



Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control Winter 2018/2019

Lecturer: Sebastian Korczak, PhD, Eng.

Lecture 8

Laplace transform.
Transfer function.
Inputs and outputs in time domain.

Materials license: only for educational purposes of Warsaw University of Technology students.

Laplace transform

Assumption: $x(t)$ - signal such that for $t < 0$ $x(t) = 0$

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Inverse Laplace transform of $x(t)$: $x(t) = L^{-1}\{X(s)\} = \frac{1}{2\pi j} \lim_{\omega \rightarrow \infty} \int_{\gamma - j\omega}^{\gamma + j\omega} X(s) e^{st} ds$

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A necessary condition for existence of the integral is that $x(t)$ must be locally integrable on t in $(-\infty, \infty)$.

Laplace transform

Example 1

Calculate Laplace transform of $x(t)$ function from definition.

$$x(t) = e^{-2t}$$

Laplace transform

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$$x(t) = e^{-2t}$$

$$X(s) = L\{e^{-2t}\} = \int_0^{\infty} e^{-2t} e^{-st} dt = \int_0^{\infty} e^{-(2+s)t} dt = \left[\frac{e^{-(2+s)t}}{-(2+s)} \right]_0^{\infty} = \frac{1}{s+2}$$

Laplace transform

$f(t), t \geq 0$	$F(s)$
$\delta(t)$ unit impulse	1
$1(t)$ unit step	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-bt}	$\frac{1}{s+b}$
$1 - e^{-bt}$	$\frac{b}{s(s+b)}$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
$a \cdot f(t)$	$a \cdot F(s)$
$x(t) + y(t)$	$X(s) + Y(s)$
$x(t) * y(t)$ convolution	$X(s) \cdot Y(s)$
$\frac{dy(t)}{dt}$	$sY(s) - y(0)$
$\frac{d^2 y(t)}{dt^2}$	$s^2 Y(s) - s y(0) - \frac{dy(0)}{dt}$
$\frac{d^n y(t)}{dt^n}$	$s^n Y(s) - \frac{d^{n-1} y(0)}{dt^{n-1}} - s \frac{d^{n-2} y(0)}{dt^{n-2}} - \dots - s^{n-1} y(0)$

*table on
the website*

Laplace transform pairs

$f(t), t \geq 0$	$F(s)$
$\delta(t)$ unit impulse	1
$1(t)$ unit step	$\frac{1}{s}$
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Properties of Laplace transform

$f(t), t \geq 0$	$F(s)$
$a \cdot f(t)$	$a \cdot F(s)$
$x(t) + y(t)$	$X(s) + Y(s)$
$x(t) * y(t)$ convolution	$X(s) \cdot Y(s)$

Properties of Laplace transform cont.

$\frac{dy(t)}{dt}$	$sY(s) - y(0)$
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Properties of Laplace transform cont.

$\int_{t=0}^{\infty} f(t) dt$	$\frac{F(s)}{s}$
$\int \int \dots \int_n f(t) dt$	$\frac{F(s)}{s^n}$
$f(t - \tau)$	$e^{-\tau s} F(s)$

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$\int_{t=0}^{\infty} f(t) dt$	$\frac{F(s)}{s}$
$\int \int \dots \int_n f(t) dt$	$\frac{F(s)}{s^n}$
$f(t - \tau)$	$e^{-\tau s} F(s)$

Laplace transform

Example 2

Solve equation for a given initial conditions using Laplace transform.

$$\frac{d^2 y(t)}{dt^2} - 3 \frac{dy(t)}{dt} + 2 y(t) = 1(t), \quad \frac{dy(0)}{dt} = 2, \quad y(0) = 3, \quad t \geq 0$$

Laplace transform

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after Laplace
transformation

$$Y(s) = \frac{1 - 7s + 3s^2}{s(s-1)(s-2)}$$

after partial fraction
decomposition/expansion

$$Y(s) = \frac{1}{2} \frac{1}{s} + 3 \frac{1}{s-1} - \frac{1}{2} \frac{1}{s-2}$$

after inverse
Laplace
transformation

$$y(t) = \frac{1}{2} 1(t) + 3e^t - \frac{1}{2} e^{2t}$$

Transfer function

Linear time-invariant SISO system for continuous-time input signals $x(t)$ and output $y(t)$ in a form:

