Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control

Winter 2018/2019

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Institute of Machine Design Fundamentals - Department of Mechanics
http://www.ipbm.simr.pw.edu.pl/

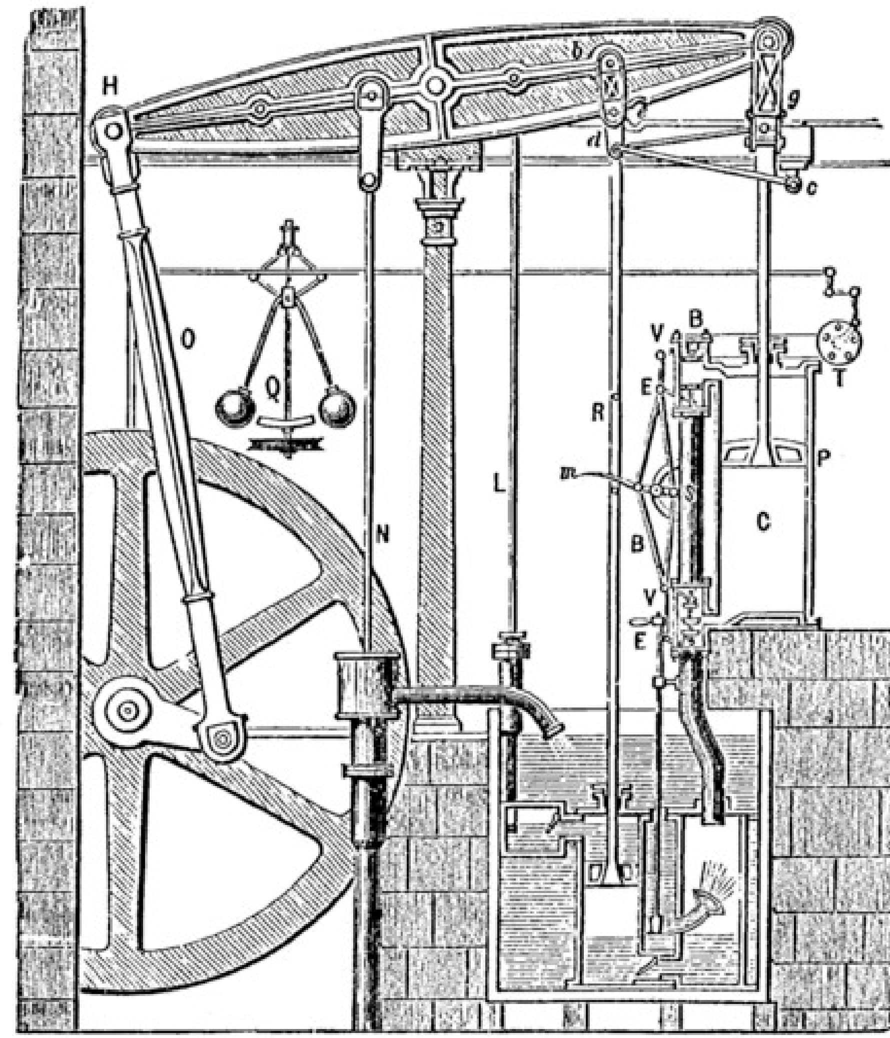
Lecture 6

Machine dynamics.
Reduction of masses and forces.
Machine equation of motion.

