



Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control Winter 2018/2019

Lecturer: Sebastian Korczak, PhD, Eng.

Lecture 8 cont.

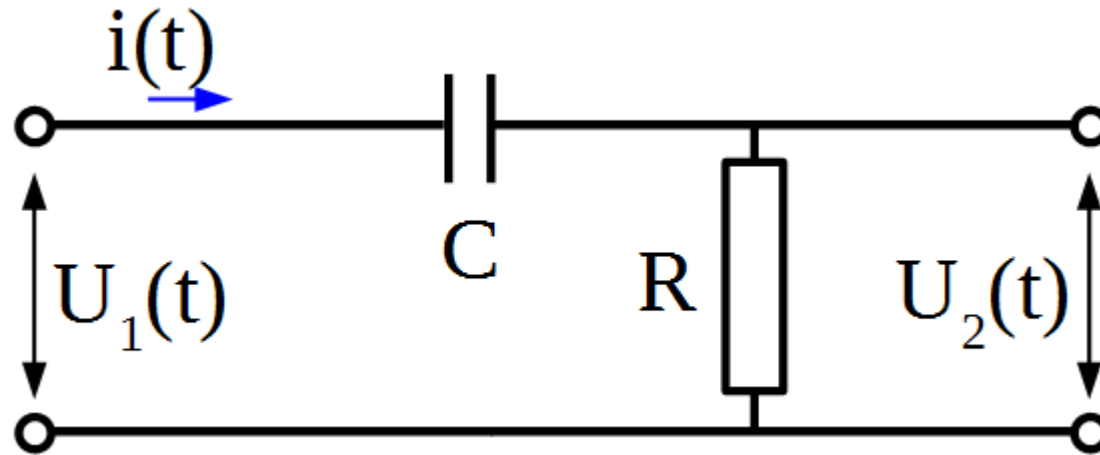
Laplace transform.

Transfer function.

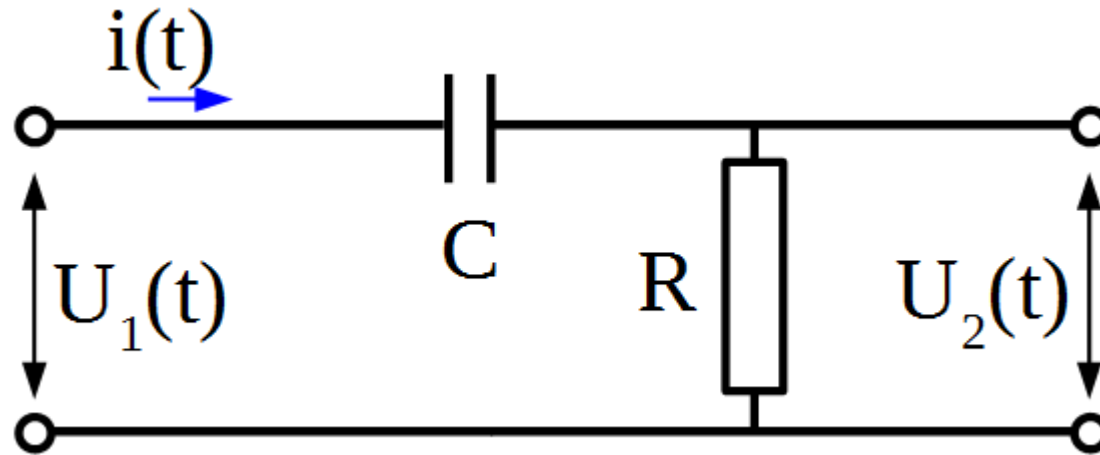
Inputs and outputs in time domain.

Materials license: only for educational purposes of Warsaw University of Technology students.

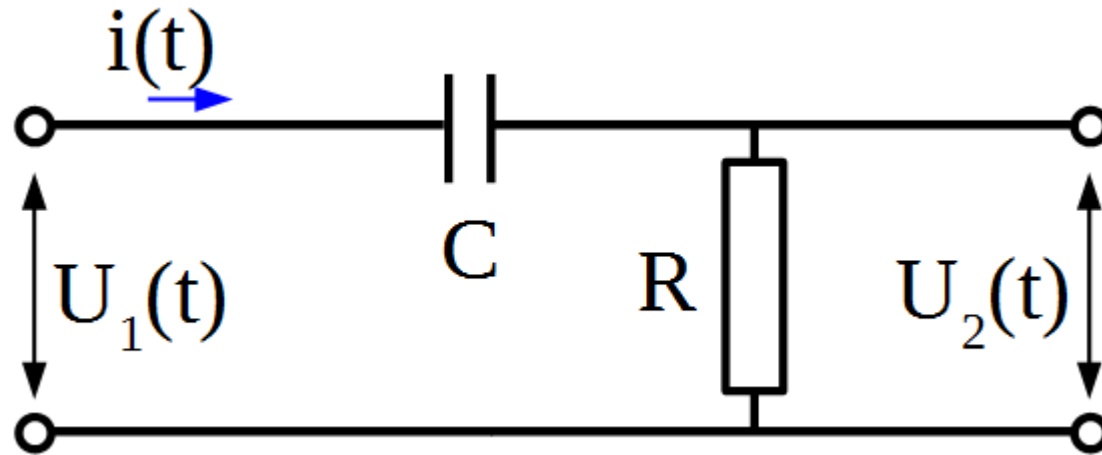
Step response – example 2



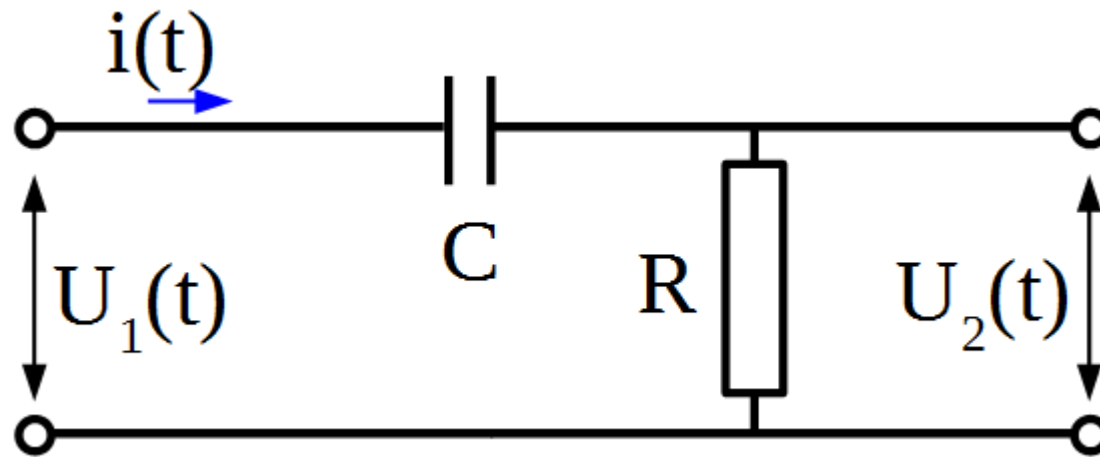
Step response – example 2



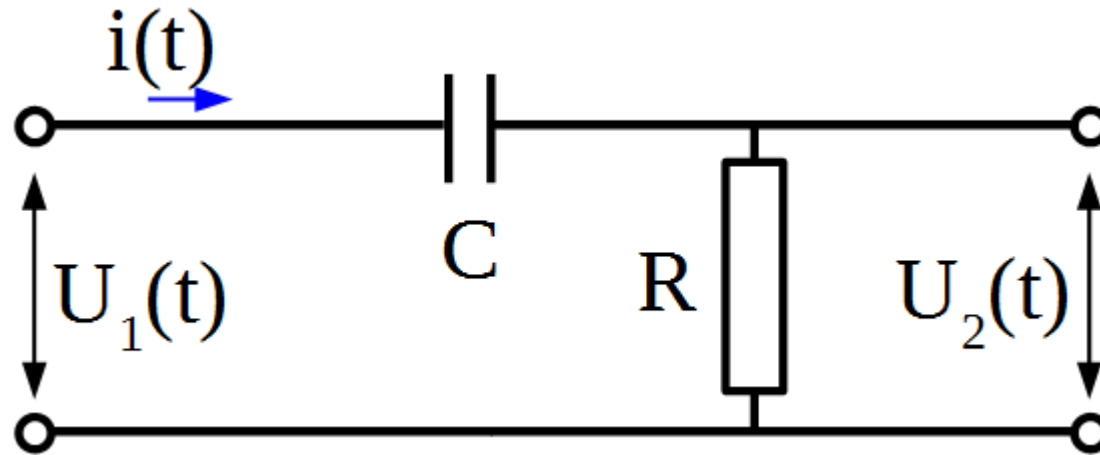
Step response – example 2



Step response – example 2



Step response – example 2

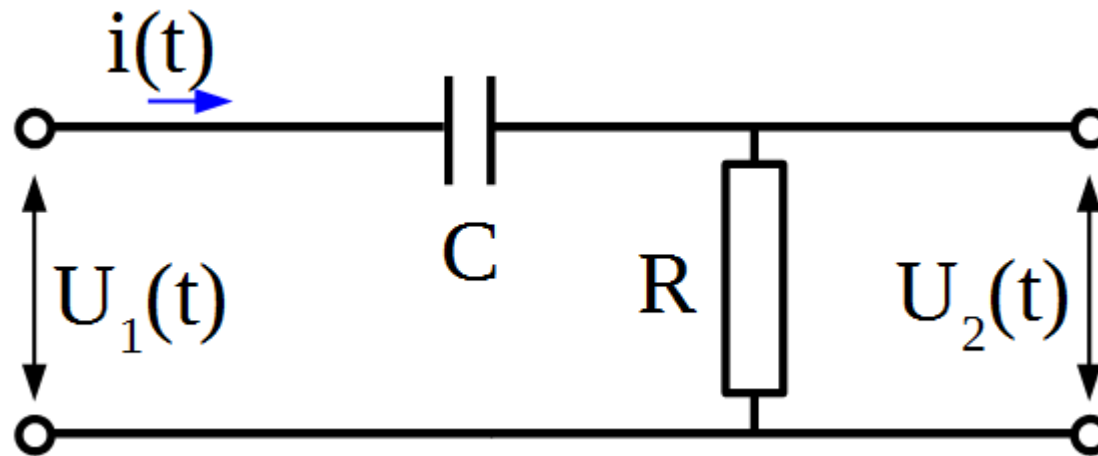


$$u_1(t) = u_C(t) + u_R(t)$$

$$u_C(t) = \frac{q(t)}{C}, \quad u_R(t) = i(t)R, \quad i = \frac{dq}{dt} \quad u_2(t) = u_R(t)$$

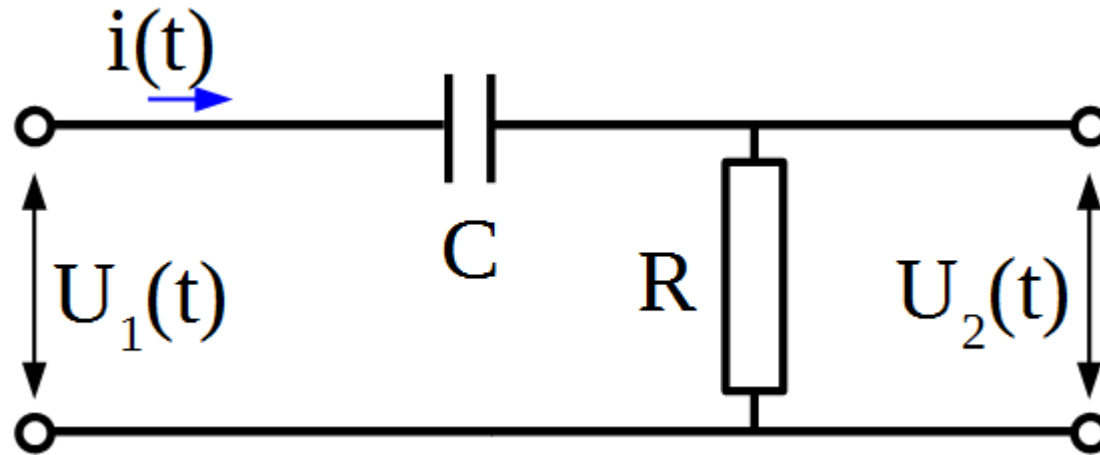
$$u_C(t) = \frac{q(t)}{C} = \frac{1}{C} \int i dt = \frac{1}{C} \int \frac{u_R}{R} dt = \frac{1}{CR} \int u_2 dt$$

Step response – example 2



$$\frac{1}{CR} \int u_2(t) dt + u_2(t) = u_1(t)$$

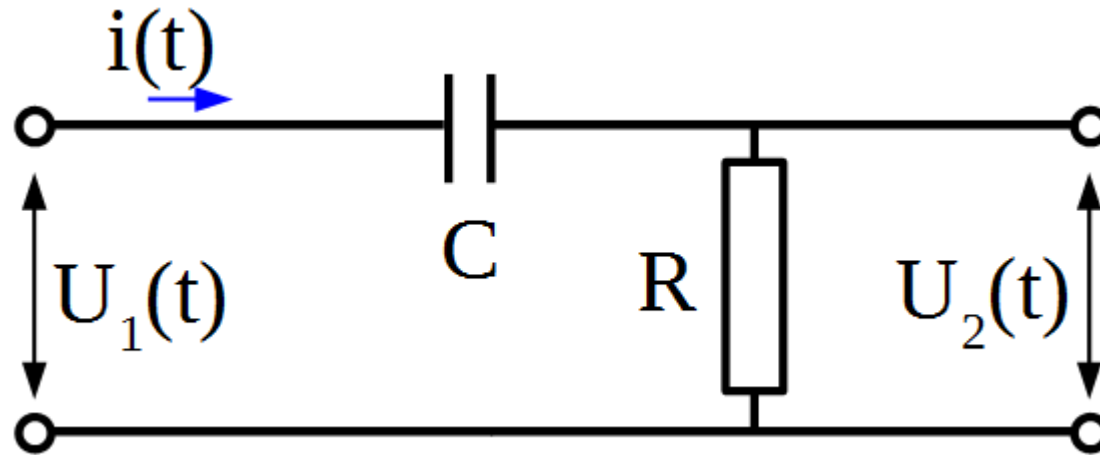
Step response – example 2



$$\frac{1}{CR} \int u_2(t) dt + u_2(t) = u_1(t)$$

$$\frac{1}{CR} u_2(t) + \frac{du_2(t)}{dt} = \frac{du_1(t)}{dt}$$

Step response – example 2

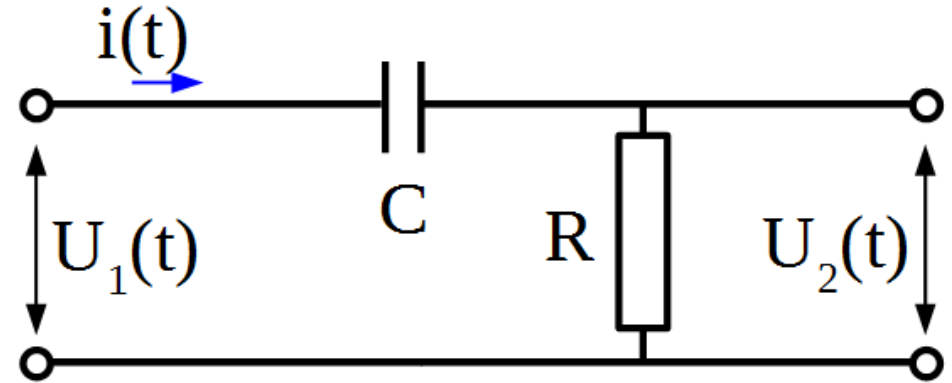


$$\frac{1}{T}U_2(s) + sU_2(s) = sU_1(s) \quad T = CR$$

$$G(s) = \frac{U_2(s)}{U_1(s)} = \frac{Ts}{1 + Ts}$$

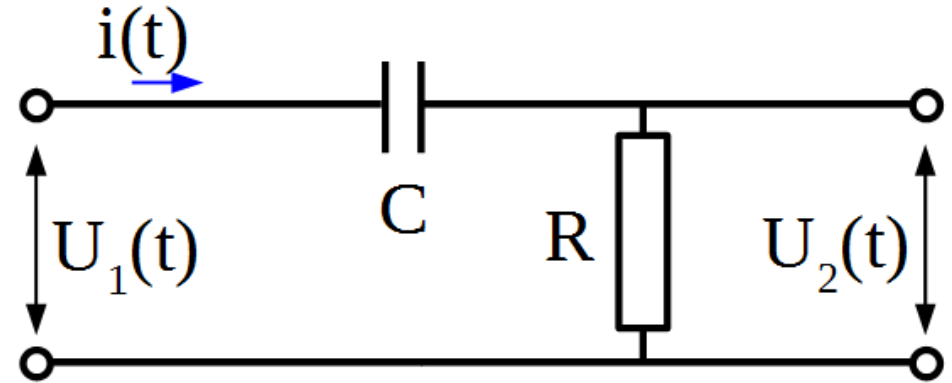
Step response – example 2

$$G(s) = \frac{Ts}{1 + Ts}$$



Step response – example 2

$$G(s) = \frac{Ts}{1 + Ts}$$



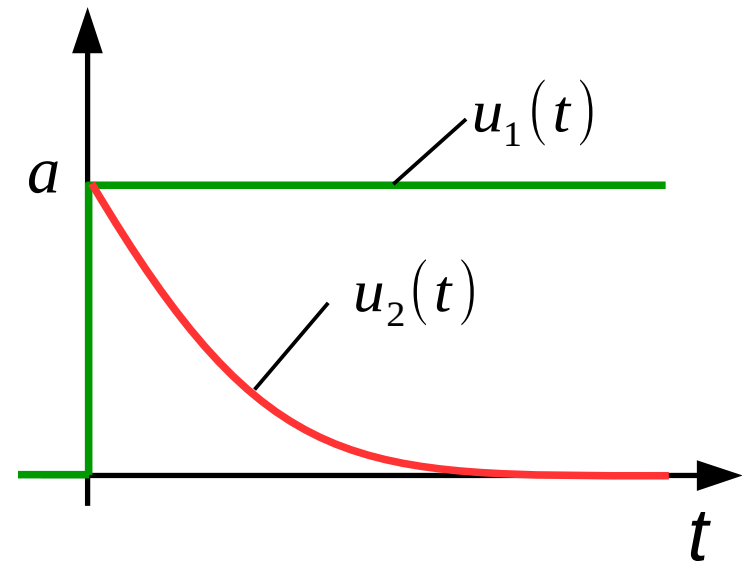
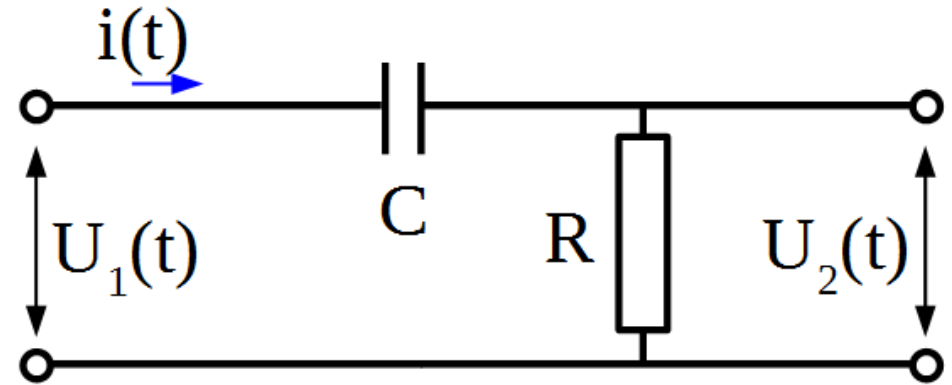
Step response – example 2

$$G(s) = \frac{Ts}{1+Ts}$$

$$u_1(t) = a \cdot 1(t),$$

$$U_2(s) = U_1(s) \cdot G(s) = a \frac{1}{s + \frac{1}{T}}$$

$$u_2(t) = L^{-1}[U_2(s)] = ae^{-\frac{t}{T}}$$



Computer methods for transfer function analysis

Exemplary computer algebra systems:

- Maxima/wxMaxima (free and open source)
- Wolfram Mathematica (<http://www.wolfram.com/mathematica/>)
- Mathcad
- Website: www.wolframalpha.com

(en.wikipedia.org/wiki/List_of_computer_algebra_systems)

Spreadsheet for graphs (Excel, LibreOffice Calc)

WolframAlpha



transfer function $(8*s+4)/(2*s^4+7*s^3+11*s^2+19*s+6)$



[Examples](#) [Random](#)

Input interpretation:

systems model

transfer function
$$\frac{4 + 8 s}{6 + 19 s + 11 s^2 + 7 s^3 + 2 s^4}$$

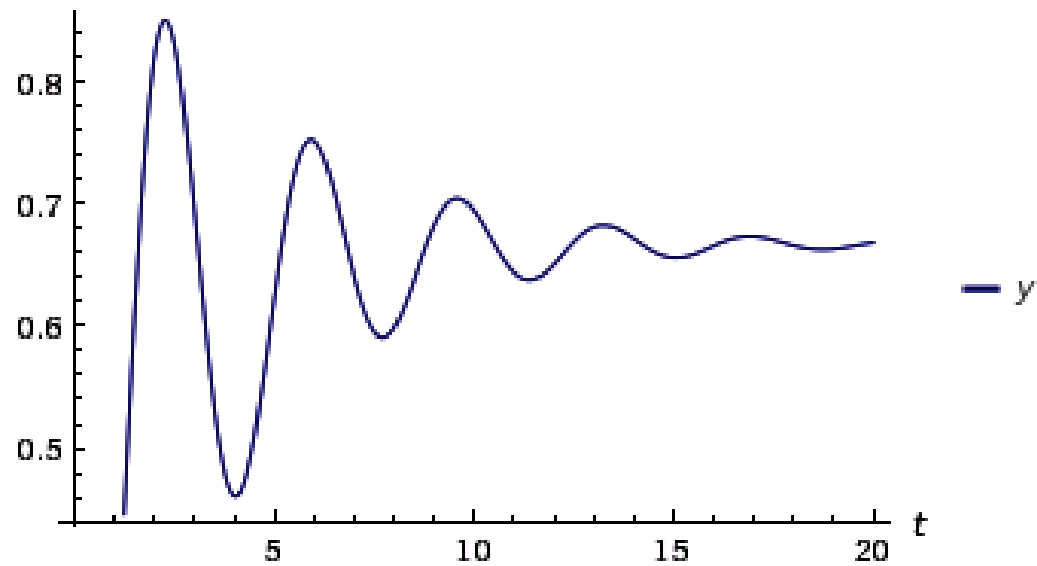
WolframAlpha

Unit step response plot:

Less time

More time

Unit step ▼



Lecture 9

Frequency response. Classification of basic automatic systems.

