OF A RIGID BODY WITH LINEAR DAMPING:
input – torque M(t)
output – angle φ(t)

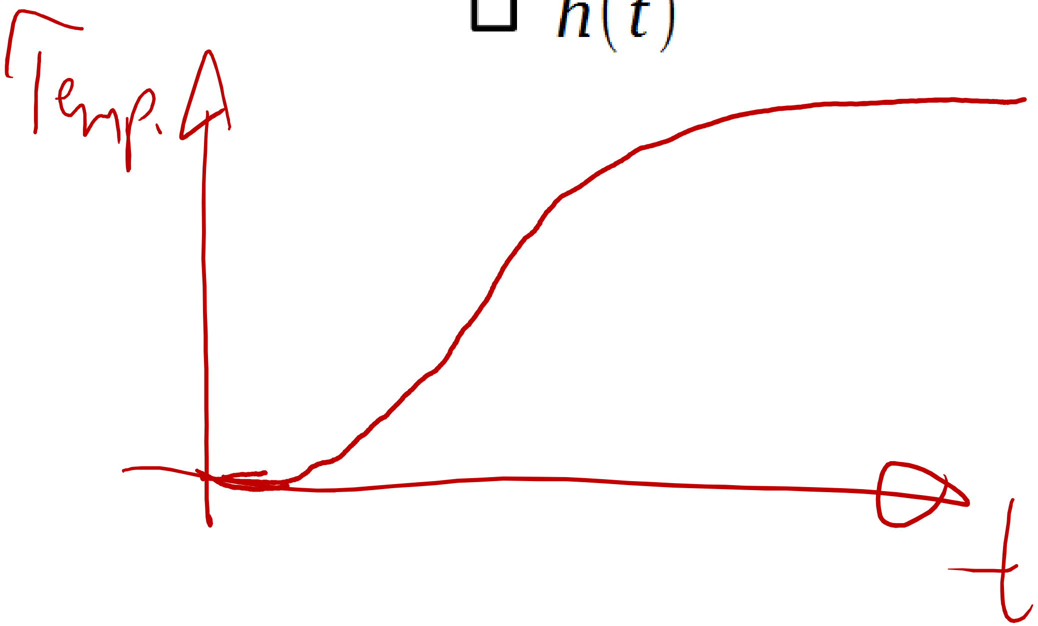
# Second-order inertial element

Examples

4

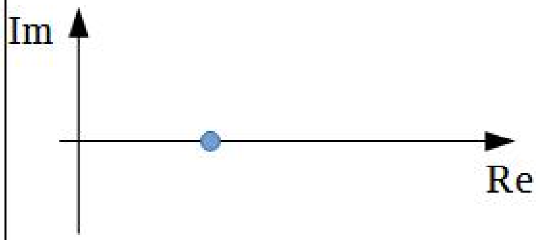
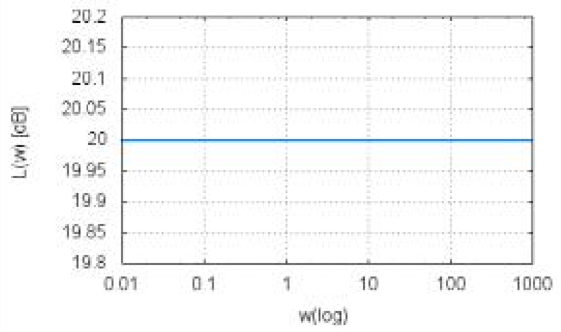
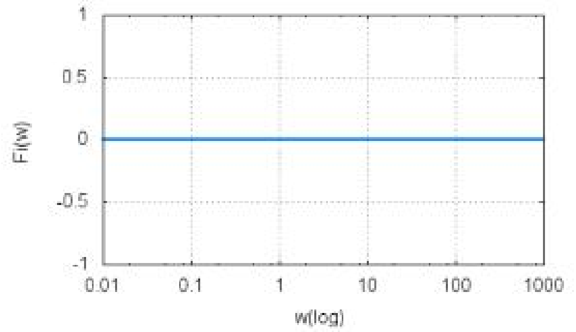
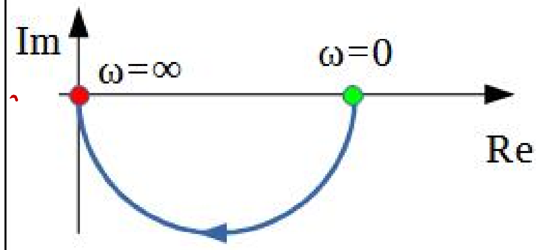