Machine dynamics

Overview

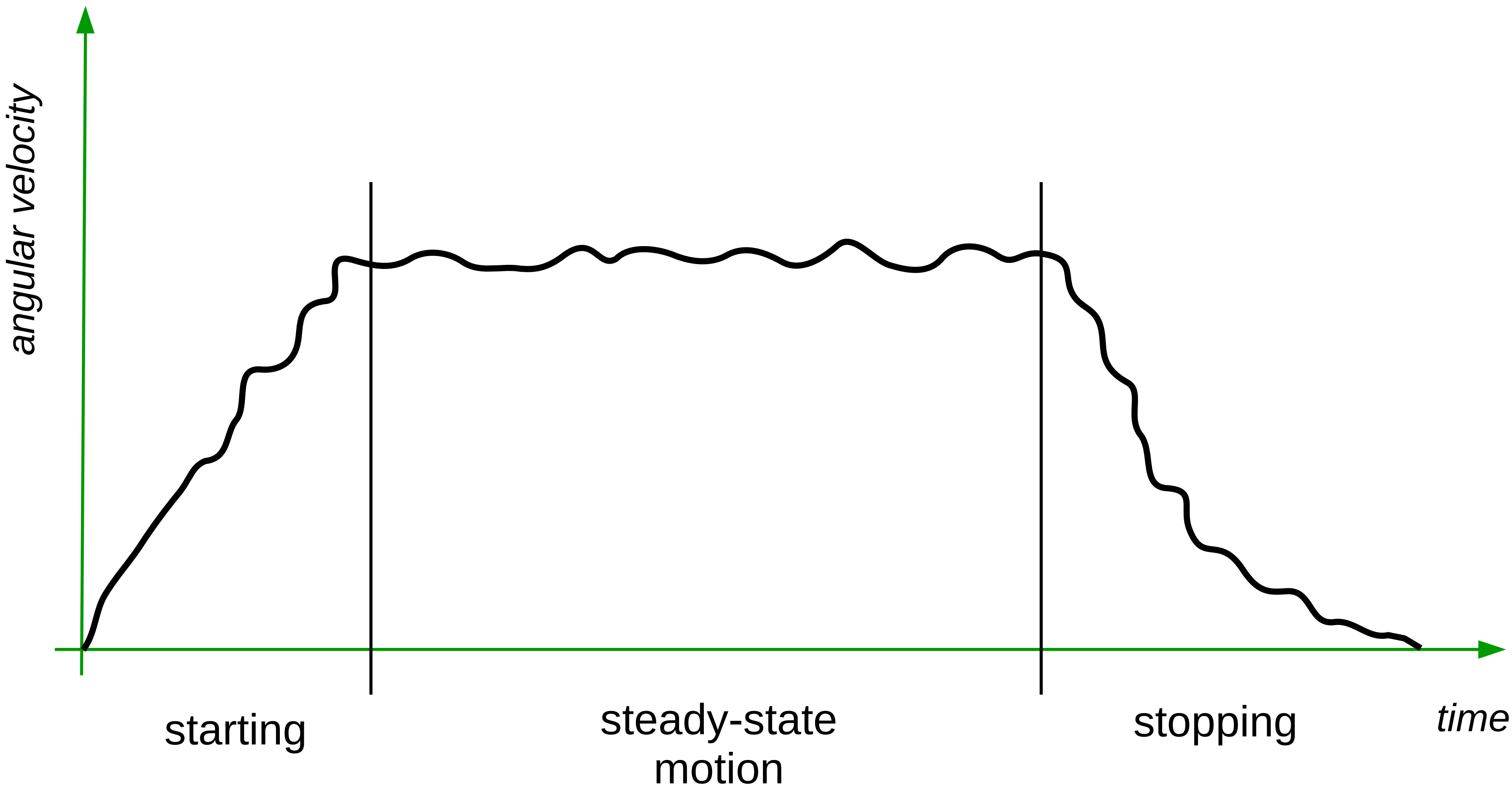
Machine – a tool containing one or more parts that uses energy to perform an intended action. Machines are assembled from components.