Materials license: only for educational purposes of Warsaw University of Technology students.

Transfer function

Linear time-invariant SISO system for continuous-time input signal $x(t)$ and output $y(t)$ in a form

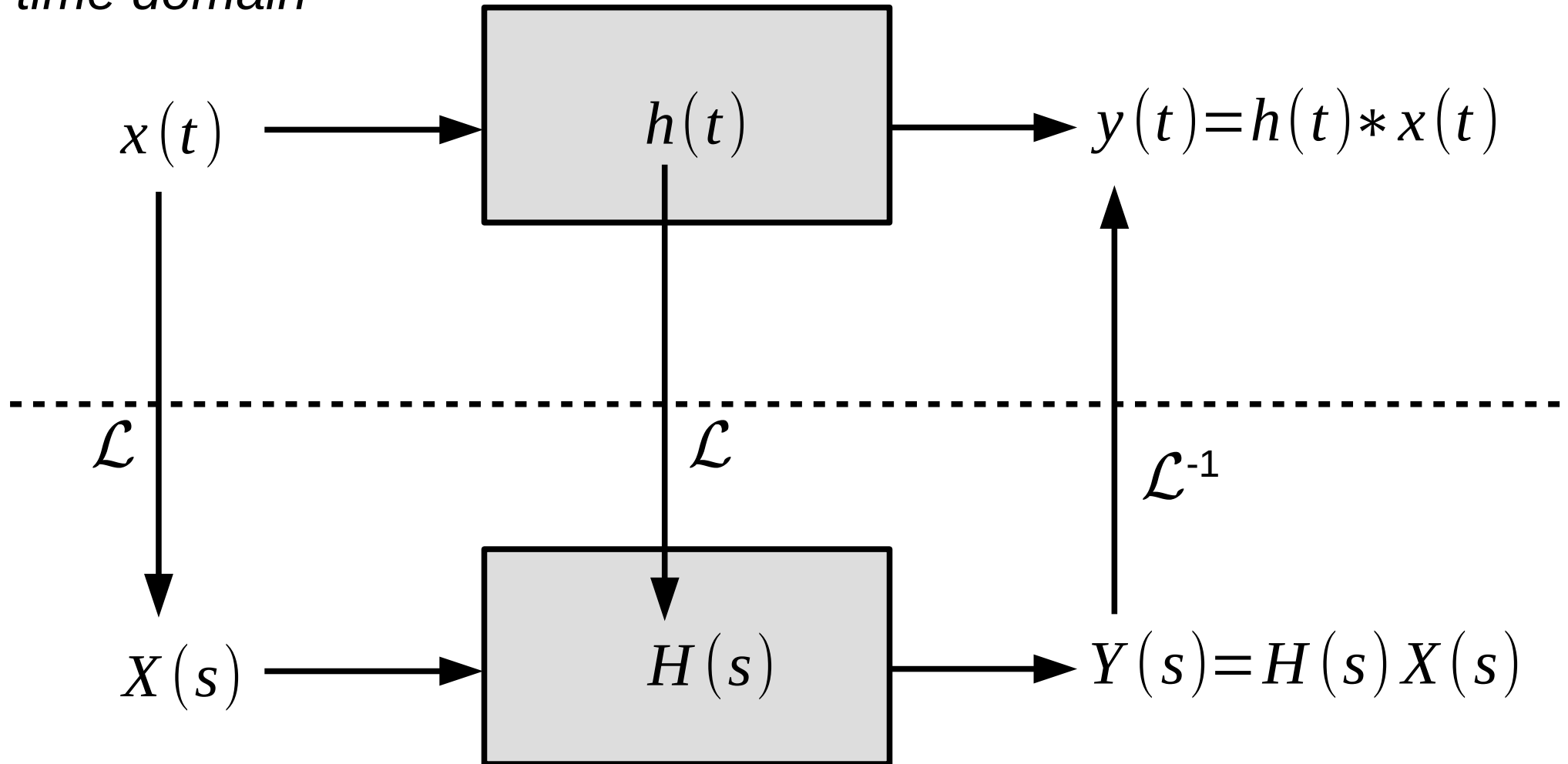
$$\text{Transfer function } H(s) = \frac{Y(s)}{X(s)}$$

$Y(s)$ - Laplace's transform of an output

$X(s)$ - Laplace's transform of an input

Input and output

time domain



complex domain

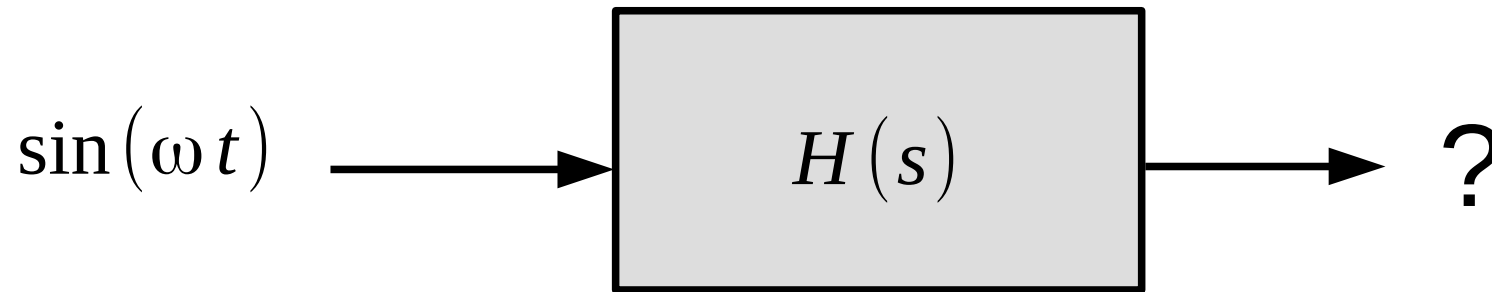
Transfer function & frequency response

Transfer function
(Laplace domain)

$$H(s)$$

Full system description
(for every possible input)

Transfer function & frequency response



Transfer function & frequency response



Transfer function & frequency response

Transfer function
(Laplace domain)

$$H(s)$$

Full system description
(for every possible input)

$s = j\omega$

Frequency response
(Fourier domain)

$$H(j\omega)$$

Description of a system in
steady state with harmonic
input

Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$
harmonic excitation *steady state!!!*

Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s = j\omega} \blacktriangleright$$

Transfer function – frequency response

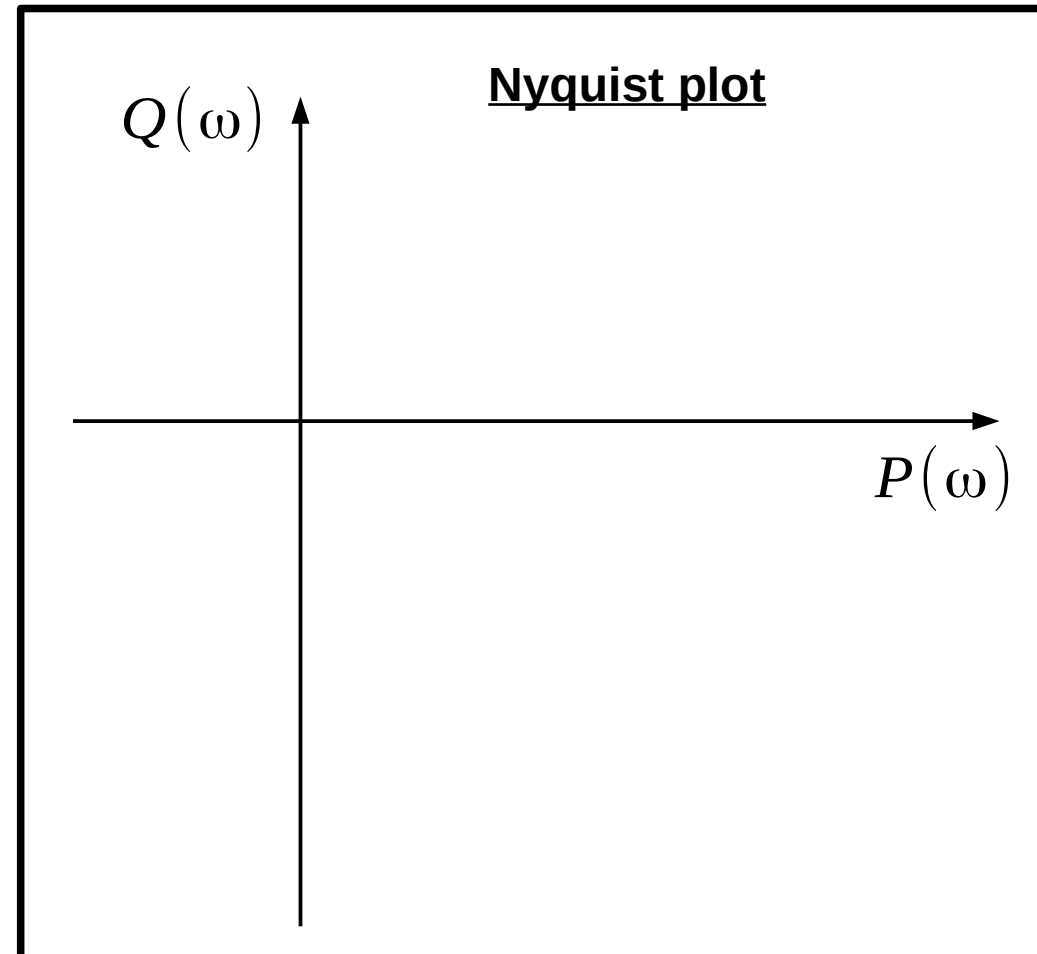
input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

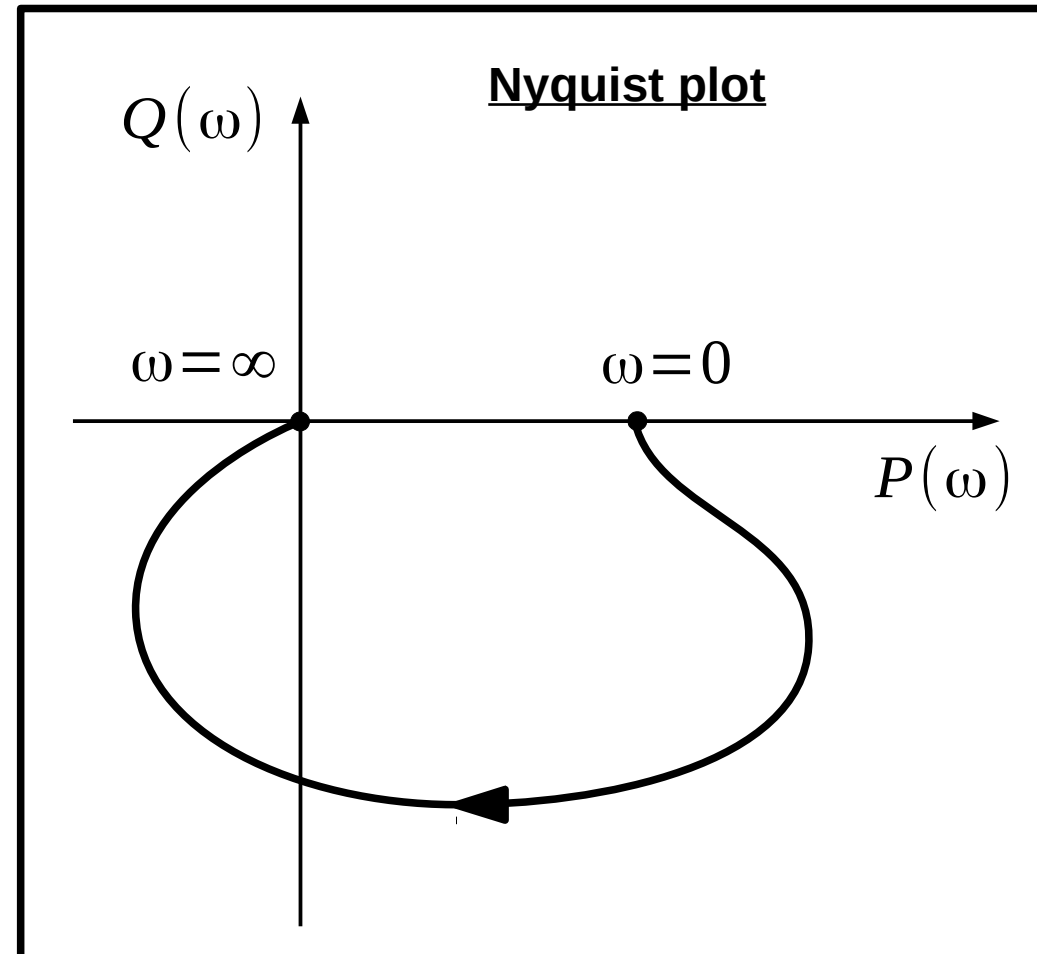
$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$



Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

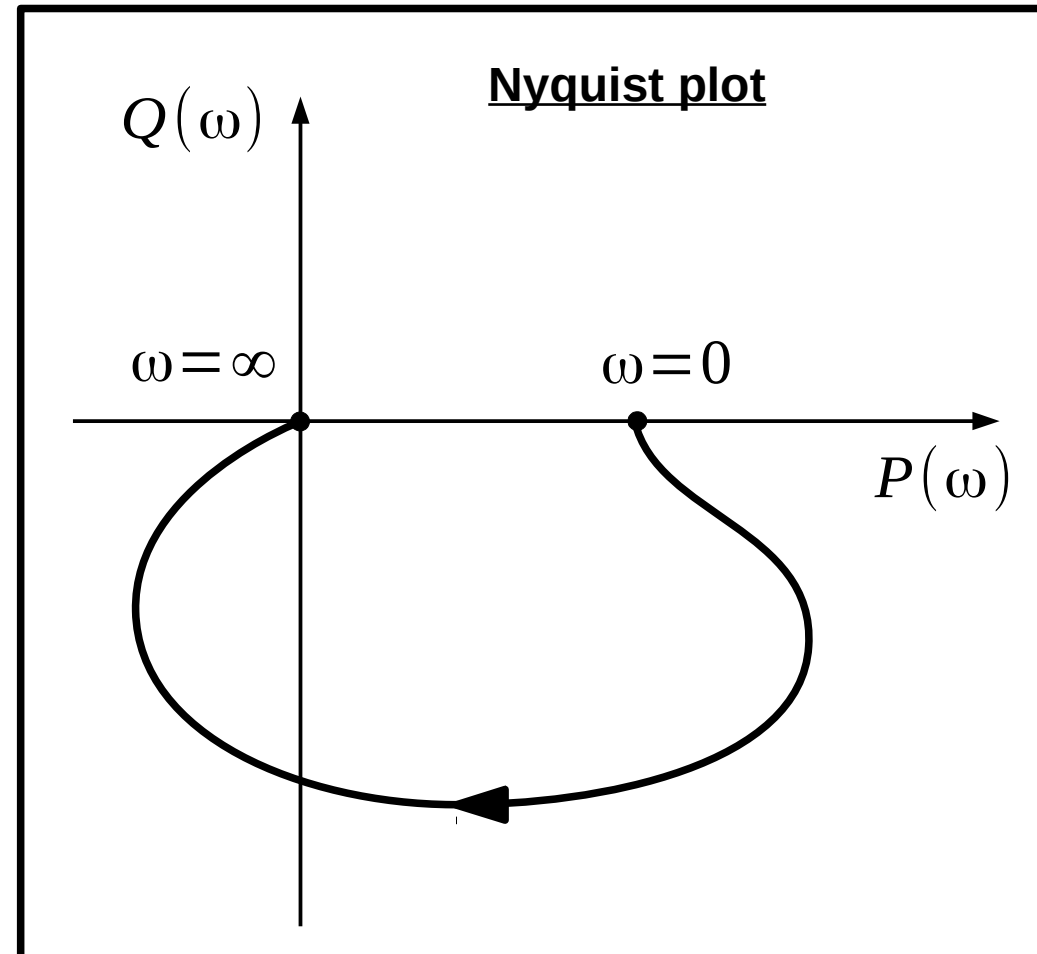


Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

$$A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$



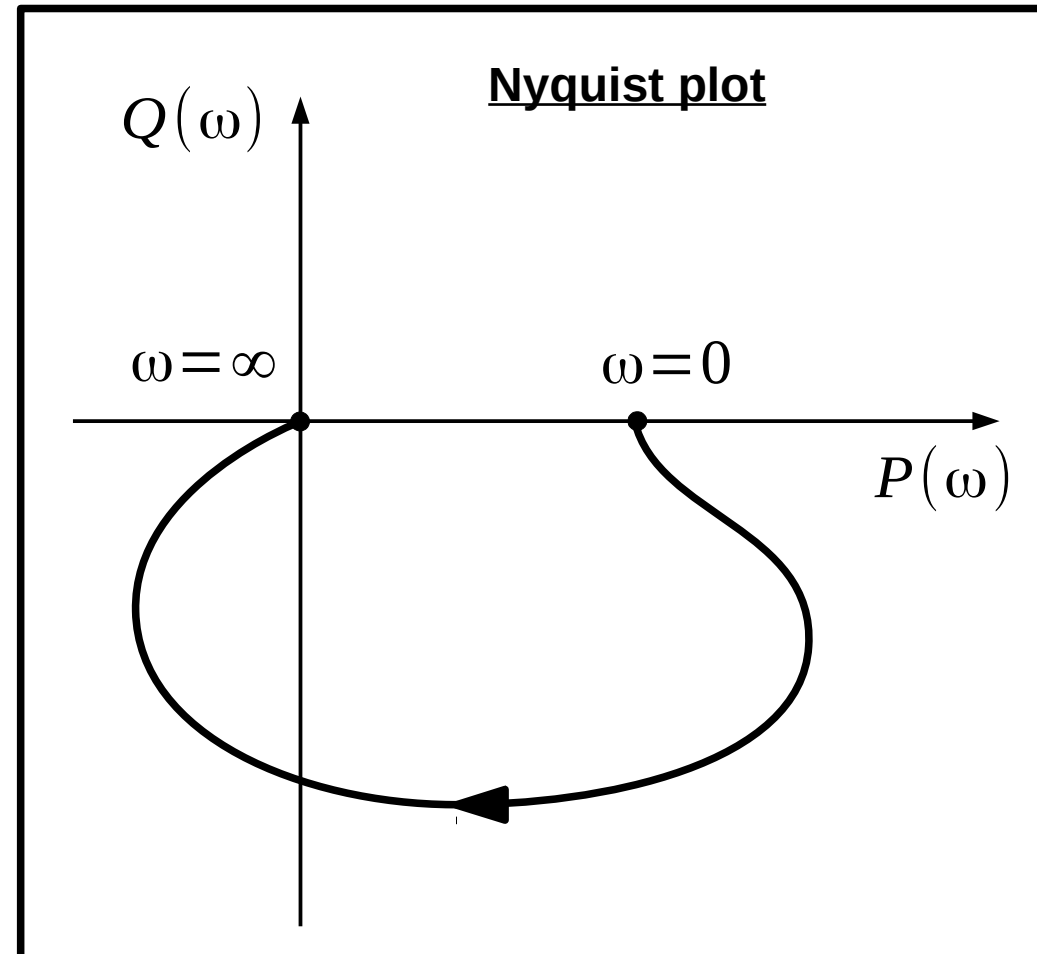
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