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1} y(t)}{dt^{n-1}} + \dots + a_{n-1} \frac{dy(t)}{dt} + a_n y(t) = \frac{d^m x(t)}{dt^m} + b_1 \frac{d^{m-1} x(t)}{dt^{m-1}} + \dots + b_{m-1} \frac{dx(t)}{dt} + b_m x(t)$$

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after Laplace transformation with zero initial conditions

$$s^n Y(s) + a_1 s^{n-1} Y(s) + \dots + a_{n-1} s Y(s) + a_n Y(s) = s^m X(s) + b_1 s^{m-1} X(s) + \dots + b_{m-1} s X(s) + b_m X(s)$$

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Transfer function	$H(s) = \frac{Y(s)}{X(s)} = \frac{s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$
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Transfer function

For LTI SISO system with continuous input $x(t)$ and output $y(t)$ for zero initial conditions, transfer function is a ratio of the output of a system to the input of a system described in complex domain by the Laplace transformation.

$$H(s) = \frac{Y(s)}{X(s)}$$

Transfer function

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

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$$H(s) = \frac{Y(s)}{X(s)} = \frac{(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

z_1, z_2, \dots, z_m - zeroes

p_1, p_2, \dots, p_n - poles

Transfer function

$$H(s) = \frac{Y(s)}{X(s)}$$

for every $s \in \mathbb{C}$ there is $H(s) \in \mathbb{C}$

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for every $s \in \mathbb{C}$ there is $H(s) \in \mathbb{C}$

for every $s = \sigma + j\omega$ there is $H(s) = |H(s)| \exp(j \arg H(s))$

Transfer function

Example

$$H(s) = \frac{2-s}{s^3+s^2-2} = \frac{s-2}{(s-1)(s+j+1)(s-j+1)}$$

Transfer function

Example

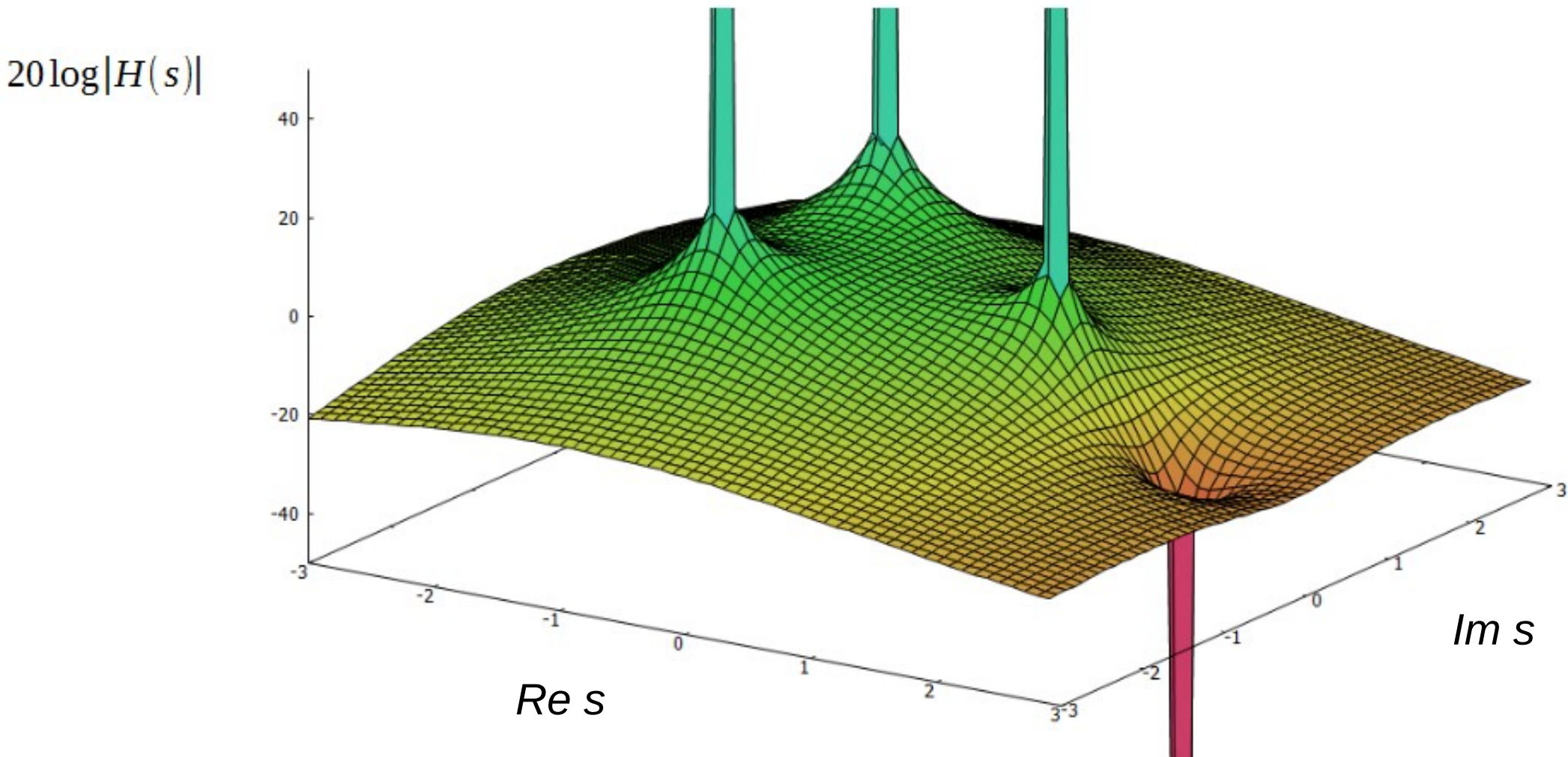
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Poles: $p_1=1$, $p_2=-1-j$, $p_3=-1+j$ Zeroes: $z_1=2$