source: wikipedia.org, The Boulton & Watt Steam Engine, 1784

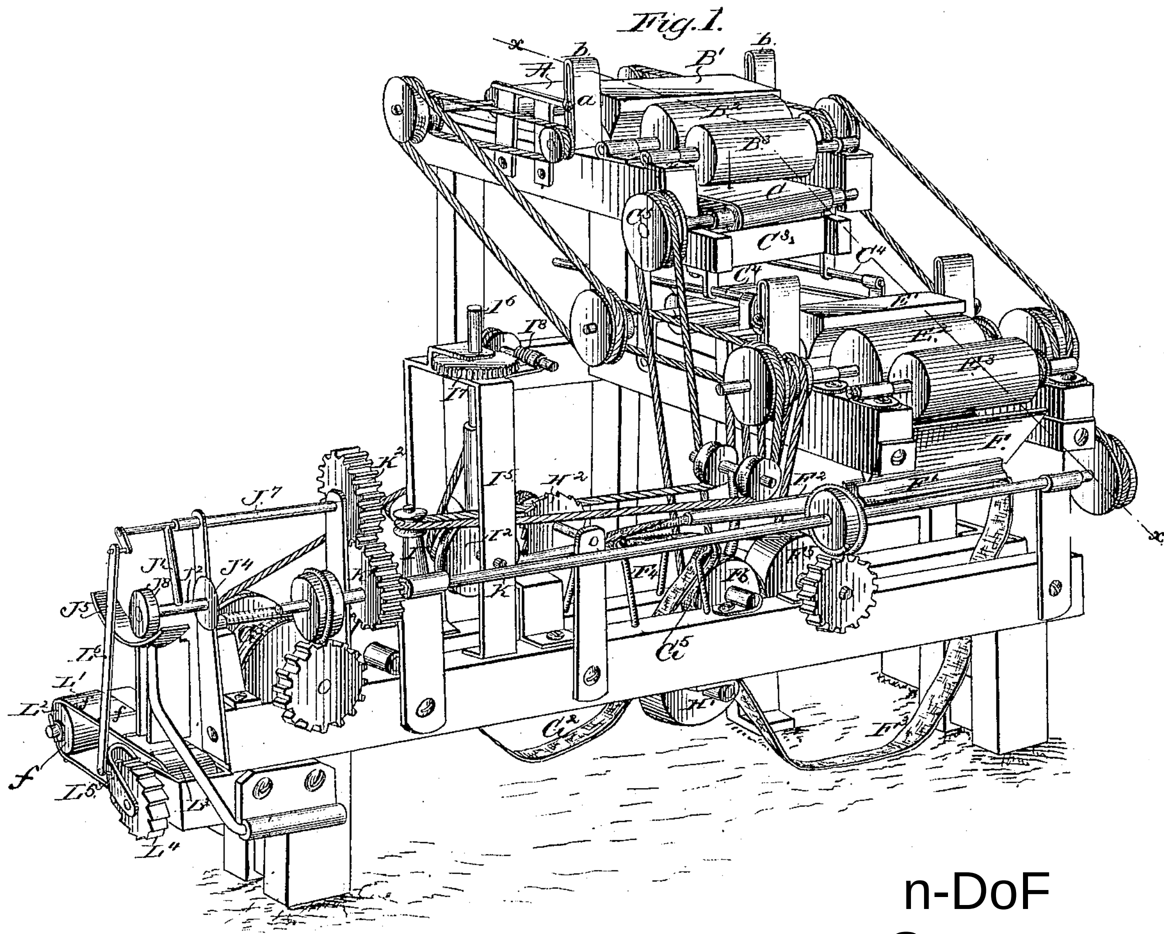
Machine dynamics

Overview



Reduction of masses and forces

Idea of reduction



Source: James Albert Bonsack (1859 – 1924) - U.S. patent 238,640
cigarette rolling machine, invented in 1880 and patented in 1881