$$A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$

$$\varphi(\omega) = \text{Arg } H(j\omega) = \arctan \frac{Q}{P}$$



Transfer function – frequency response

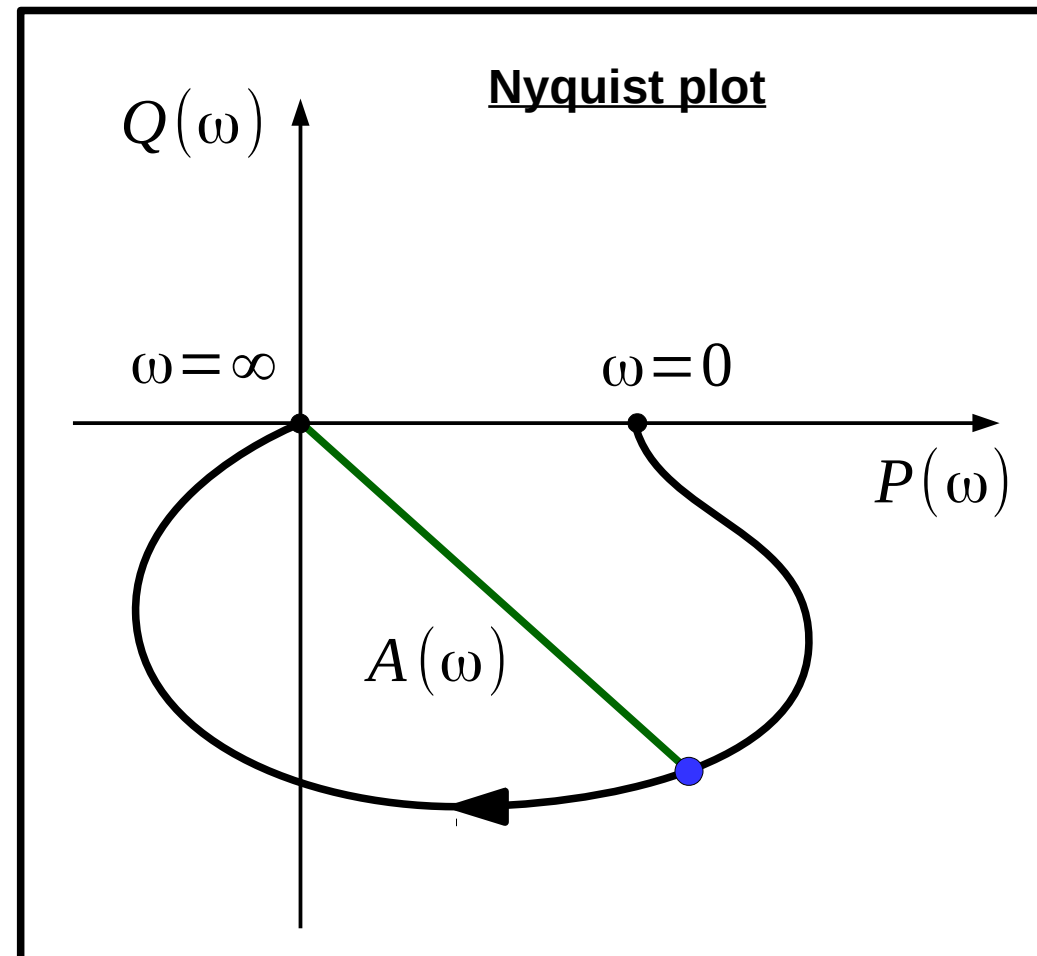
input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

$$A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$

GAIN

$$\varphi(\omega) = \text{Arg } H(j\omega) = \arctan \frac{Q}{P}$$



Transfer function – frequency response

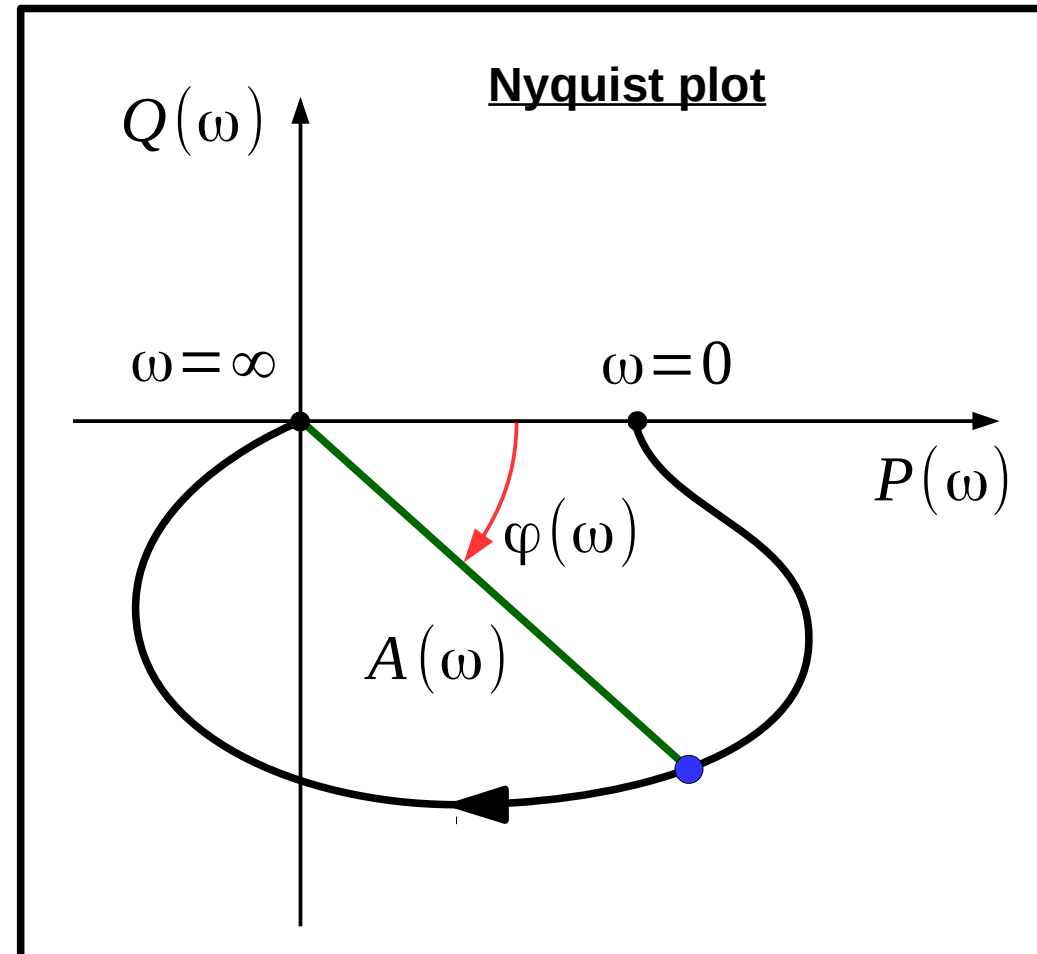
input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

$$H(s) \xrightarrow{s=j\omega} H(j\omega) = P(\omega) + jQ(\omega)$$

$$A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)}$$

$$\varphi(\omega) = \text{Arg } H(j\omega) = \arctan \frac{Q}{P}$$

DELAY



Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

Bode Plot



Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$

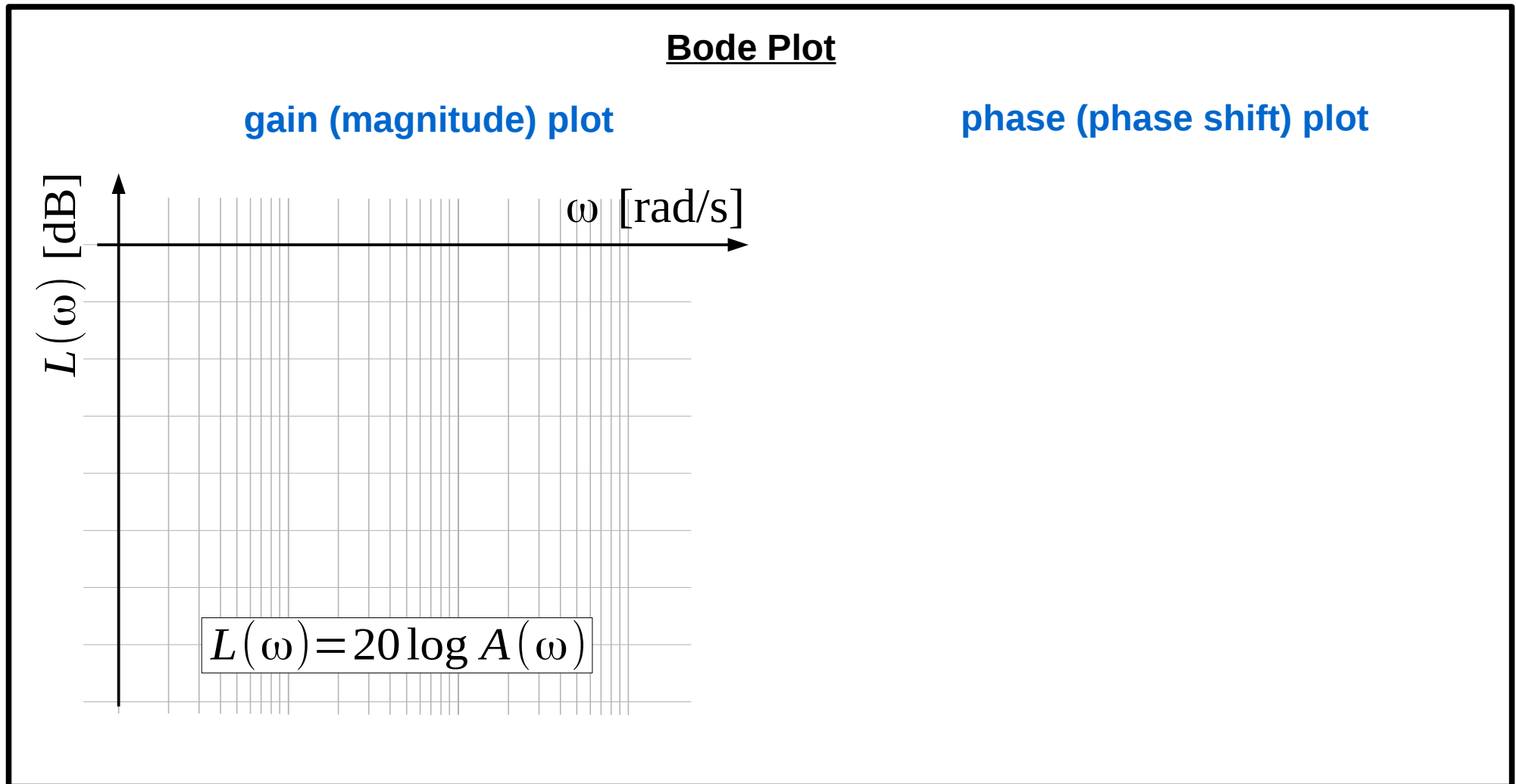
Bode Plot

gain (magnitude) plot

phase (phase shift) plot

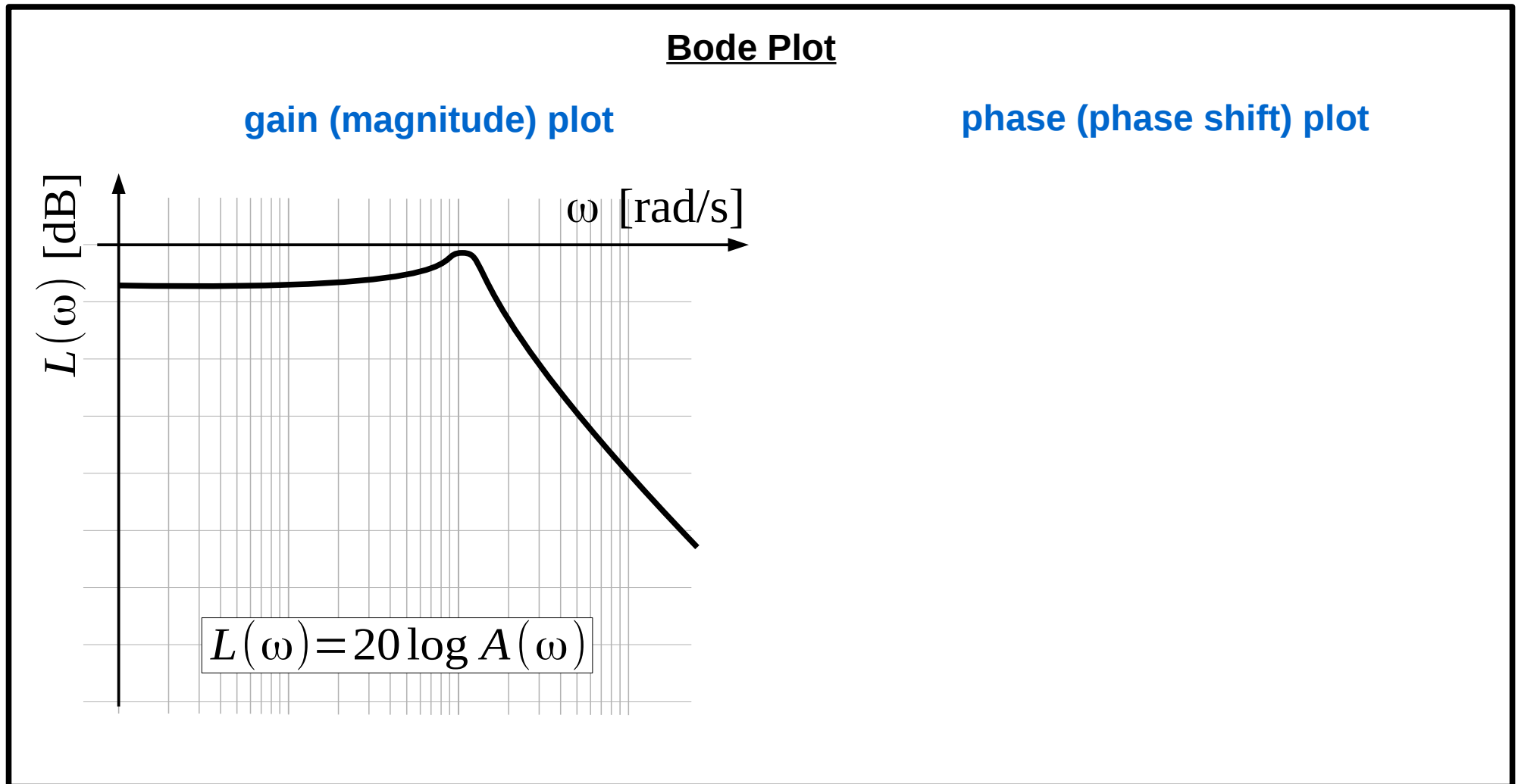
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$



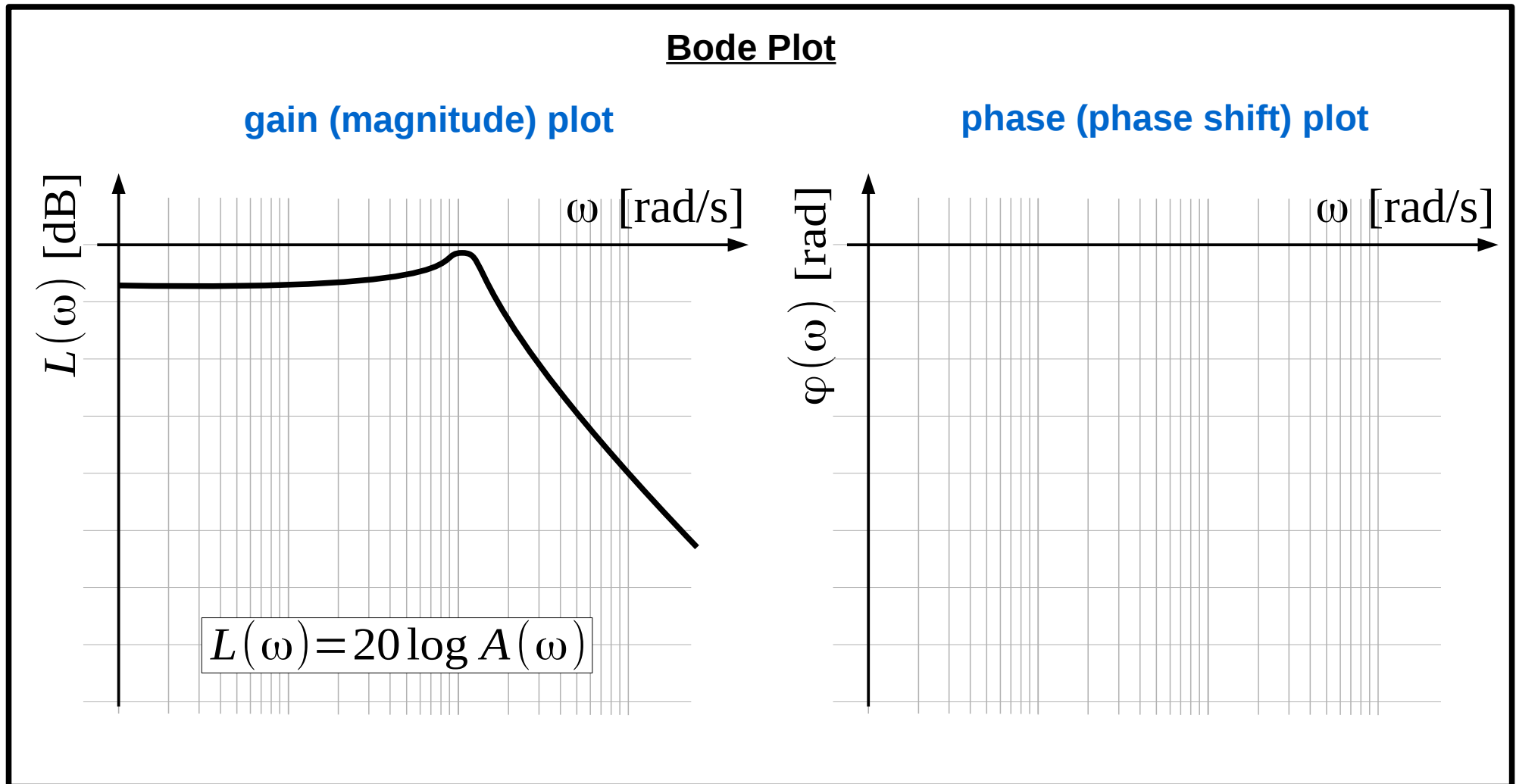
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$



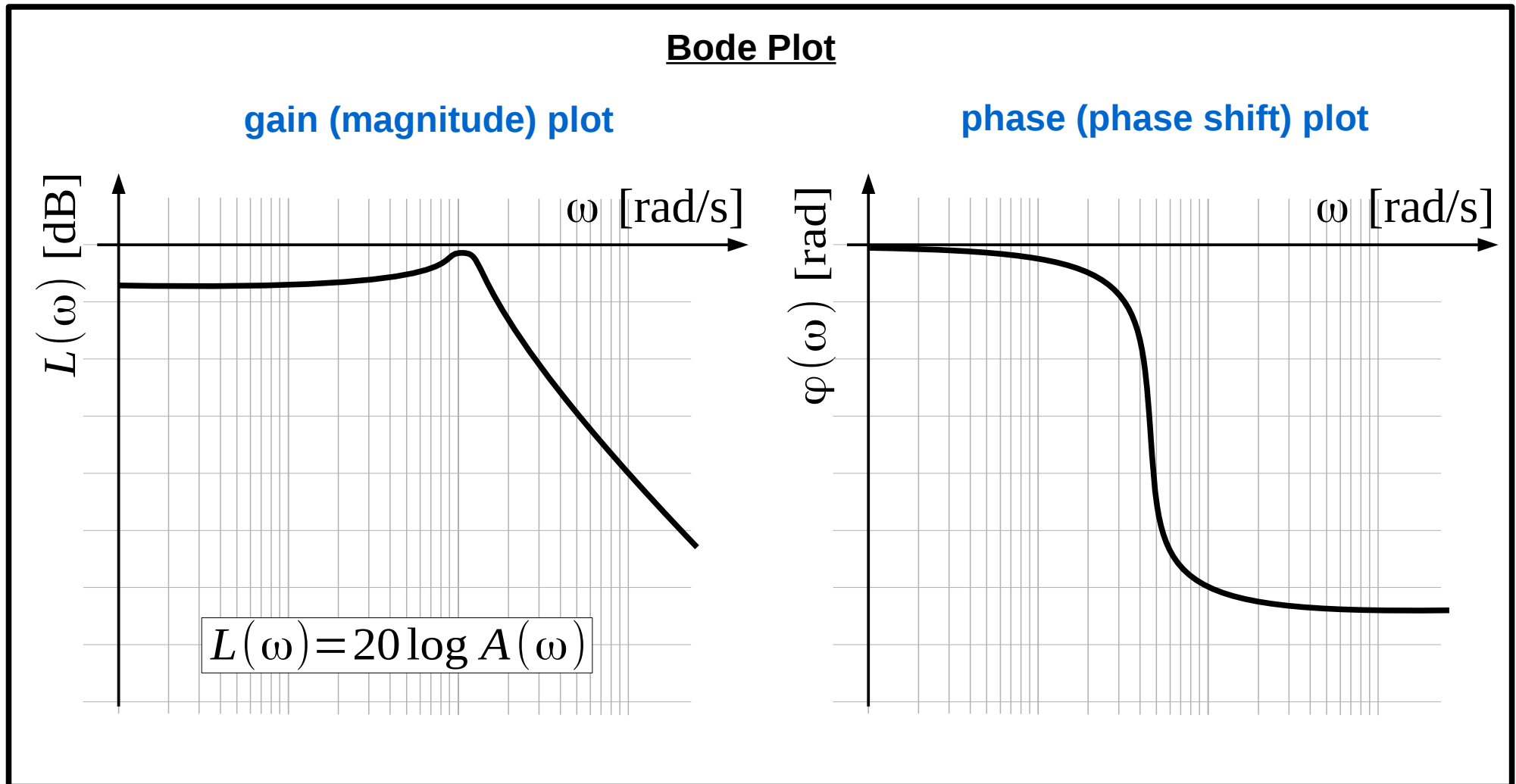
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$