Transfer function

Example

Poles: $p_1=1$, $p_2=-1-j$, $p_3=-1+j$ Zeroes: $z_1=2$



Input and output

Transfer function: $H(s) = \frac{Y(s)}{X(s)}$

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Laplace transform of output: $Y(s) = H(s)X(s)$

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$$y(t) = L^{-1}\{H(s)X(s)\} = L^{-1}\{H(s)\} * L^{-1}\{X(s)\} = h(t) * x(t)$$

Input and output

Transfer function: $H(s) = \frac{Y(s)}{X(s)}$

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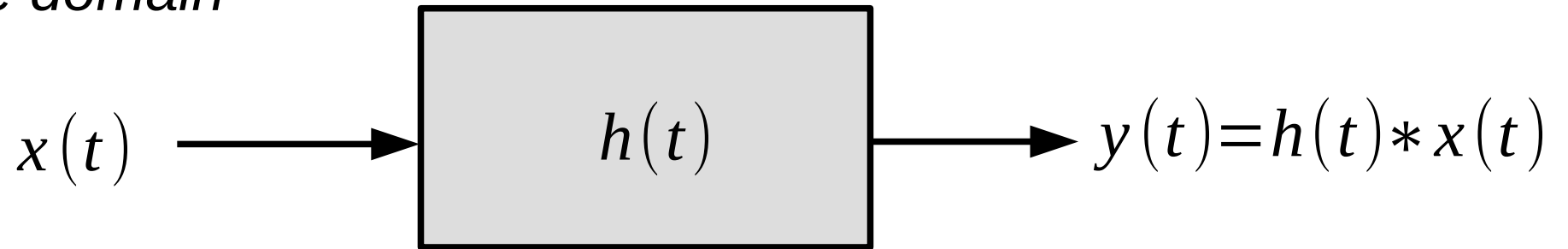
$h(t)$ - system impulse response ($y(t)$ when $x(t) = \delta(t)$)