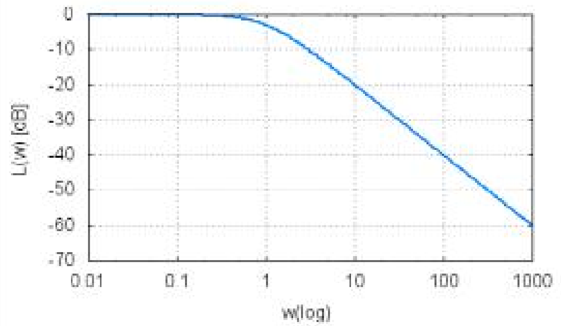
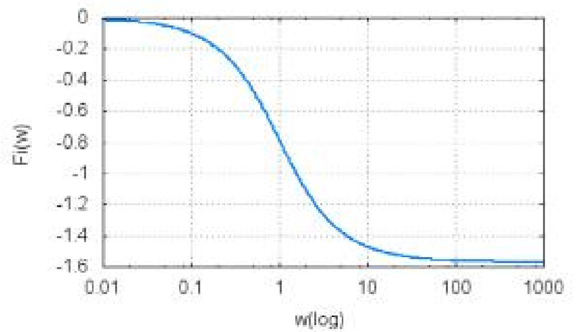
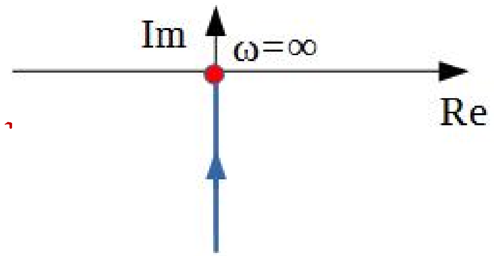
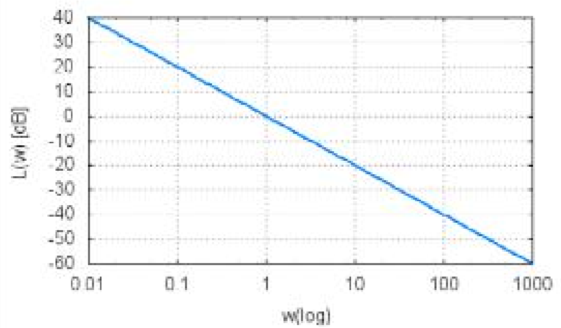
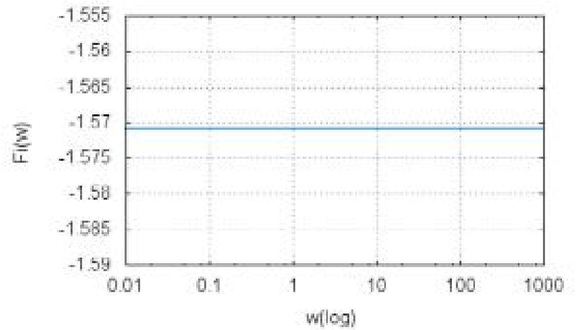
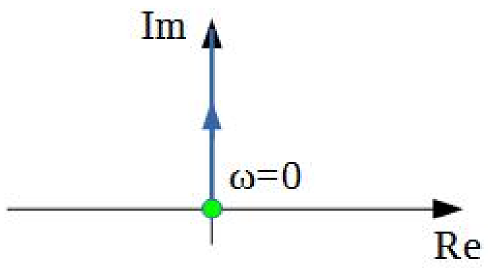
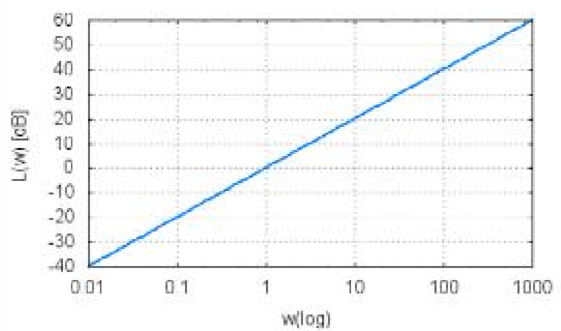
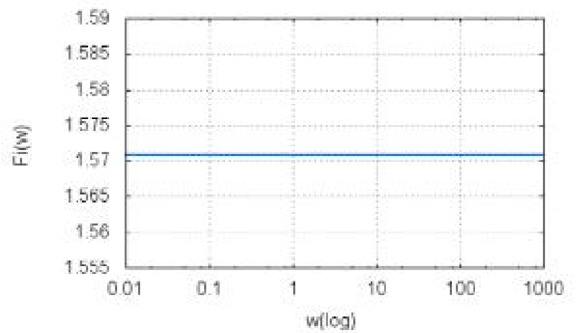