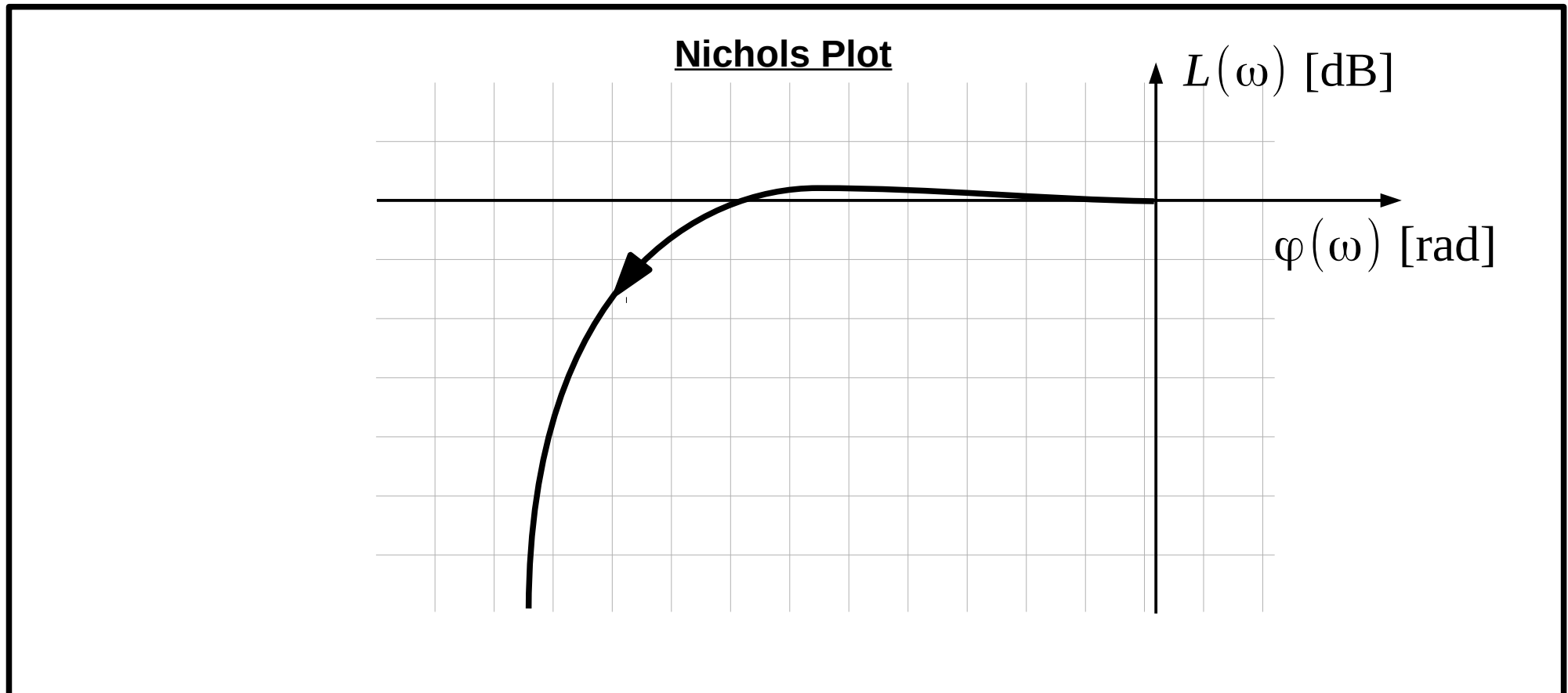
Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$



Transfer function – frequency response

input: $x(t) = \sin(\omega t)$ transfer function: $H(s)$ output: $y(t) = A \sin(\omega t + \varphi)$



Transfer function – frequency response

A (gain)	20logA [dB]
1000	60
100	40
10	20
1	0
0.1	-20
0.01	-40
0.001	-60

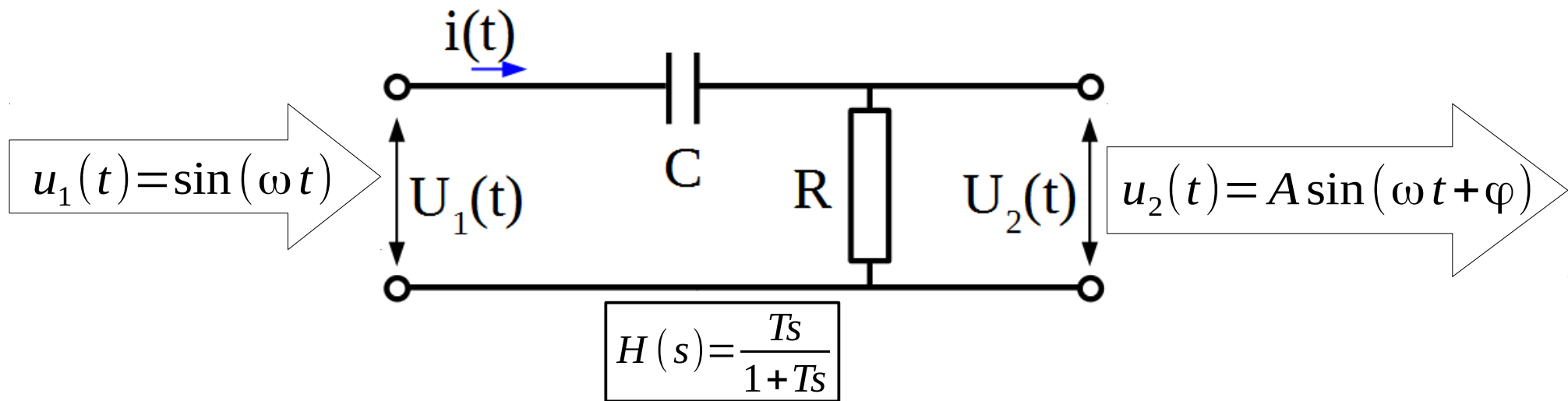
Transfer function – frequency response

Useful for:

- audio components (amplifiers, microphones, loudspeakers, cables)
- wireless components (antennas, amplifiers)
- vibrating systems (suspensions, drivetrains)
- control systems (regulators, objects)

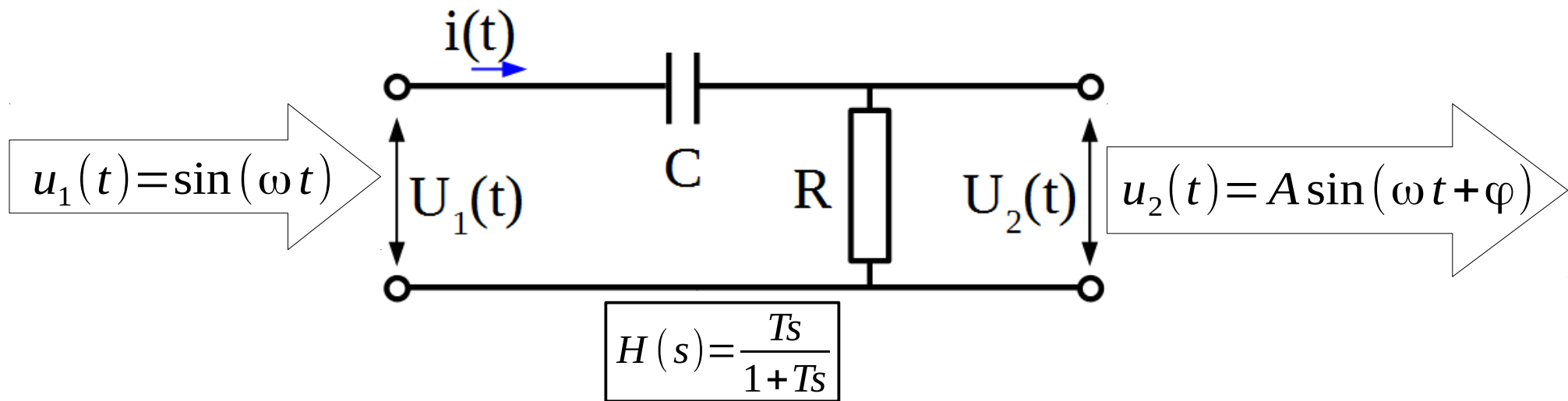
Transfer function – frequency response

RC circuit example



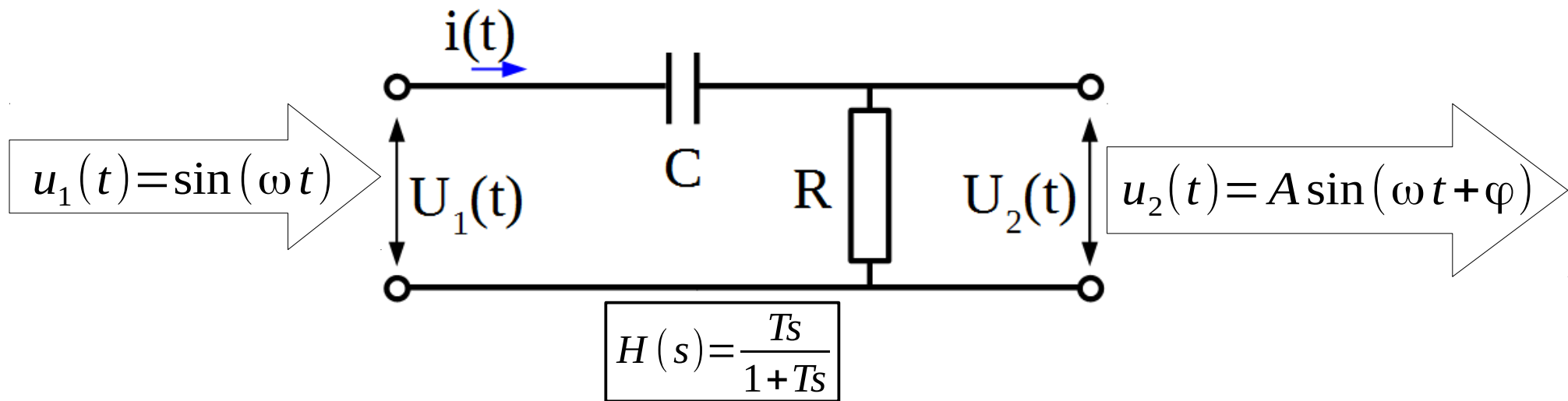
Transfer function – frequency response

RC circuit example



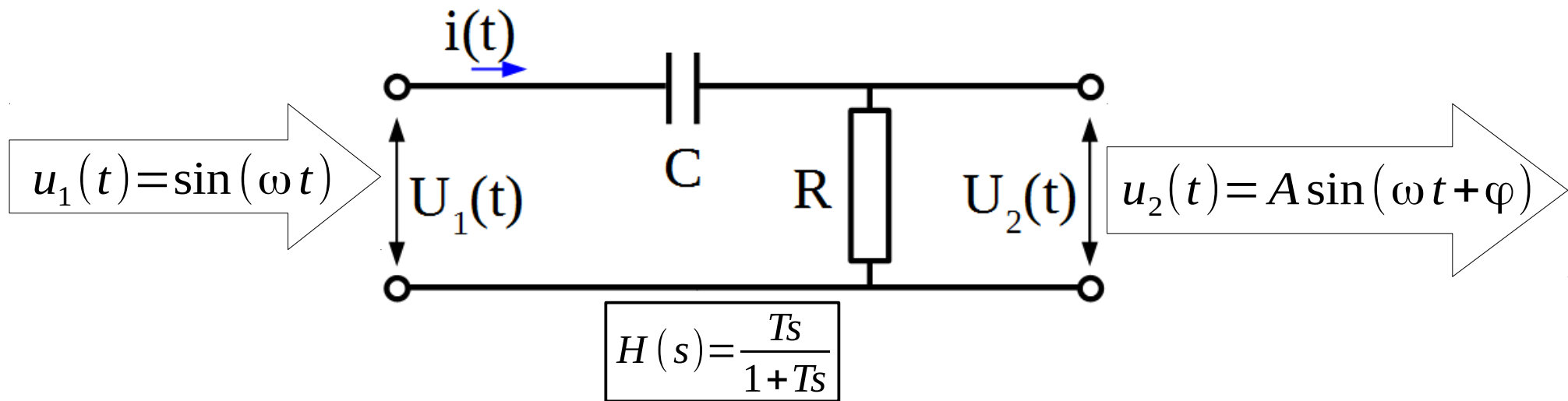
Transfer function – frequency response

RC circuit example



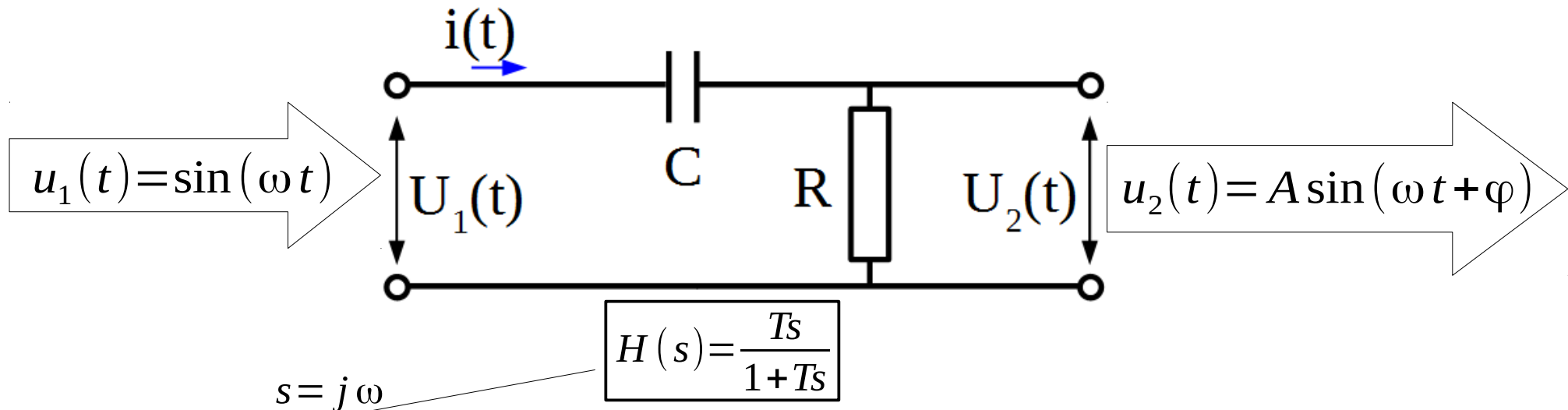
Transfer function – frequency response

RC circuit example



Transfer function – frequency response

RC circuit example



$$H(j\omega) = \frac{Tj\omega}{1 + Tj\omega} = \frac{Tj\omega}{1 + Tj\omega} \cdot \frac{1 - Tj\omega}{1 - Tj\omega} = \frac{Tj\omega - T^2j^2\omega^2}{1^2 - T^2j^2\omega^2} = \frac{Tj\omega + T^2\omega^2}{1^2 + T^2\omega^2} = \frac{T^2\omega^2}{1^2 + T^2\omega^2} + j \frac{T\omega}{1^2 + T^2\omega^2}$$

$$P(\omega) = \frac{T^2\omega^2}{1 + T^2\omega^2} \quad Q(\omega) = \frac{T\omega}{1 + T^2\omega^2} \quad A(\omega) = |H(j\omega)| = \sqrt{P^2(\omega) + Q^2(\omega)} = \frac{T\omega}{\sqrt{T^2\omega^2 + 1}}$$

$$L(\omega) = 20 \log A(\omega) = 20 \log \frac{T\omega}{\sqrt{T^2\omega^2 + 1}} = 20 \log T\omega - 20 \log \sqrt{T^2\omega^2 + 1}$$

$$\varphi(\omega) = \arg H(j\omega) = \arctan \frac{Q}{P} = \arctan \left(\frac{1}{T\omega} \right)$$

Transfer function – frequency response

RC circuit example

Nyquist plot

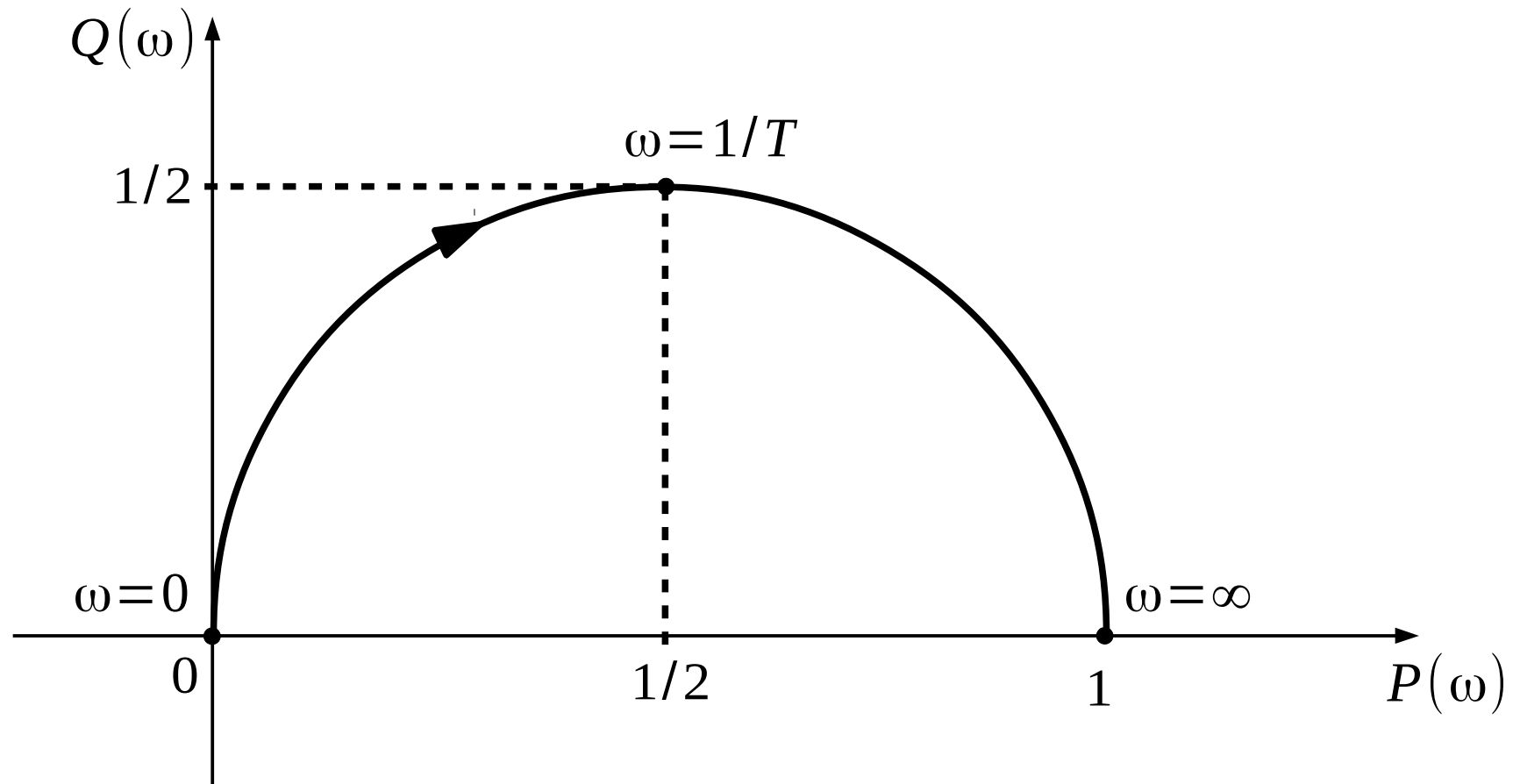
$$P(\omega) = \frac{T^2 \omega^2}{1 + T^2 \omega^2} \quad Q(\omega) = \frac{T \omega}{1 + T^2 \omega^2}$$

Transfer function – frequency response

RC circuit example

Nyquist plot

$$P(\omega) = \frac{T^2 \omega^2}{1 + T^2 \omega^2} \quad Q(\omega) = \frac{T \omega}{1 + T^2 \omega^2}$$



Transfer function – frequency response

RC circuit example

$$A(\omega) = \frac{T\omega}{\sqrt{T^2\omega^2 + 1}}$$

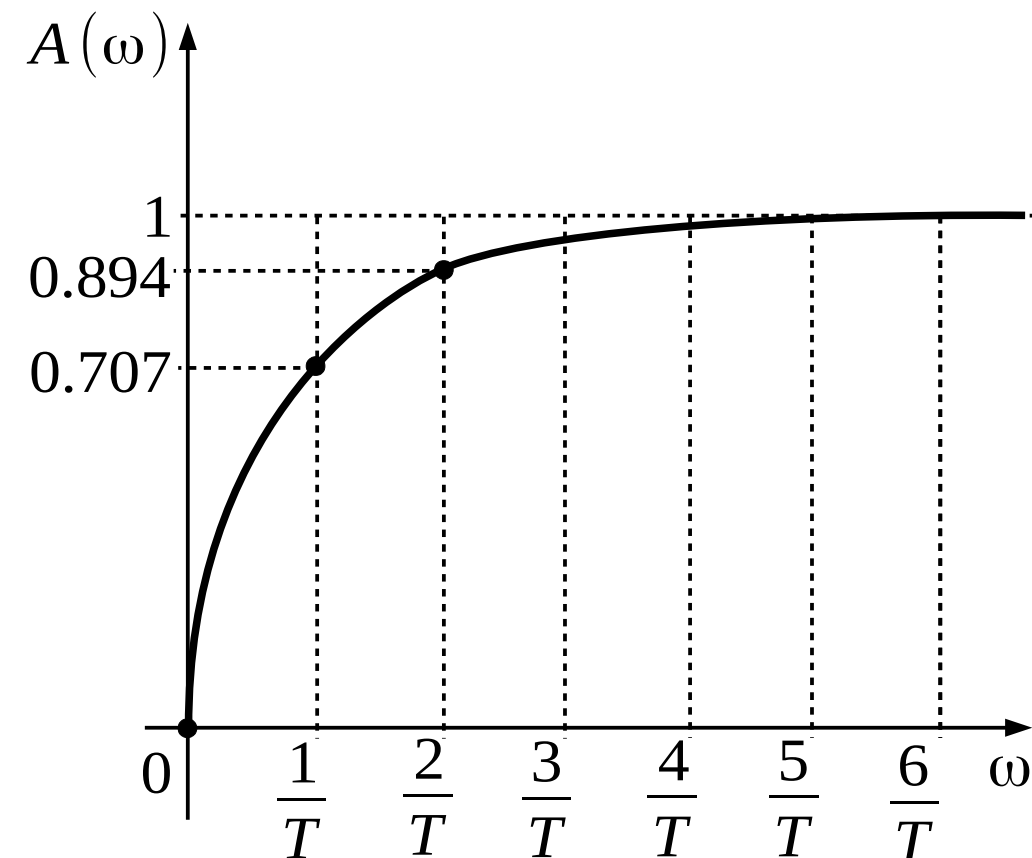
$$\varphi(\omega) = \arctan\left(\frac{1}{T\omega}\right)$$

Transfer function – frequency response

RC circuit example

$$A(\omega) = \frac{T\omega}{\sqrt{T^2\omega^2 + 1}}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T\omega}\right)$$