Input and output

Convolution $h(t) * x(t) = \int_0^{\infty} x(\tau) h(t - \tau) d\tau$

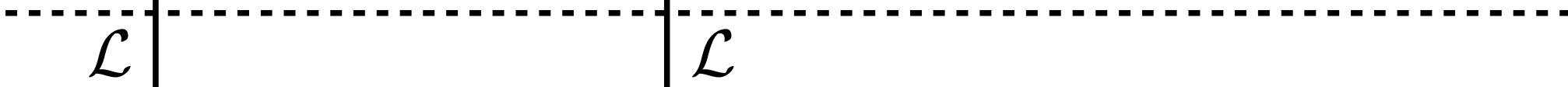
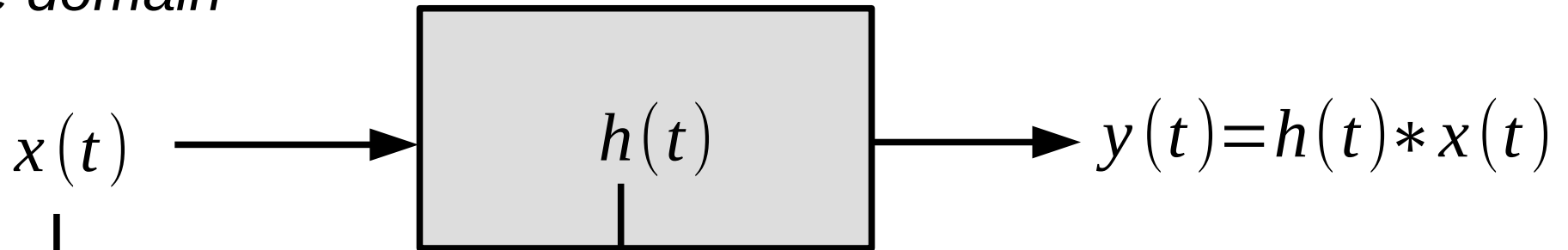
Input and output