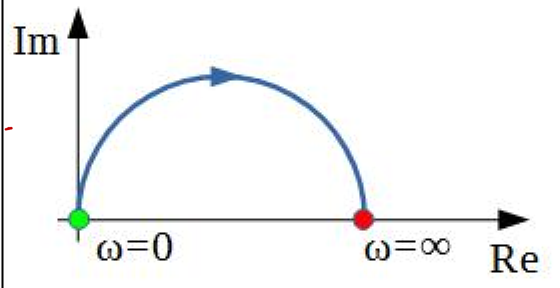
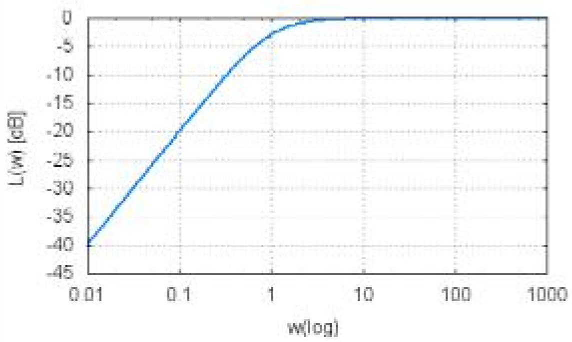
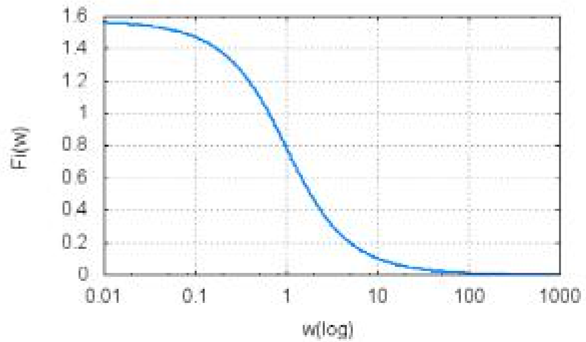
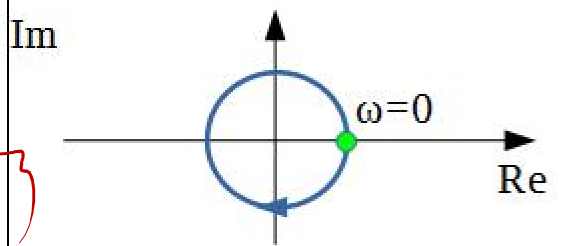
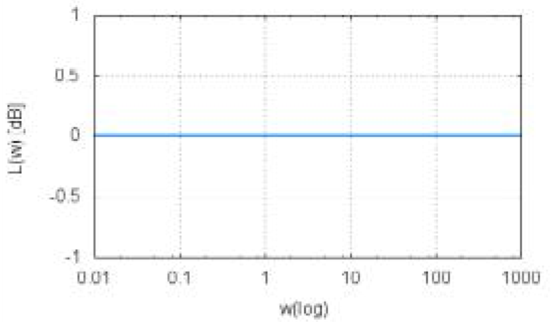
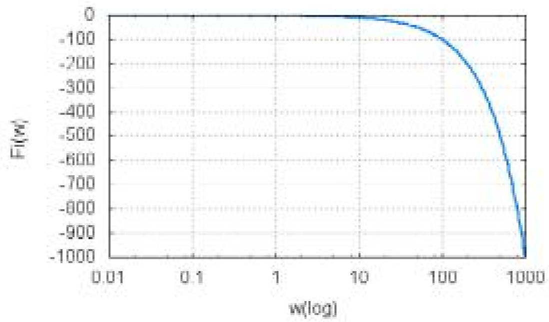
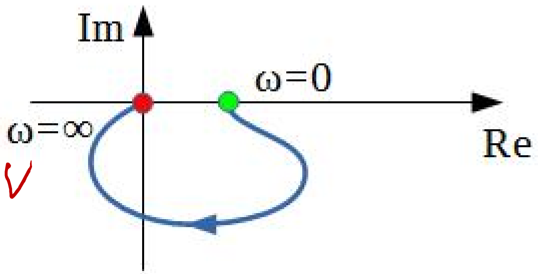
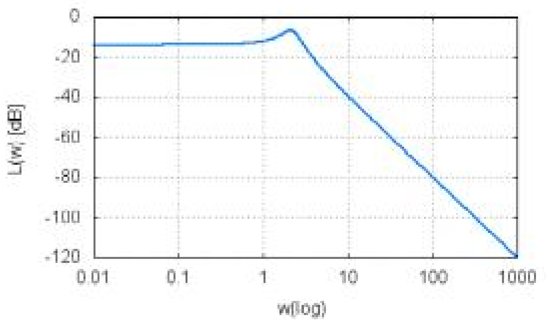
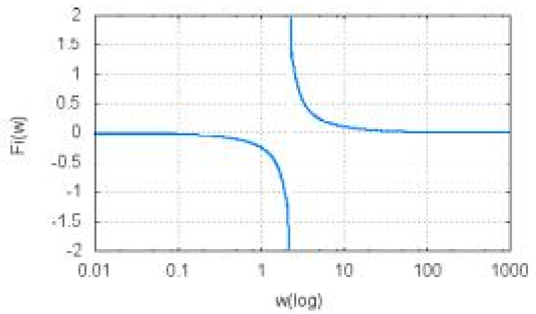

HEATED OBJECT WITH HIGH INERTIA:
input – heater power h(t)
output – object temperature $T_i(t)$



# Classification of basic automatic systems

| Element name | Transfer function |
| --- | --- |
| proportional | $k$ |
| first order (inertial) | $\frac{k}{Ts+1}$ |
| integrator | $\frac{k}{s}$ |
| differentiator | $ks$ |
| differentiator with inertia | $\frac{ks}{Ts+1}$ |
| delay | $e^{-\tau s}$ |
| second order (oscillator) | $\frac{k}{T_1^2 s^2 + T_2 s + 1}$ |

<table>
<tr><th></th><th>Nyquist plot</th><th colspan="2">Bode Plot (example)</th></tr>
<tr><td>Prop.</td><td></td><td></td><td></td></tr>
<tr><td>Iner.</td><td></td><td></td><td></td></tr>
<tr><td>Integ.</td><td></td><td></td><td></td></tr>
<tr><td>Diff.</td><td></td><td></td><td></td></tr>
</table>

| | Nyquist plot | Bode Plot (example) | |
|---|---|---|---|
| Real. Deriv. |  |  |  |
| delay |  |  |  |
| Ideal |  |  |  |