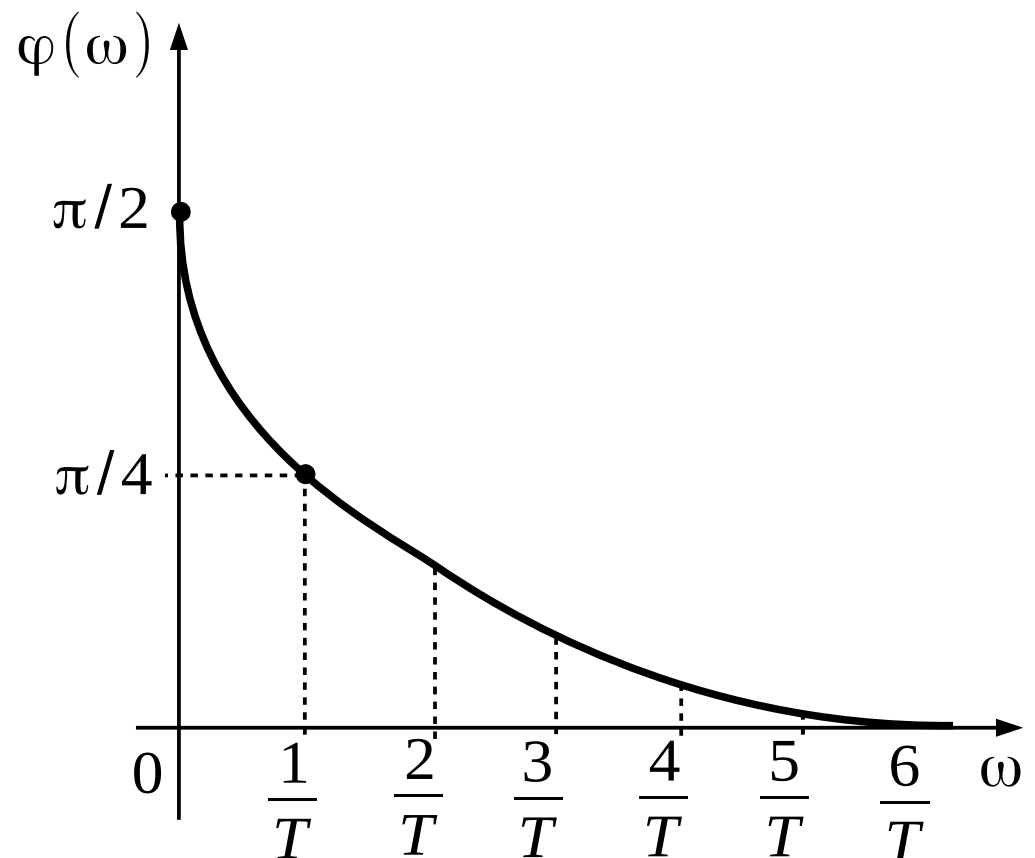
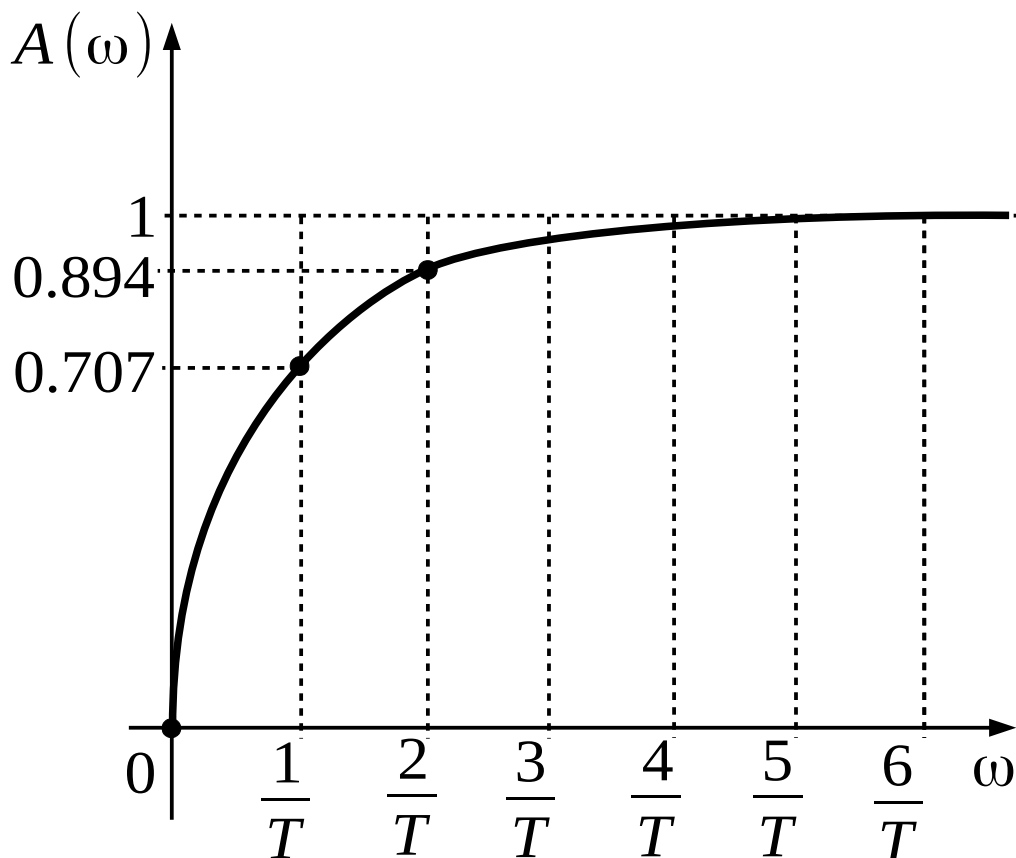


Transfer function – frequency response

RC circuit example

$$A(\omega) = \frac{T\omega}{\sqrt{T^2\omega^2 + 1}}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T\omega}\right)$$



Transfer function – frequency response

RC circuit example

$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

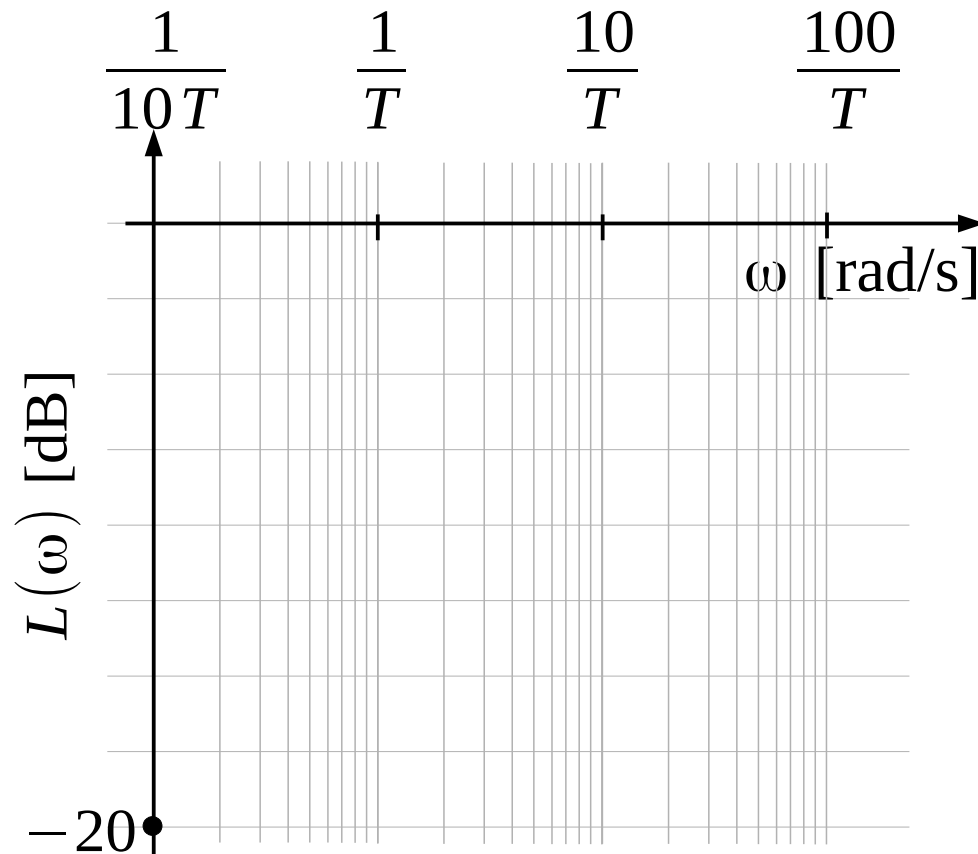
$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$

Transfer function – frequency response

RC circuit example

$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$

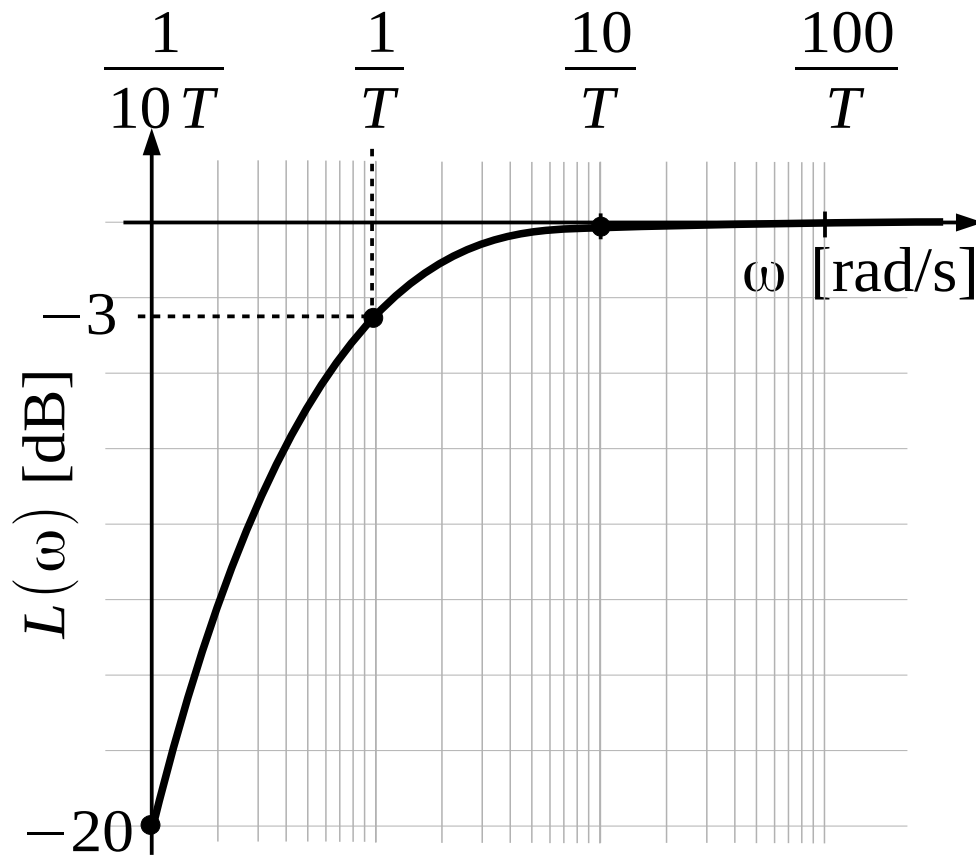


Transfer function – frequency response

RC circuit example

$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$

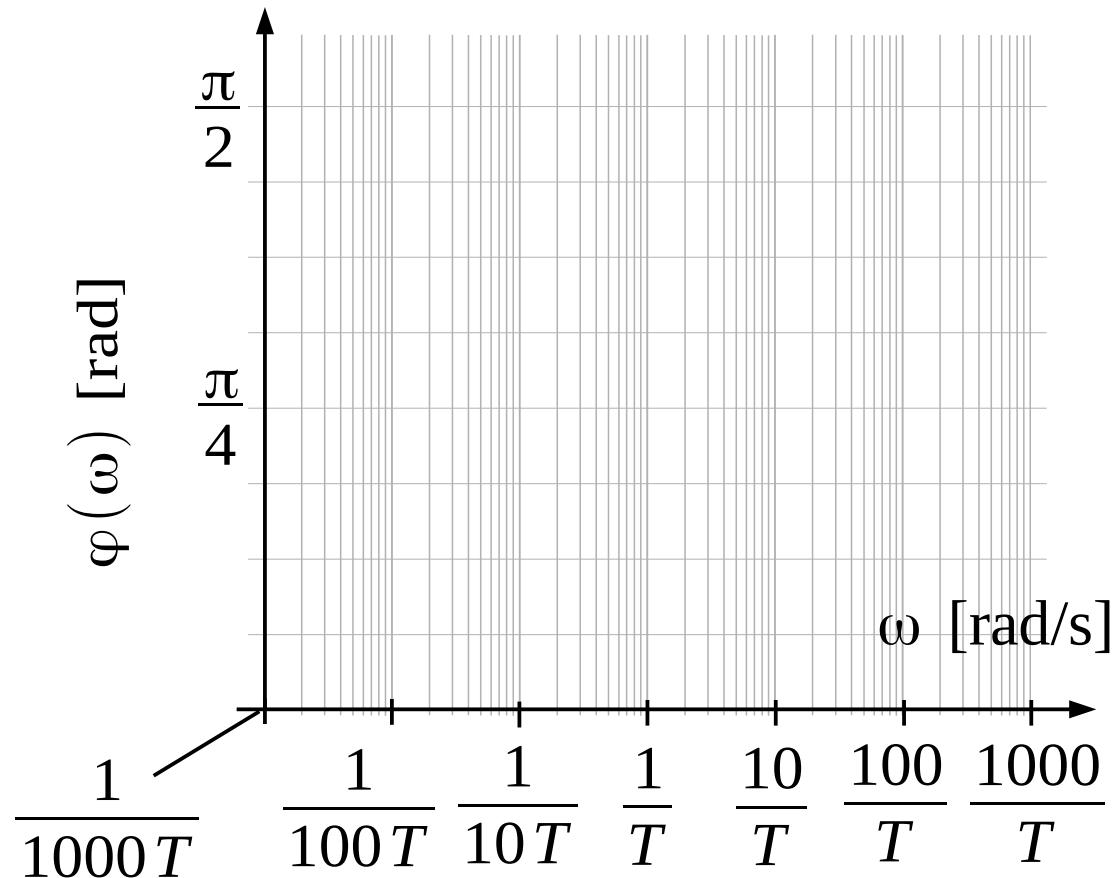


Transfer function – frequency response

RC circuit example

$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$

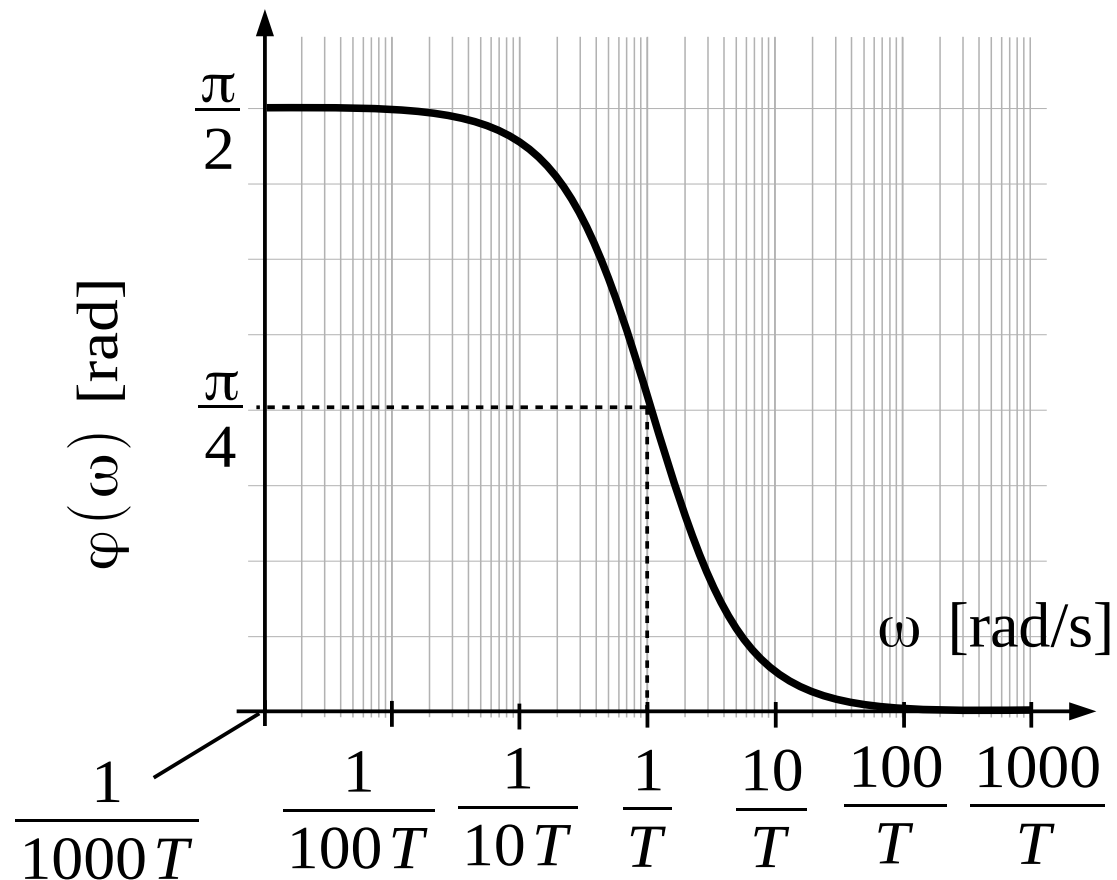


Transfer function – frequency response

RC circuit example

$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$

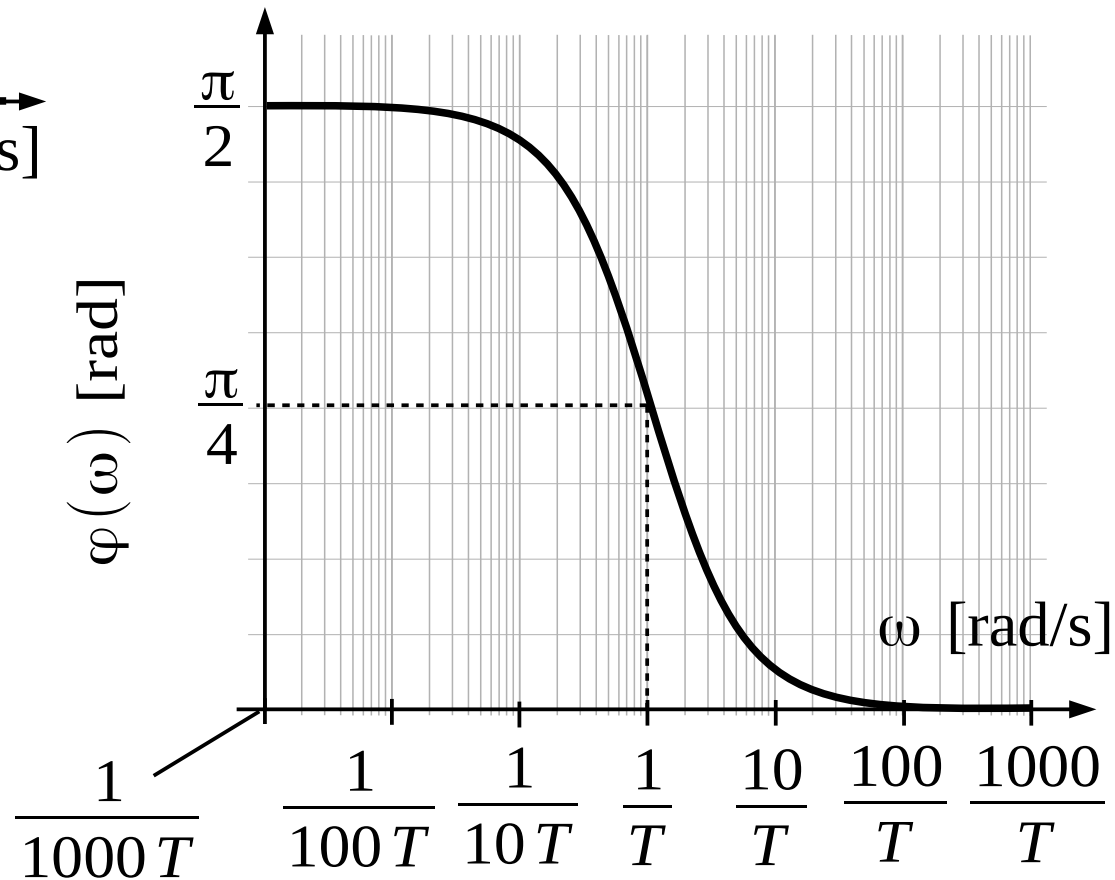
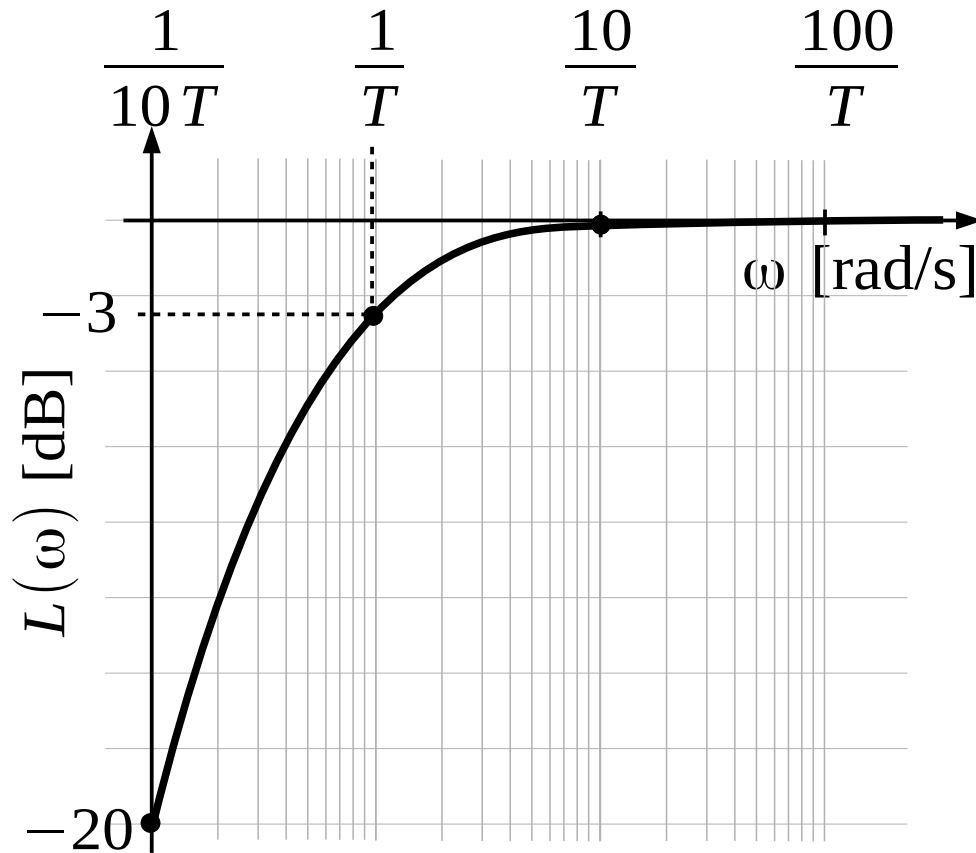