n-DoF
System

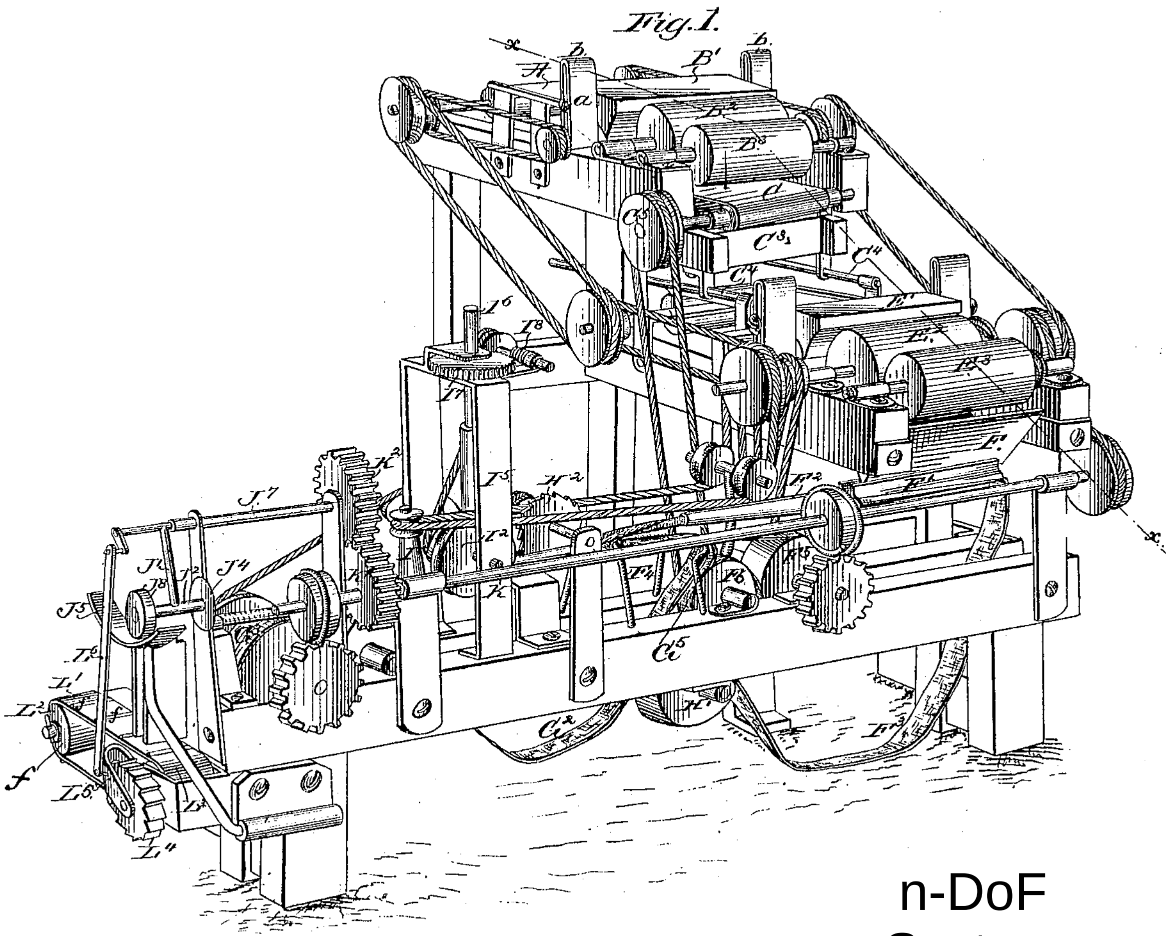
complicated

$$\ddot{x}_1(t)=F_1(x_1,x_2,...,t)$$
$$\ddot{x}_2(t)=F_2(x_1,x_2,...,t)$$
$$...$$
$$\ddot{x}_n(t)=F_n(x_1,x_2,...,t)$$