time domain

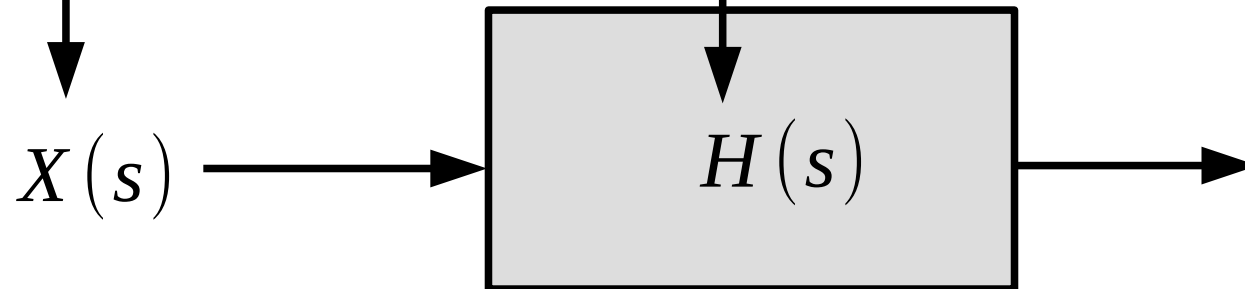


Input and output

time domain

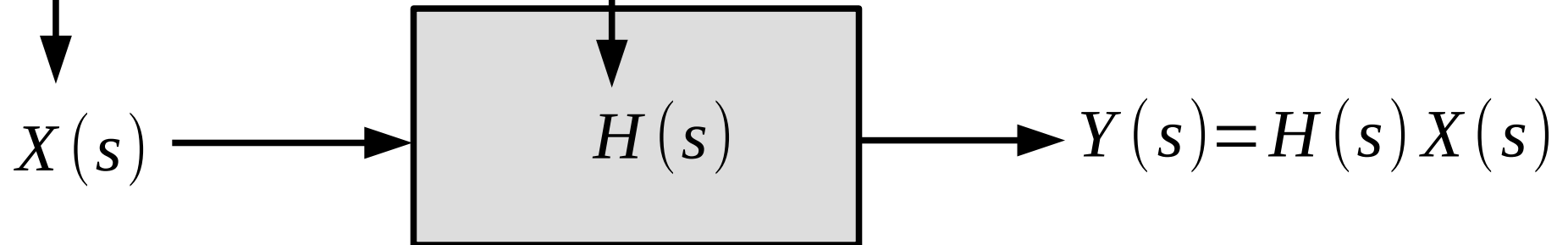
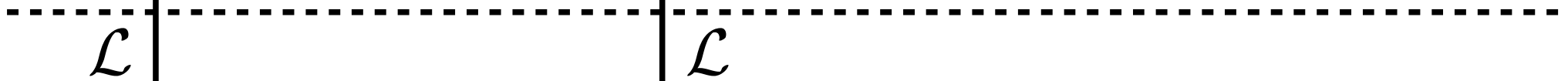
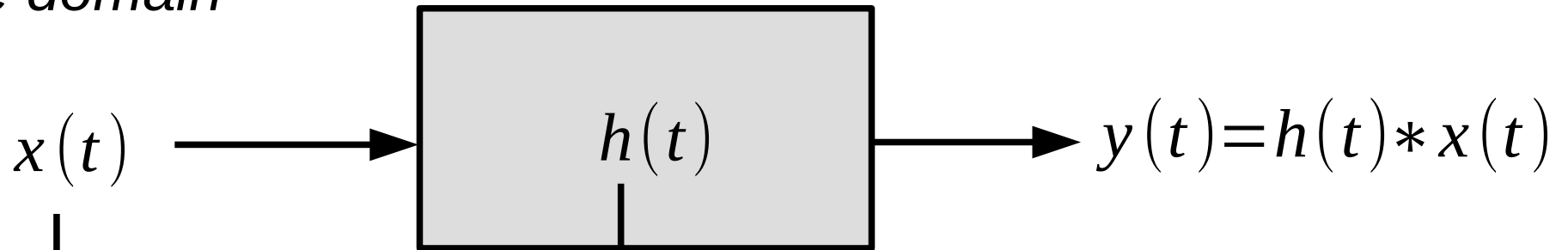