Transfer function – frequency response

RC circuit example

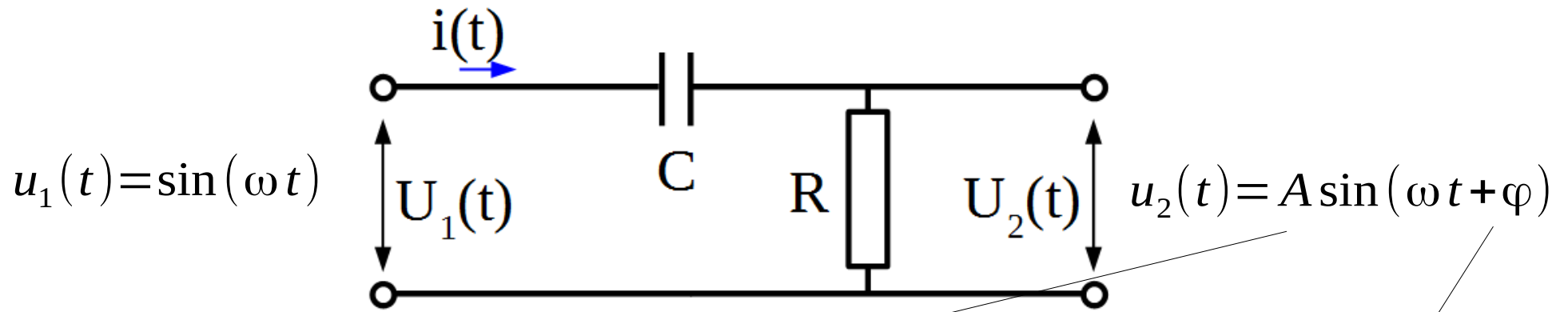
$$L(\omega) = 20 \log T \omega - 20 \log \sqrt{T^2 \omega^2 + 1}$$

$$\varphi(\omega) = \arctan\left(\frac{1}{T \omega}\right)$$



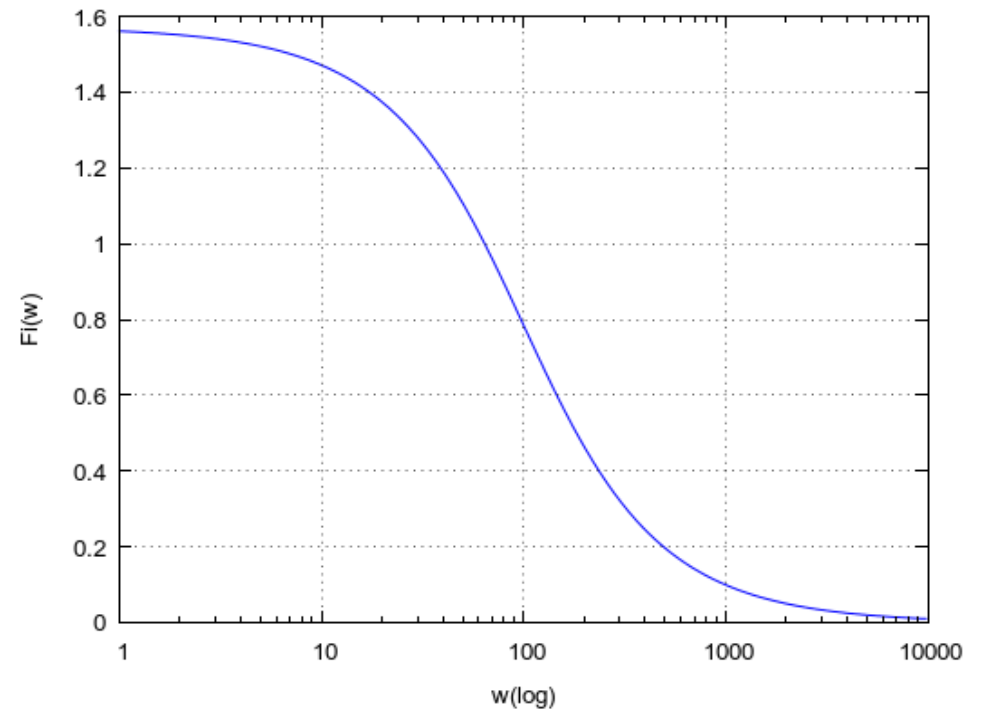
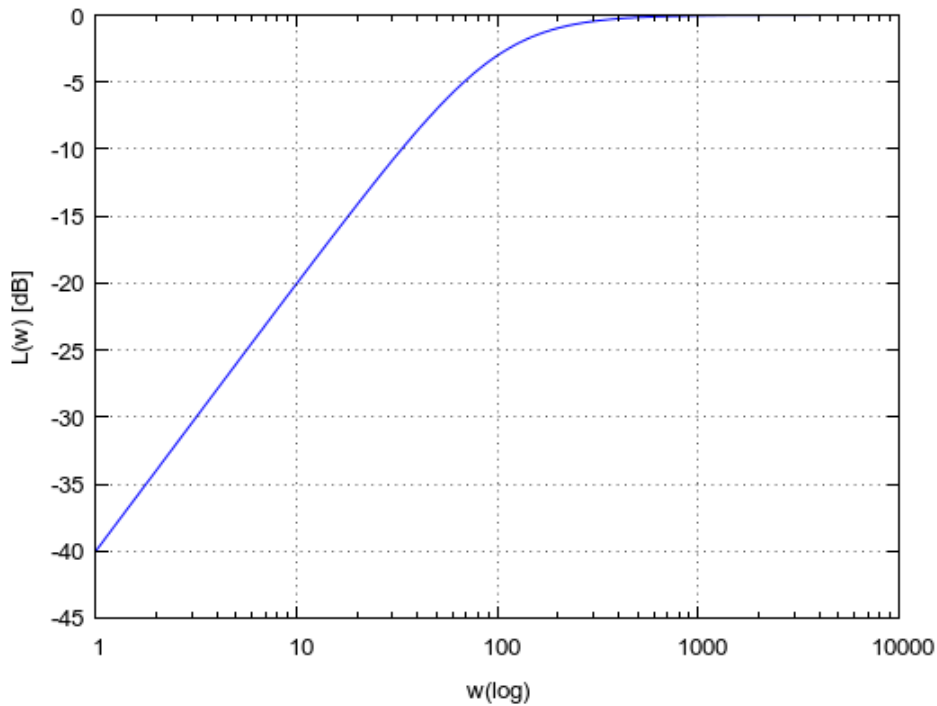
Transfer function – frequency response

RC circuit example



Example:

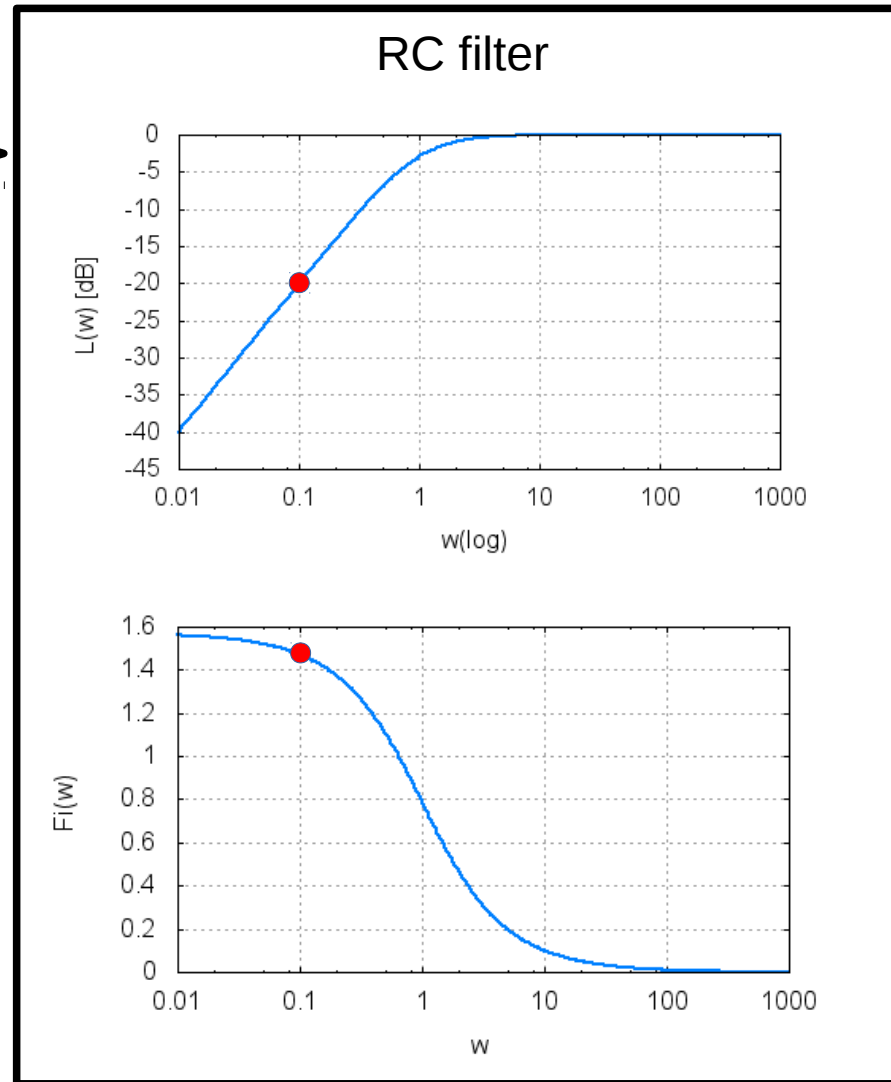
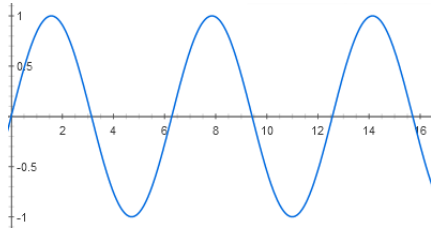
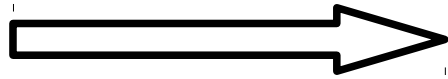
$R = 1\text{ k}\Omega$, $C = 10\mu\text{ F}$



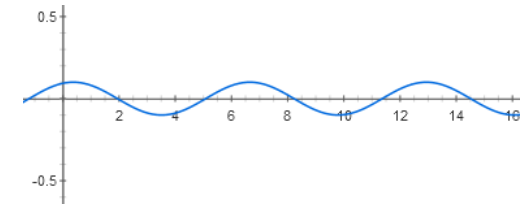
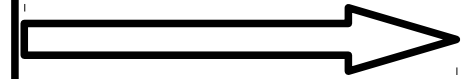
Transfer function – frequency response

RC circuit example

$$u_1(t) = \sin(\omega t)$$



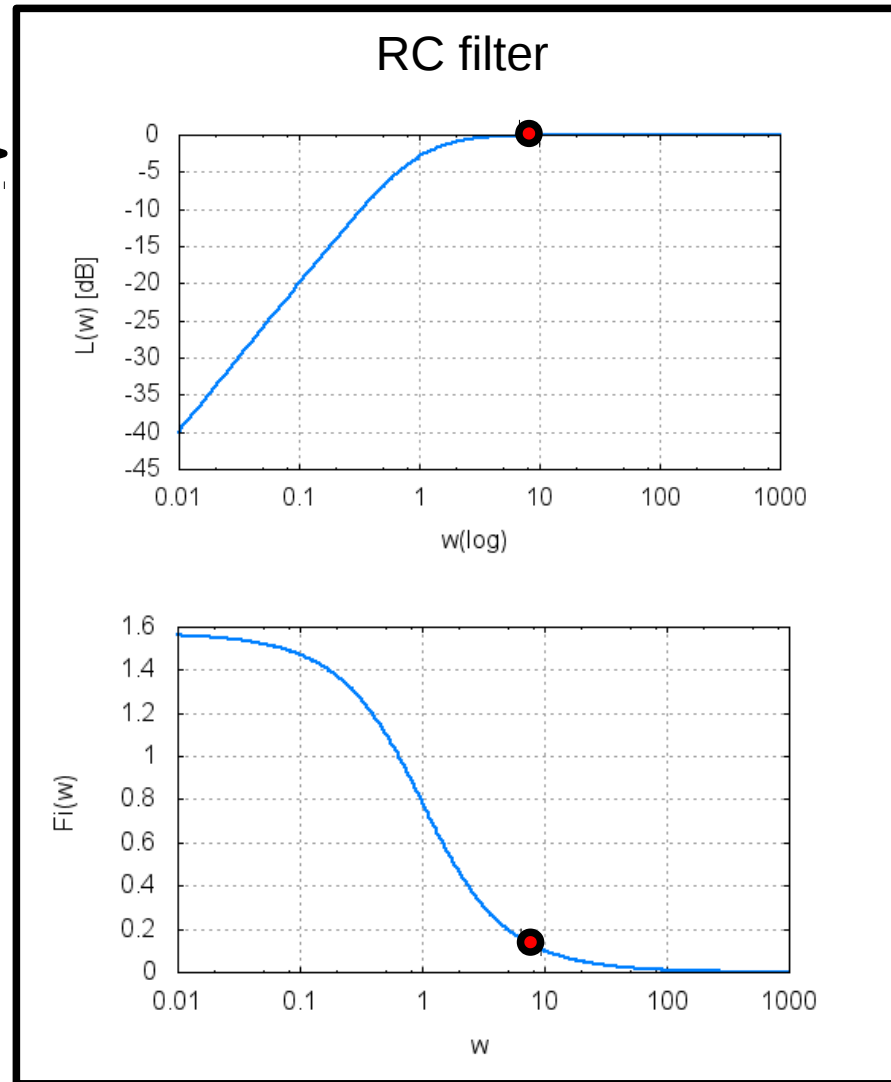
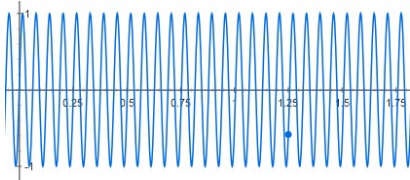
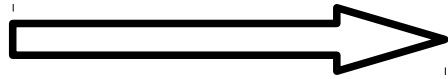
$$u_2(t) = A \sin(\omega t + \varphi)$$



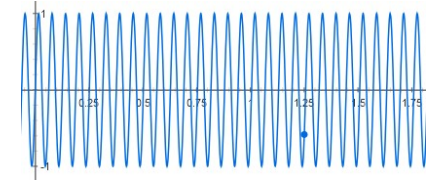
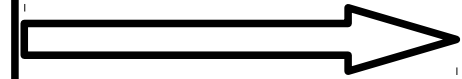
Transfer function – frequency response

RC circuit example

$$u_1(t) = \sin \omega t$$

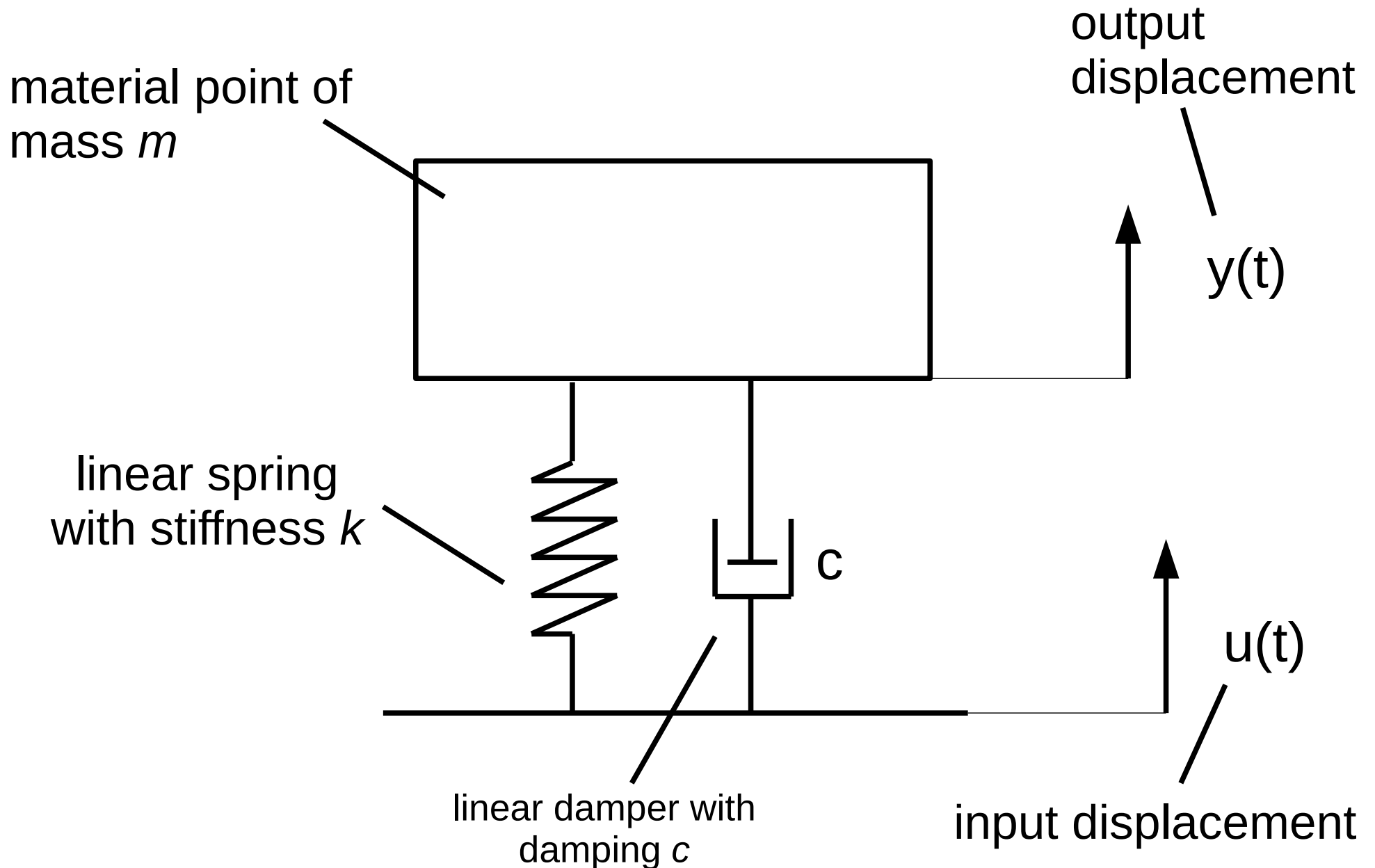


$$u_2(t) = A \sin(\omega t + \phi)$$



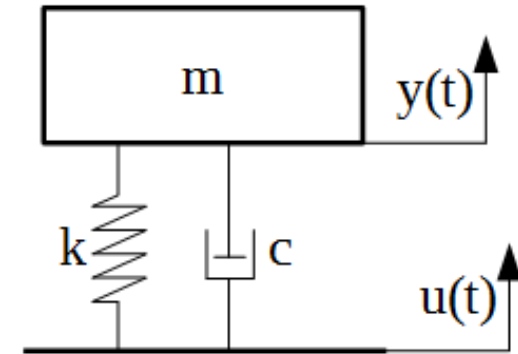
Transfer function – frequency response

Example 2 - vibrating system



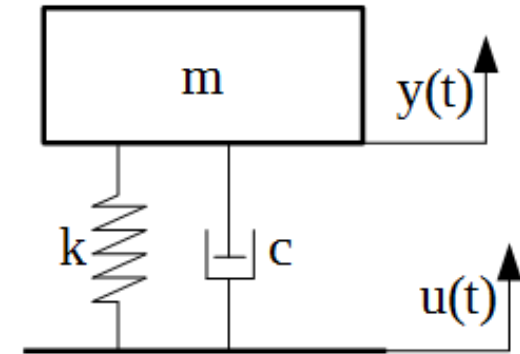
Transfer function – frequency response

Example 2 - vibrating system



Transfer function – frequency response

Example 2 - vibrating system



$$m \ddot{y}(t) + c \dot{y}(t) + k y(t) = c \dot{u}(t) + k u(t)$$

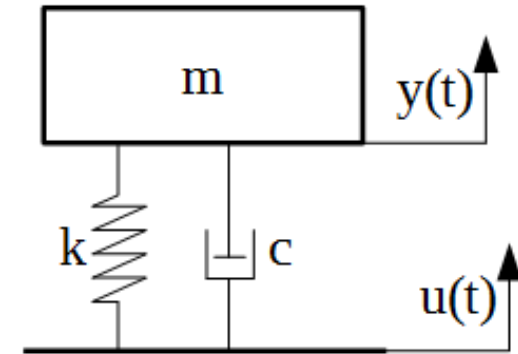
$$H(s) = \frac{cs + k}{ms^2 + cs + k}$$

$$P(\omega) = \frac{k^2 + c^2 \omega^2 - km\omega^2}{(k - m\omega^2)^2 + c^2 \omega^2}, \quad Q(\omega) = \frac{-cm\omega^3}{(k - m\omega^2)^2 + c^2 \omega^2}$$