+constraints
+ limitations

Reduction of masses and forces

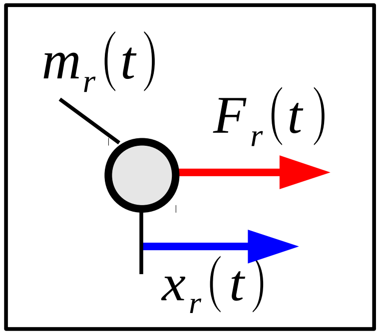
Idea of reduction



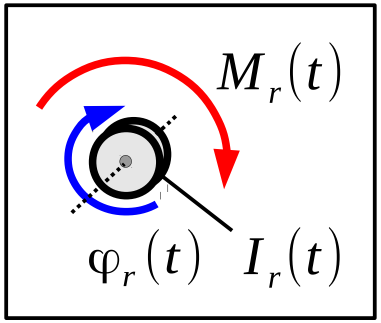
n-DoF
System

simpler, but not always possible

1-DoF
System

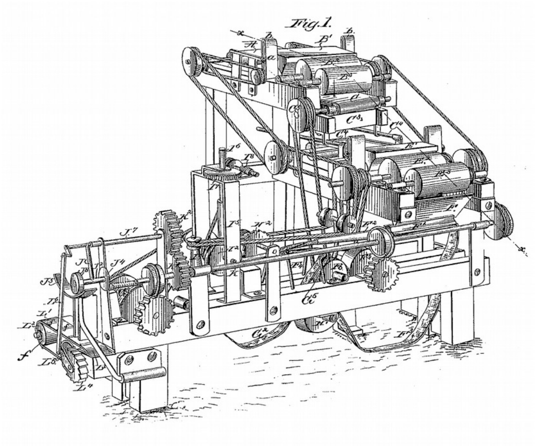


or



Reduction of masses

Kinetic energy

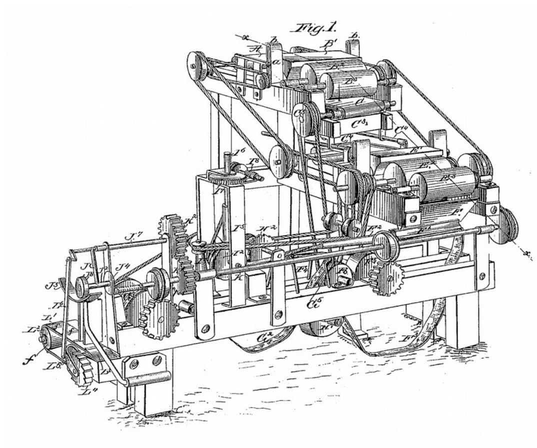


Total kinetic energy

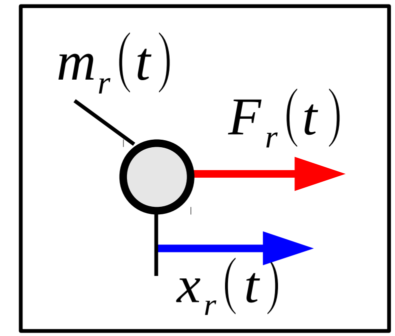
$$T=\sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2+\frac{1}{2}I_i\omega_i^2\right)$$

Reduction of masses

Kinetic energy



$$T = \frac{1}{2}\, m_r\, v_r^2$$



reduced mass

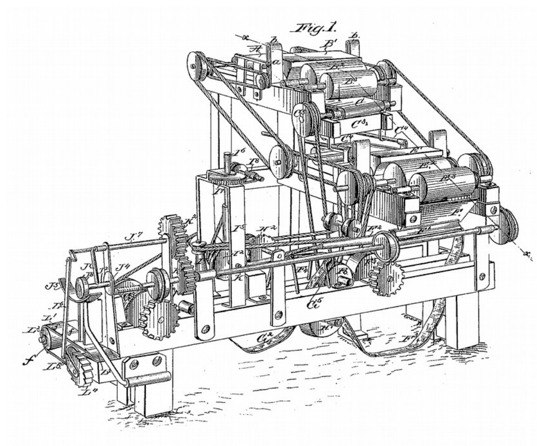
$$v_r = \frac{dx_r(t)}{dt}$$

Total kinetic energy

$$T = \sum_{i=1}^{n} \left(\frac{1}{2} m_i v_i^2 + \frac{1}{2} I_i \omega_i^2 \right)$$