complex domain



Input and output

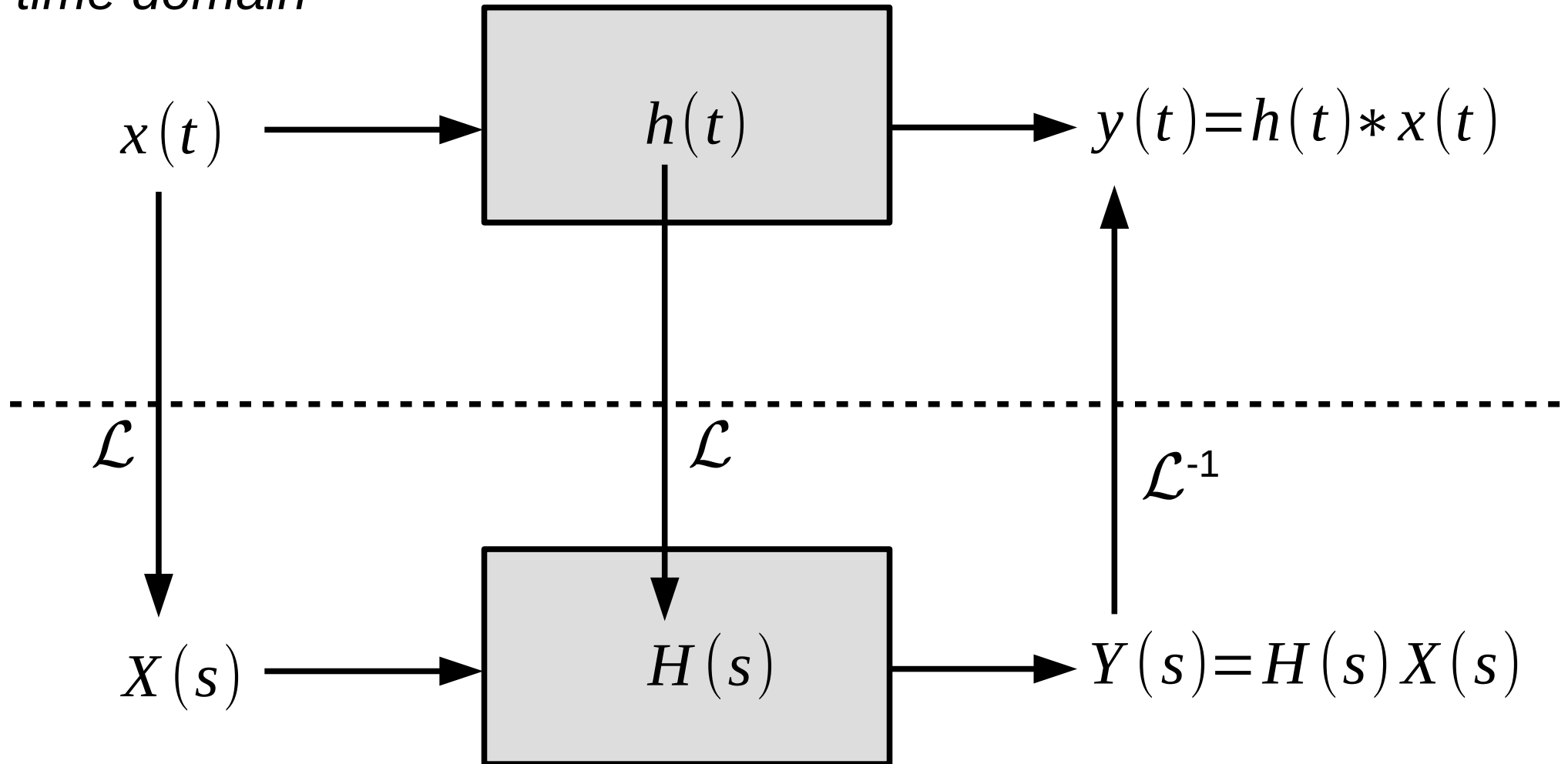
time domain



complex domain

Input and output

time domain

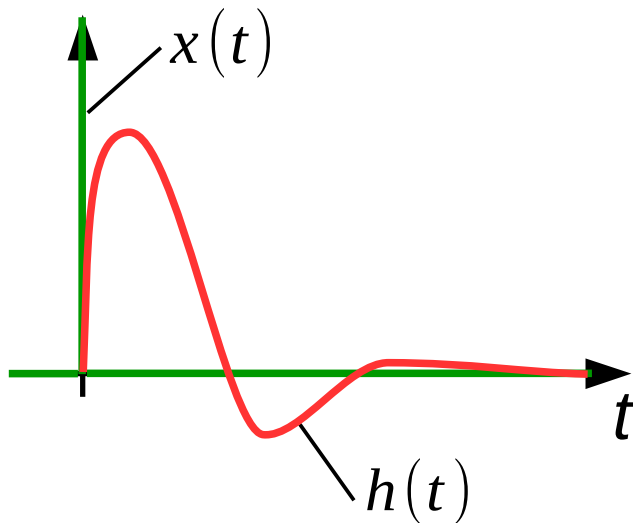


complex domain

Input and output

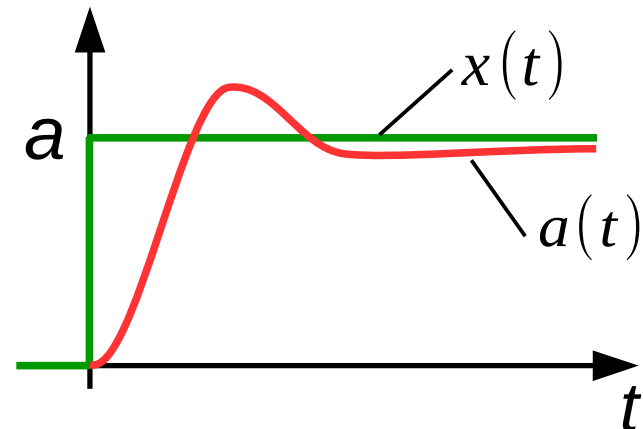
$h(t)$

impulse response
 $y(t)$ for $x(t) = \delta(t)$



$a(t)$

step response
 $y(t)$ for $x(t) = 1(t)$



$$\frac{d a(t)}{d t} = h(t)$$

Exemplary input signals

No input: $x(t) = 0$

Unit impulse (Dirac delta pseudofunction): $\delta(t) = \begin{cases} 0, & t < 0 \\ \infty, & t = 0 \\ 0, & t > 0 \end{cases}$

Unit step function (Heviside step function): $1(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$
 $H(t)$ or $1_+(t)$