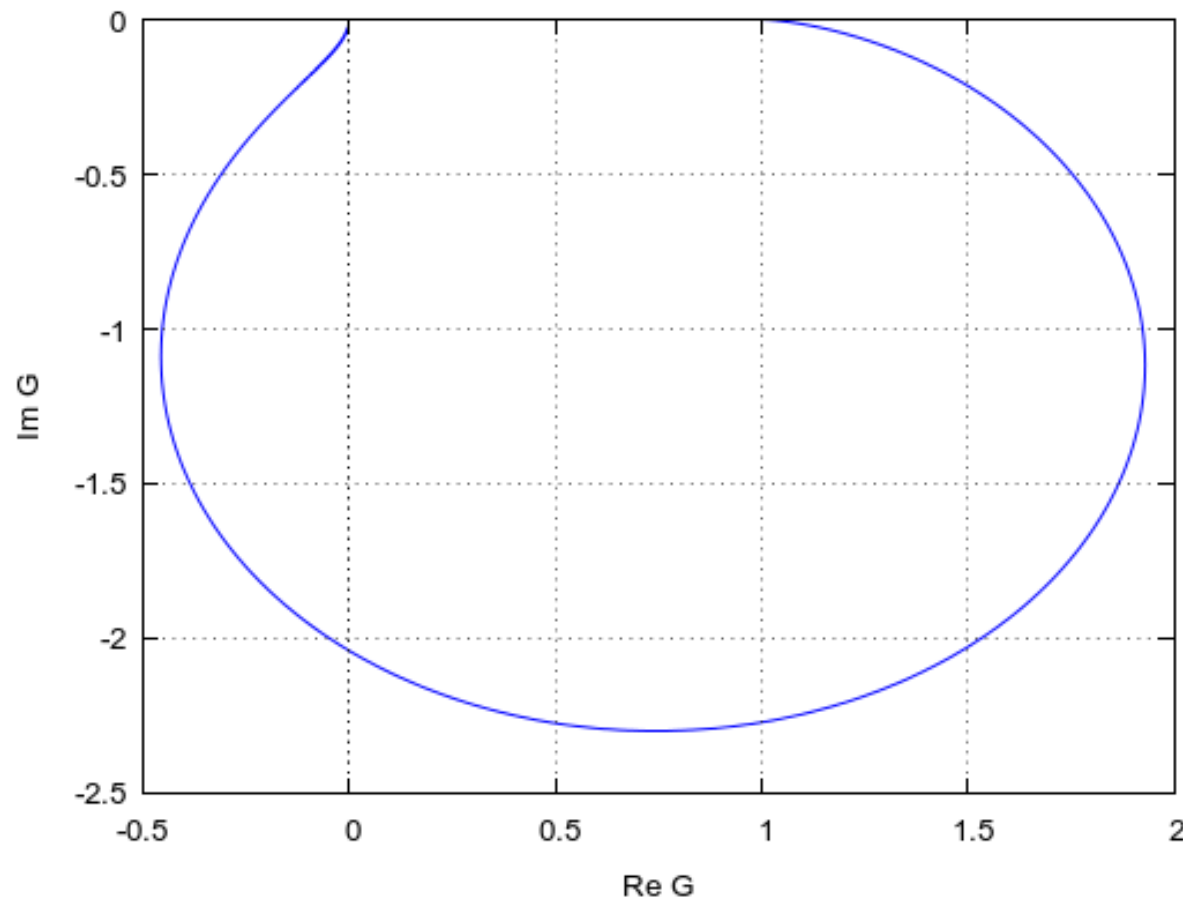
Transfer function – frequency response

Example 2 - vibrating system

Plots for: $m = 300 \text{ kg}$, $c = 800 \frac{\text{Ns}}{\text{m}}$, $k = 11000 \frac{\text{N}}{\text{m}}$

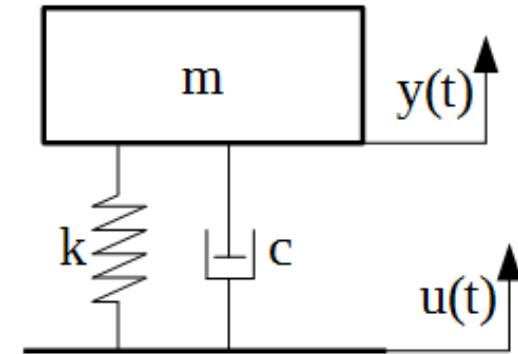


Nyquist plot

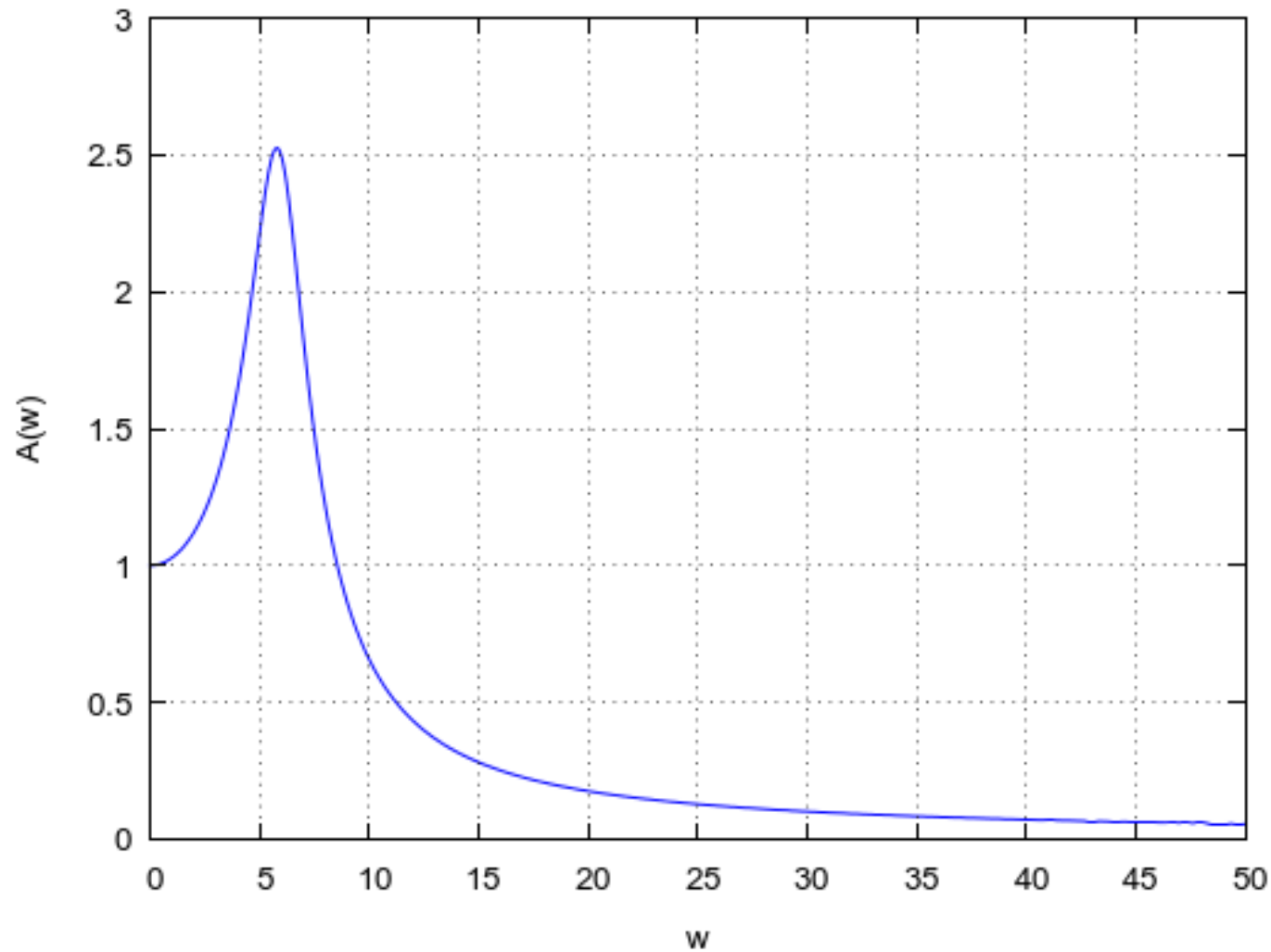


Transfer function – frequency response

Example 2 - vibrating system

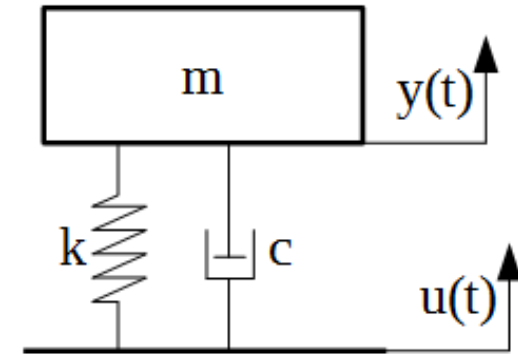


gain plot

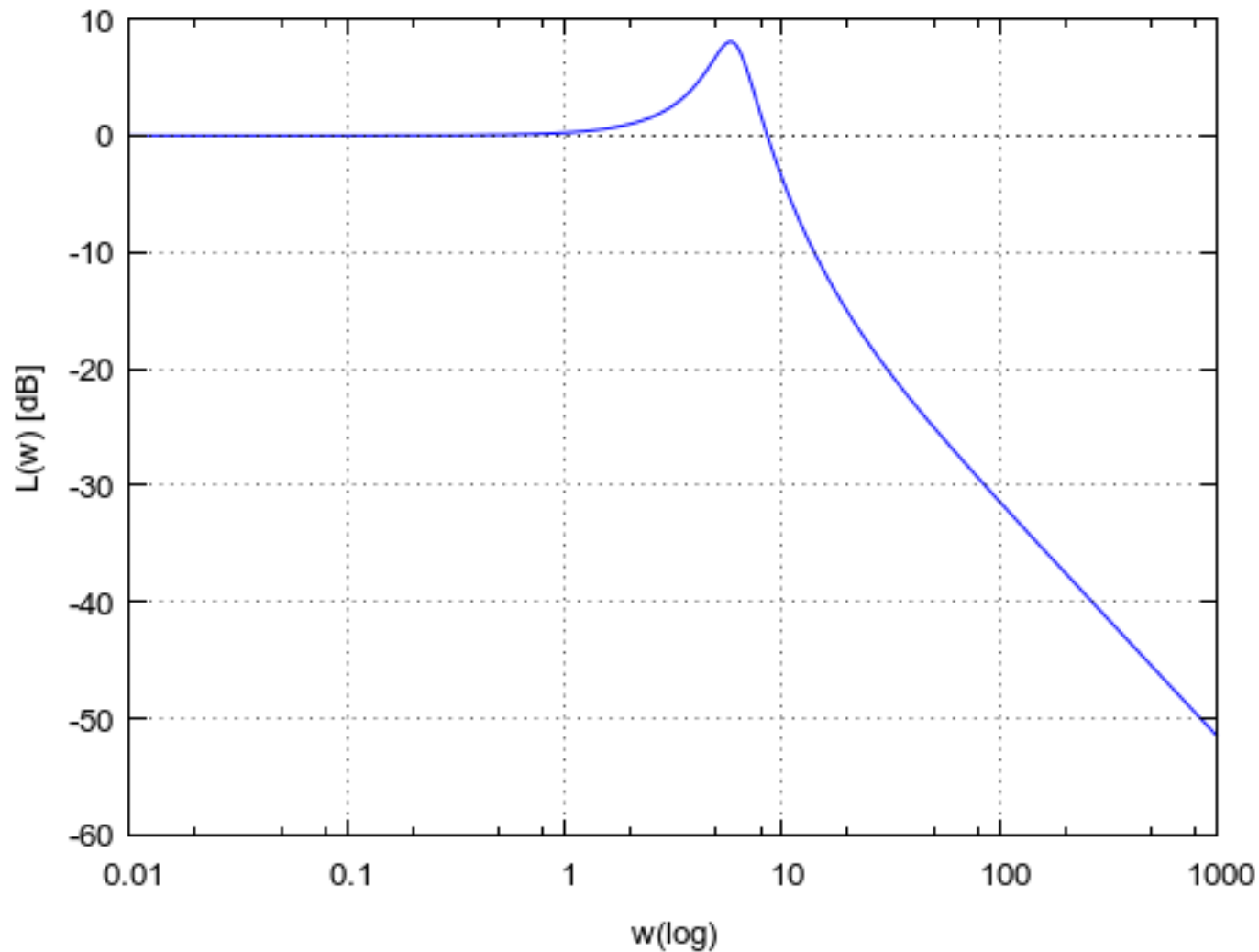


Transfer function – frequency response

Example 2 - vibrating system

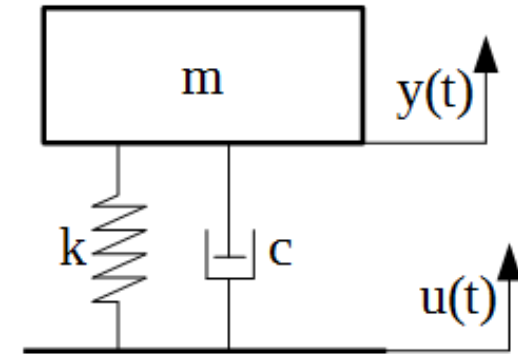


gain plot

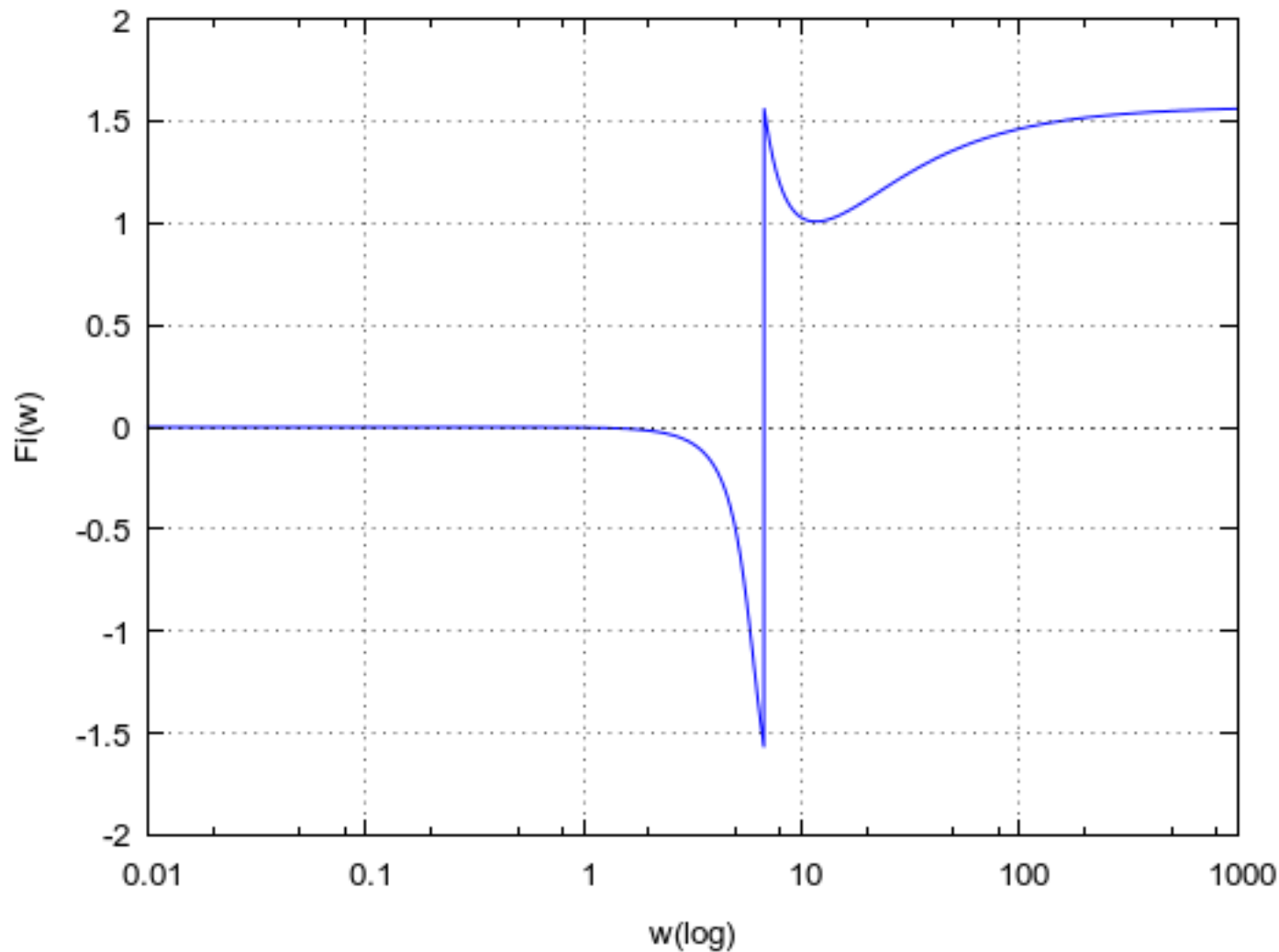


Transfer function – frequency response

Example 2 - vibrating system

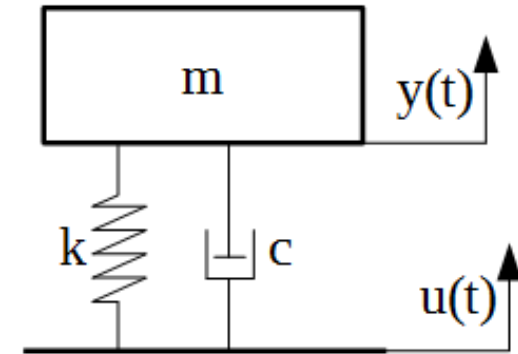


phase plot with „atan”

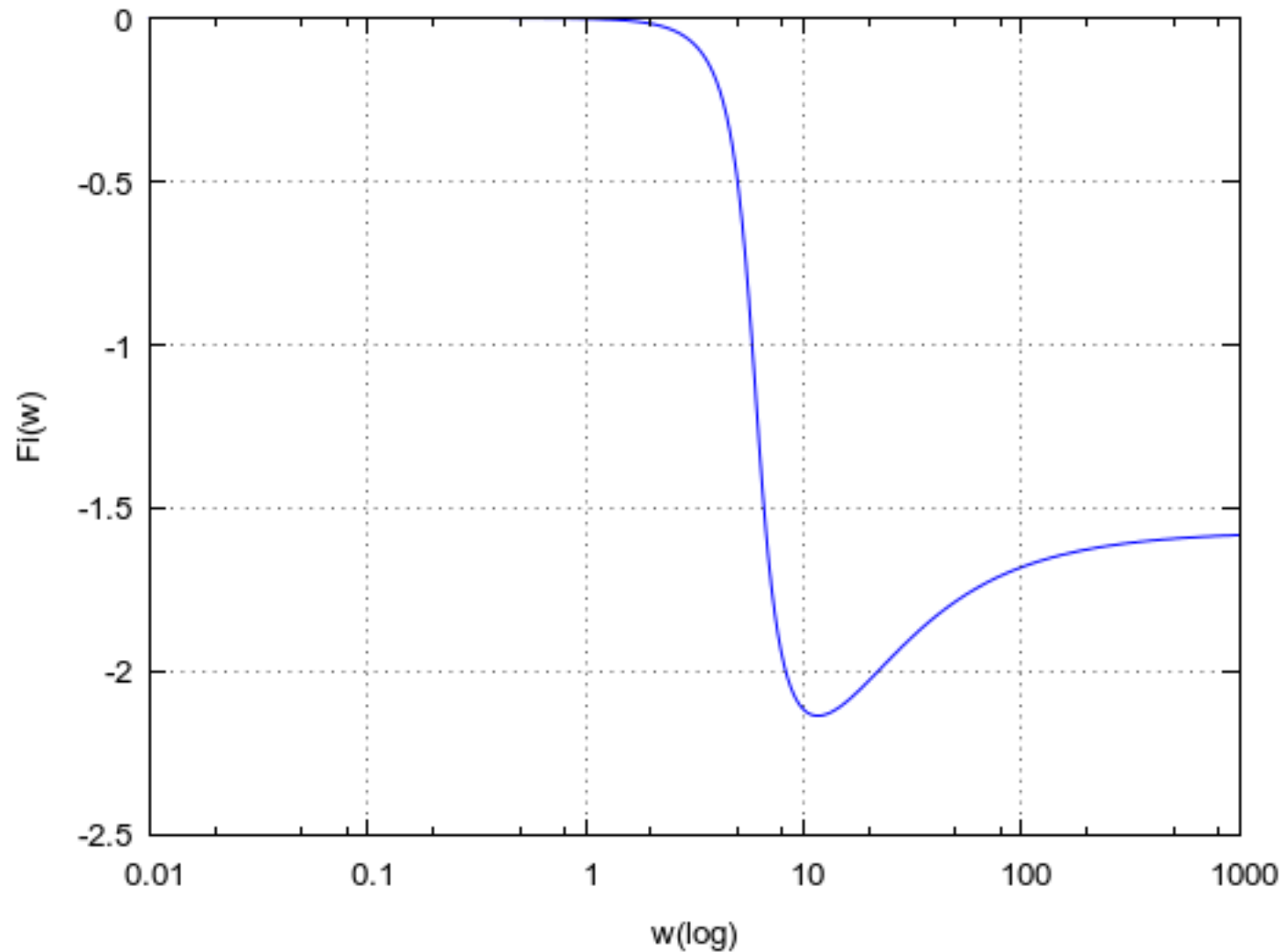


Transfer function – frequency response

Example 2 - vibrating system

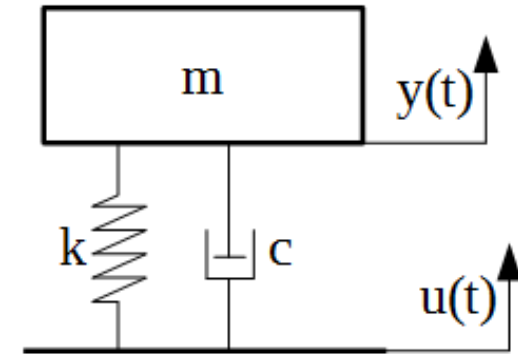


phase plot with „atan2”

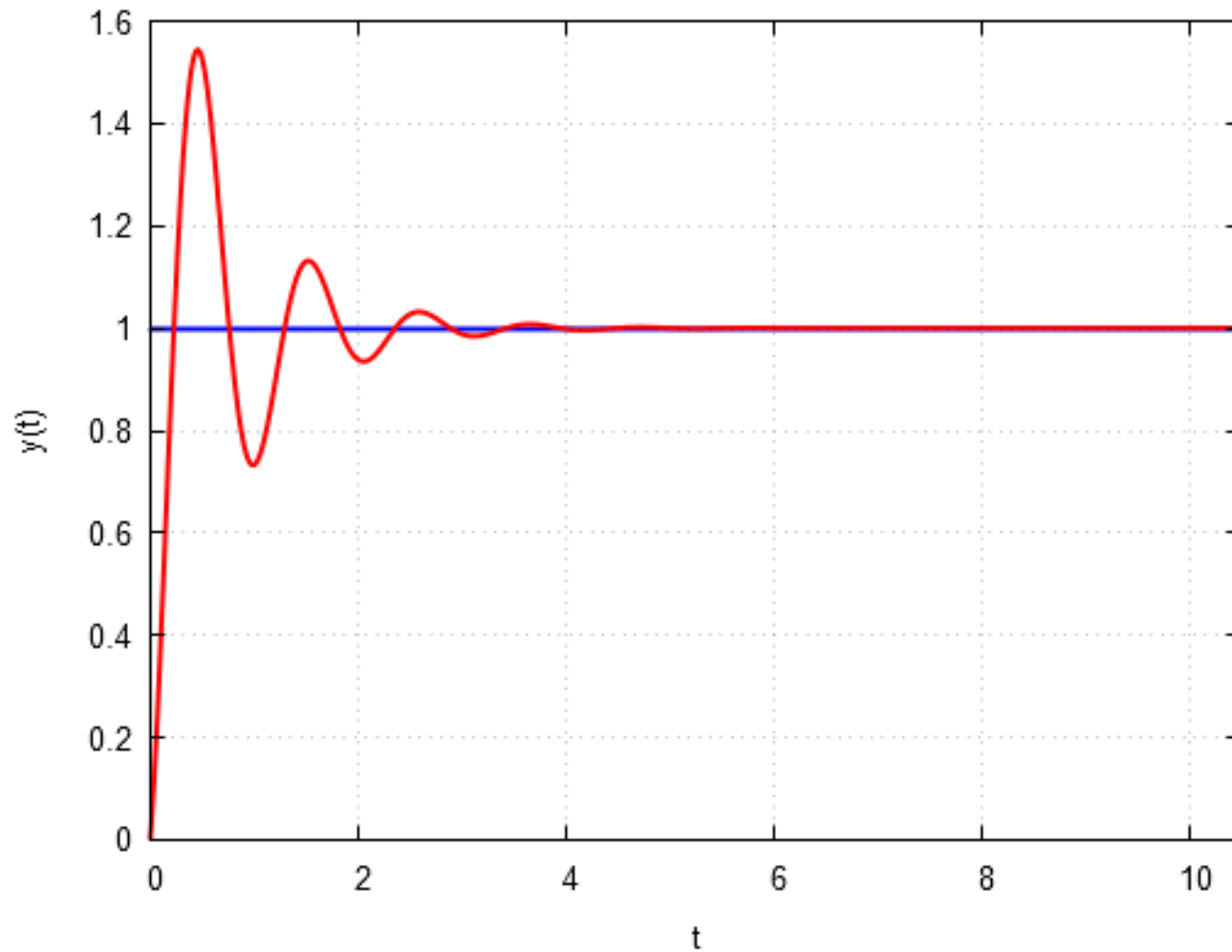


Transfer function – frequency response

Example 2 - vibrating system



Step response



Classification of basic automatic systems

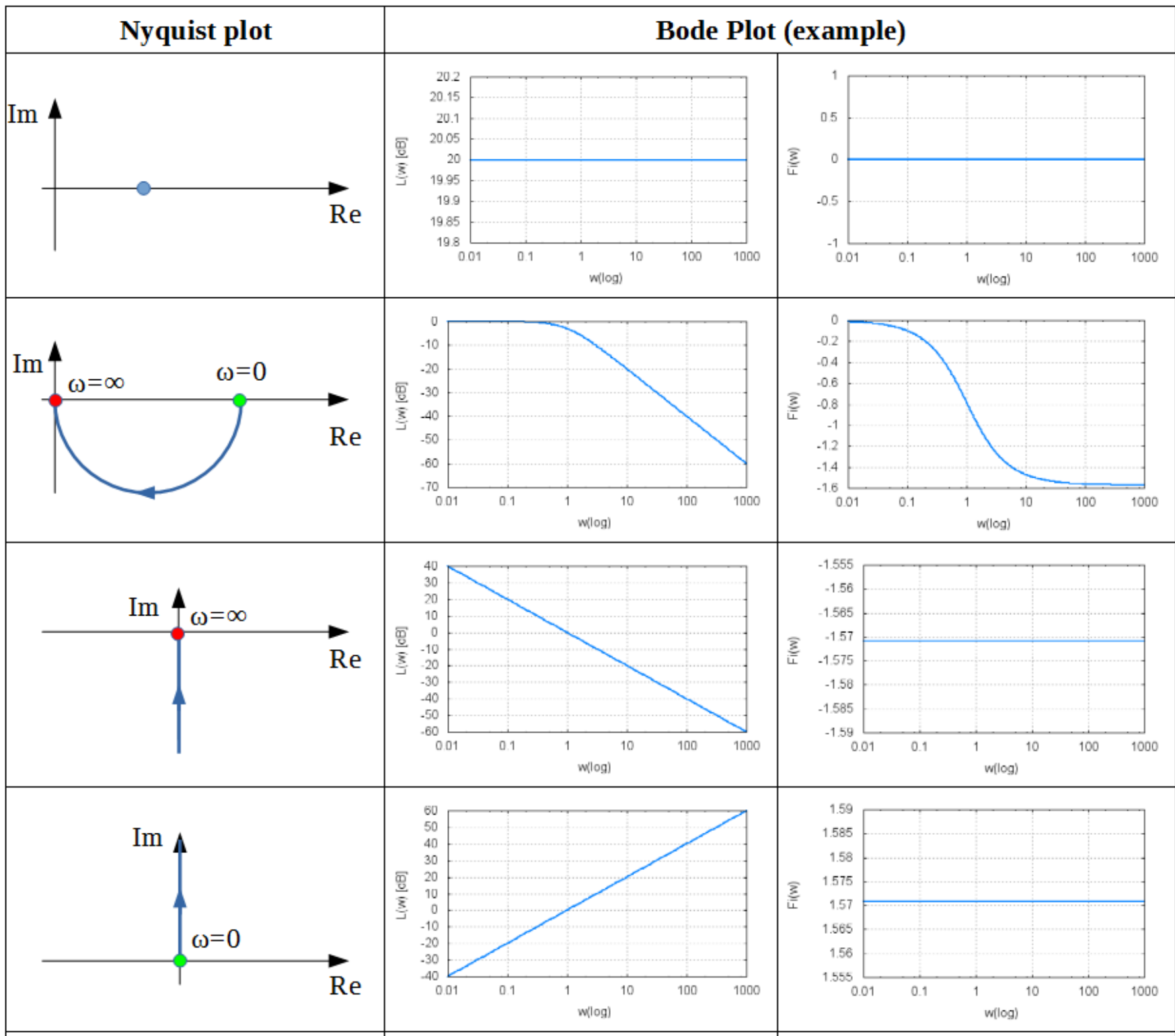
Element name	Equation	Transfer function
proportional	$y(t) = ku(t)$	k
first order (inertial)	$T \frac{dy(t)}{dt} + y(t) = ku(t)$	$\frac{k}{Ts + 1}$
integrator	$y(t) = k \int_0^t u(t) dt$ <p style="text-align: center;">or</p> $\frac{dy(t)}{dt} = ku(t)$	$\frac{k}{s}$

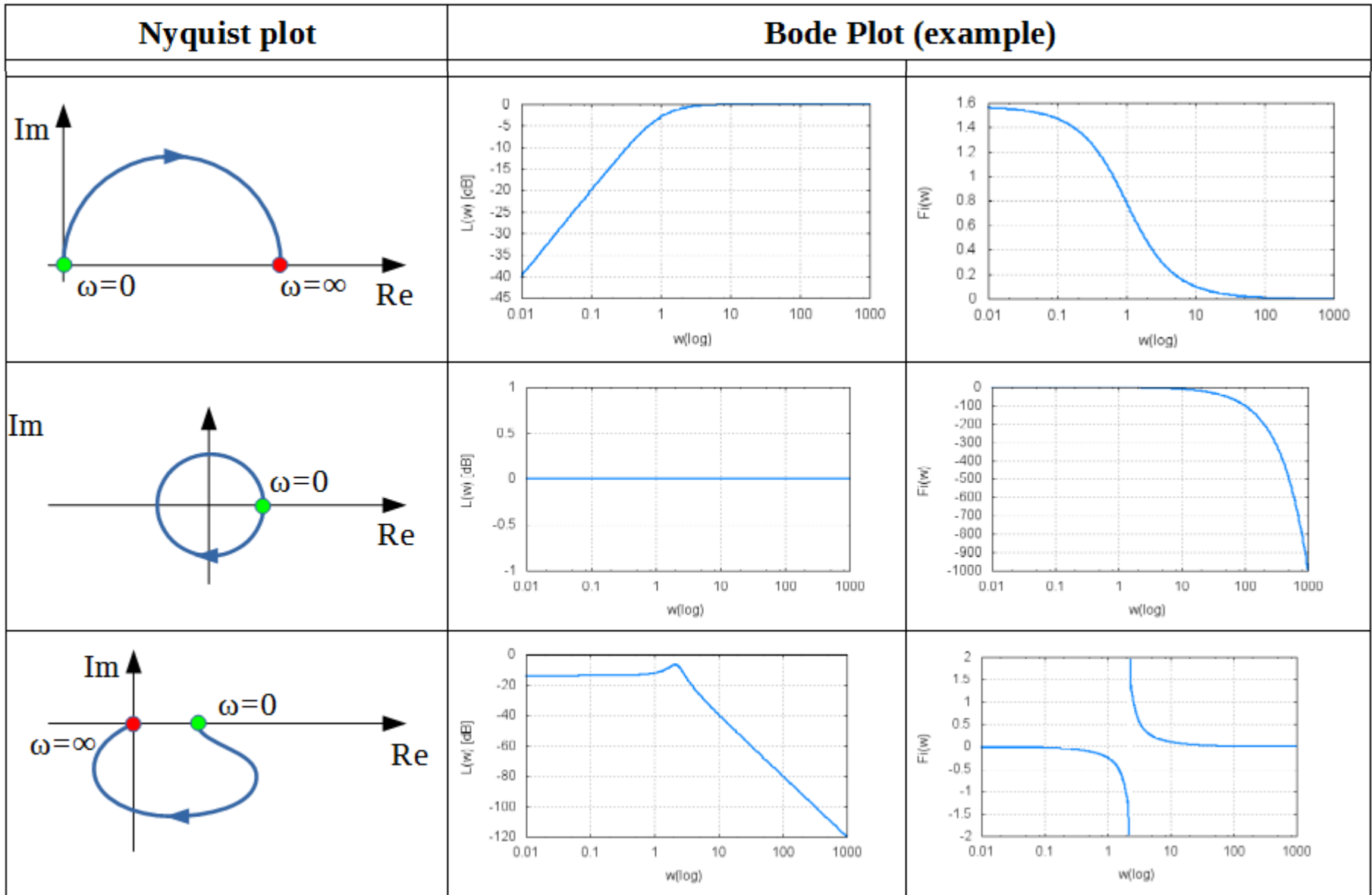
Classification of basic automatic systems

Element name	Equation	Transfer function
derivative	$y(t) = k \frac{du(t)}{dt}$	ks
derivative with inertia	$T \frac{dy(t)}{dt} + y(t) = k \frac{du(t)}{dt}$	$\frac{ks}{Ts + 1}$

Classification of basic automatic systems

Element name	Equation	Transfer function
delay	$y(t) = u(t - \tau)$	$e^{-\tau s}$
second order (oscillator)	$T_1^2 \frac{d^2 y(t)}{dt^2} + T_2 \frac{dy(t)}{dt} + y(t) = ku(t)$	$\frac{k}{T_1^2 s^2 + T_2 s + 1}$





WolframAlpha



transfer function $(8*s+4)/(2*s^4+7*s^3+11*s^2+19*s+6)$



[Examples](#) [Random](#)

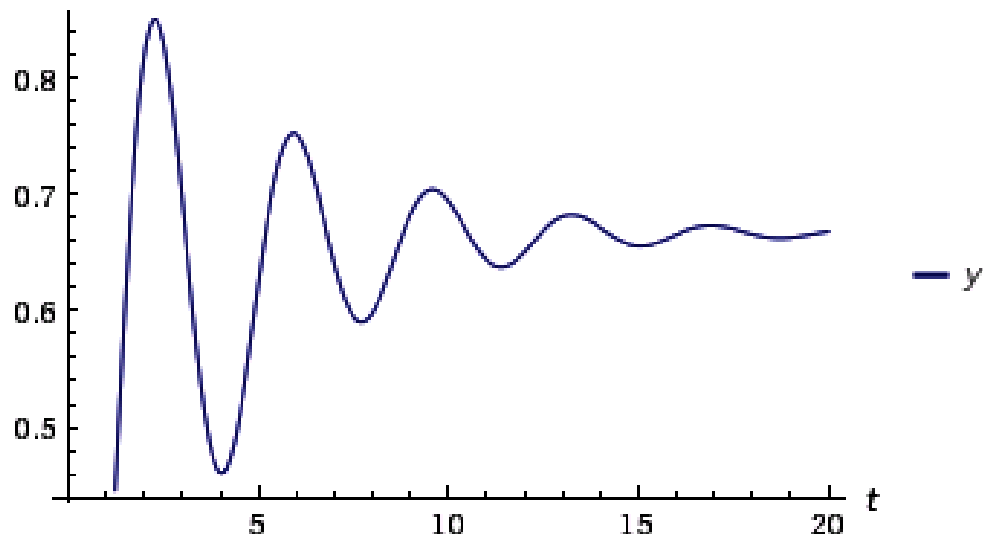
Input interpretation:

systems model

transfer function
$$\frac{4 + 8 s}{6 + 19 s + 11 s^2 + 7 s^3 + 2 s^4}$$

WolframAlpha

Unit step response plot:



Less time

More time

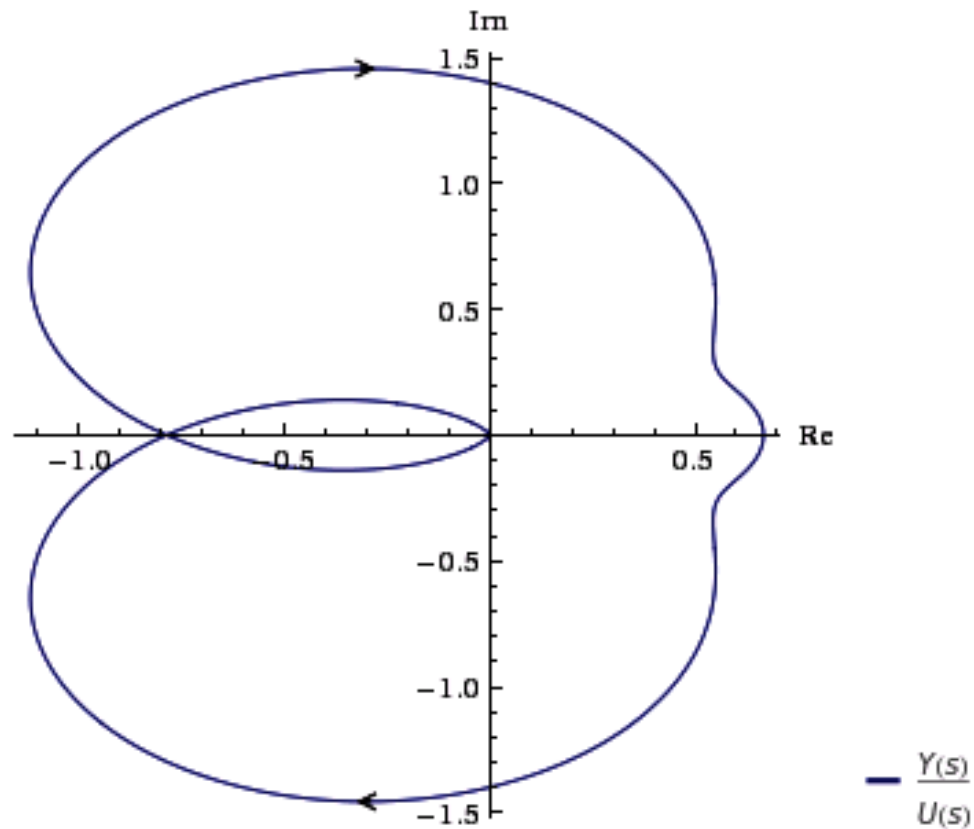
Unit step ▼

WolframAlpha

Nyquist plot:

Show Nyquist grid

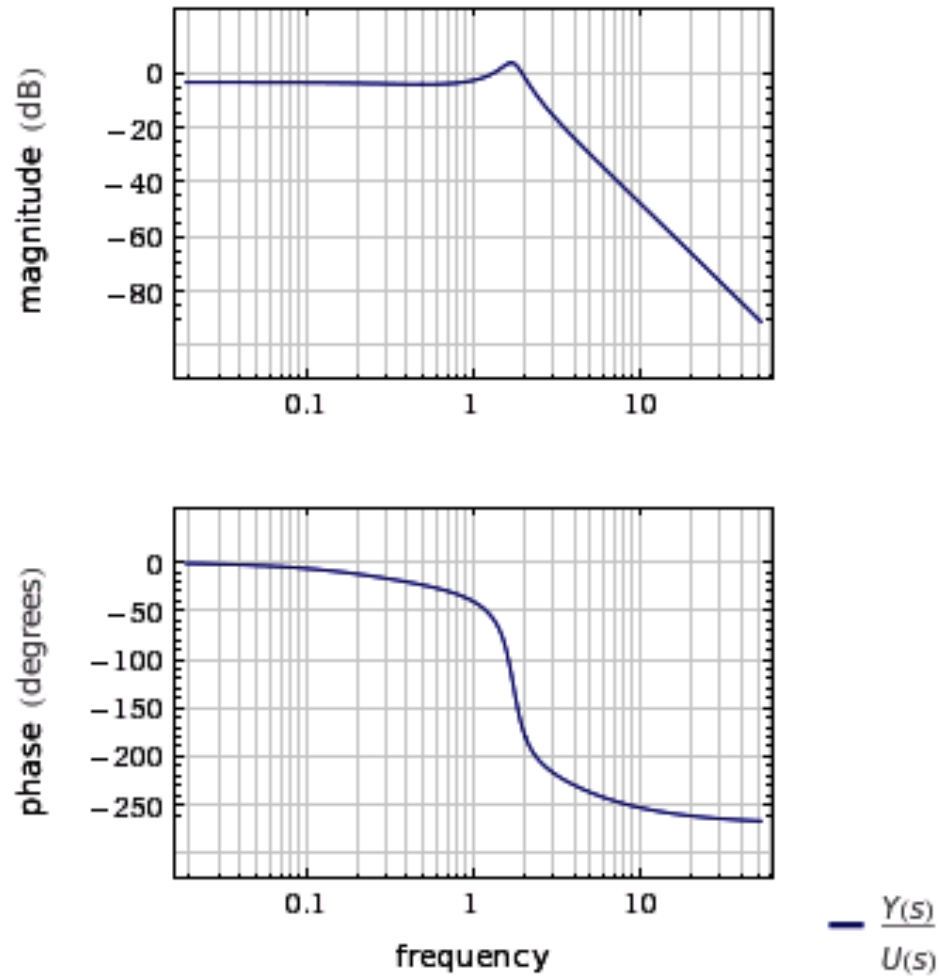
Show stability margins



WolframAlpha

Bode plot:

Show stability margins



WolframAlpha



partial fraction decomposition $s/(s^3+4s^2+5s+2)$



[Examples](#) [Random](#)

Assuming "s" is a variable | Use as a [unit](#) instead

Input:

partial fractions

$$\frac{s}{s^3 + 4s^2 + 5s + 2}$$

Result:

[Step-by-step solution](#)

$$\frac{s}{s^3 + 4s^2 + 5s + 2} = -\frac{2}{s + 2} + \frac{2}{s + 1} - \frac{1}{(s + 1)^2}$$

WolframAlpha



inverse laplace transform $s/(s^3 + 4s^2 + 5s + 2)$



Examples Random

Assuming "s" is a variable | Use as a [unit](#) instead

Input:

$$\mathcal{L}_s^{-1}\left[\frac{s}{s^3 + 4s^2 + 5s + 2}\right](t)$$

$\mathcal{L}_s^{-1}[f(s)](t)$ is the inverse Laplace transform of $f(s)$ with real variable t

Result:

$$-e^{-2t} (e^t t - 2e^t + 2)$$