Ramp function: $x(t) = \begin{cases} 0, & t < 0 \\ t, & t > 0 \end{cases}$

Harmonic function: $x(t) = a \sin(\omega t)$

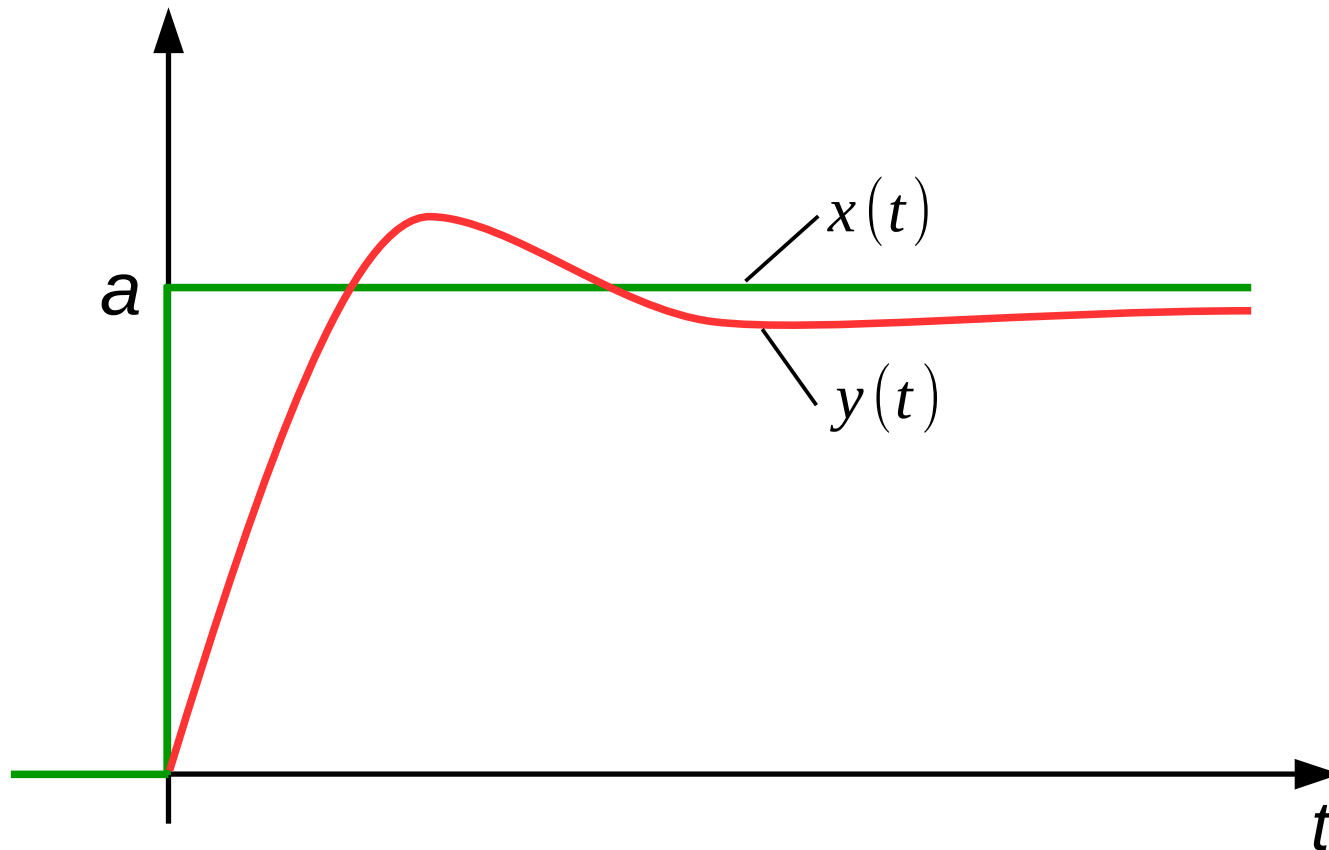
System step response

input: $x(t) = a \cdot 1(t)$ transfer function: $H(s)$ output: $y(t) = ?$

$$X(s) = L\{x(t)\} = a \cdot \frac{1}{s}$$

$$Y(s) = X(s) \cdot H(s)$$

$$y(t) = L^{-1}\{Y(s)\}$$



Step response – example 1

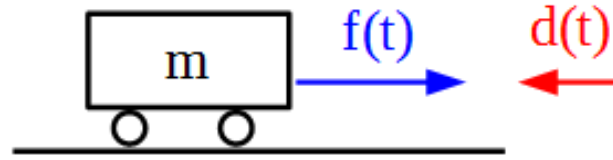
Car on a flat surface

m – mass,

$f(t)$ – driving force,

$d(t)=c*v(t)$ – air resistance,

$v(t)$ – velocity



$$m \frac{dv(t)}{dt} = f(t) - d(t)$$

Step response – example 1

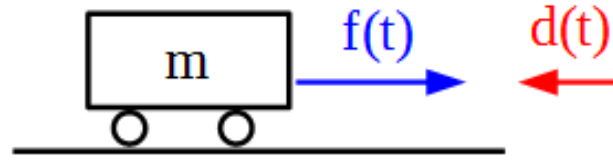
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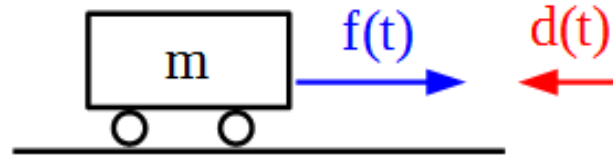
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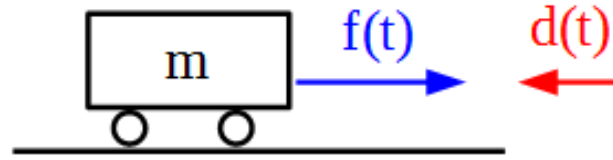
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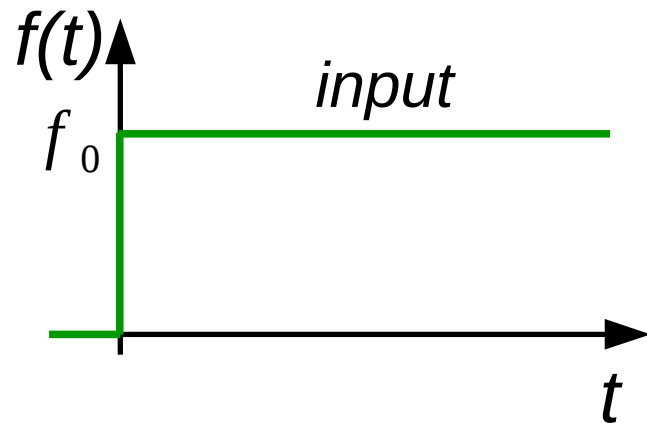
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$$m \frac{dv(t)}{dt} = f(t) - d(t)$$



$$f(t) = f_0 1(t)$$

$$F(s) = f_0 \frac{1}{s}$$

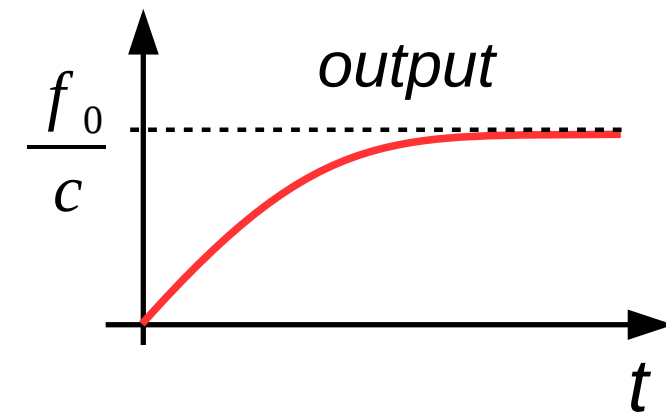
$$m \frac{dv(t)}{dt} = f(t) - c v(t)$$

$$m s V(s) = F(s) - c V(s)$$

$$H(s) = \frac{V(s)}{F(s)} = \frac{1}{ms + c}$$

$$V(s) = H(s) F(s) = \frac{1}{ms + c} f_0 \frac{1}{s} = \frac{f_0}{s(ms + c)}$$

$$v(t) = L^{-1} \left\{ \frac{f_0}{s(ms + c)} \right\} = L^{-1} \left\{ \frac{f_0}{c} \frac{c/m}{s(s + c/m)} \right\} = \frac{f_0}{c} \left(1 - e^{-\frac{c}{m}t} \right)$$



Step response – example 2