Reduction of masses

Kinetic energy



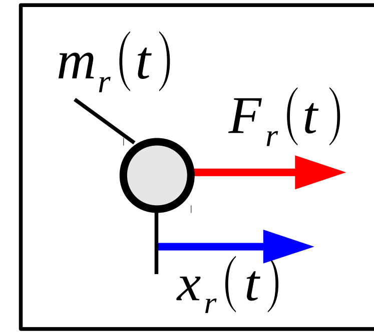
Total kinetic energy

$$T=\sum_{i=1}^{n}\left(\frac{1}{2}m_i v_i^2+\frac{1}{2}I_i\omega_i^2\right)$$

or

$$T=\frac{1}{2}m_r\,v_r^2$$

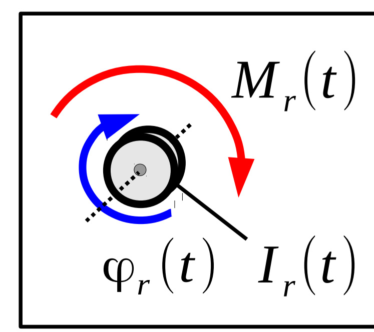
reduced mass



$$v_r=\frac{dx_r(t)}{dt}$$

$$T=\frac{1}{2}I_r\,\omega_r^2$$

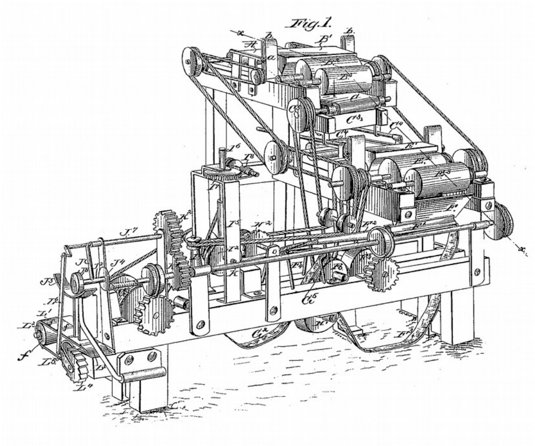
reduced moment of inertia



$$\omega_r=\frac{d\varphi_r(t)}{dt}$$

Reduction of forces

System power

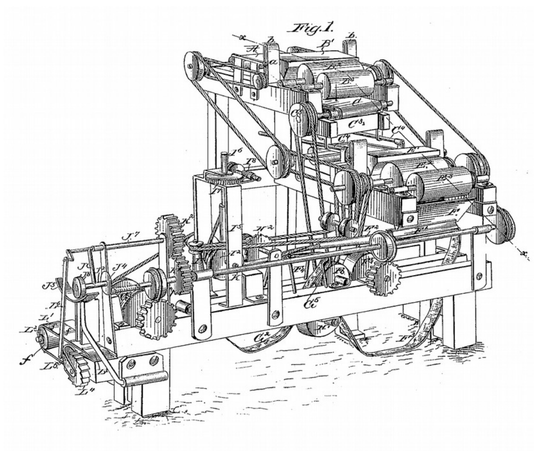


Total system's power

$$P(F_i, M_i, \omega_i, v_i, \ldots)$$

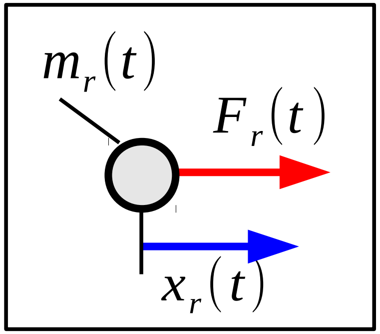
Reduction of forces

System power



$$P = F_r v_r$$

reduced force



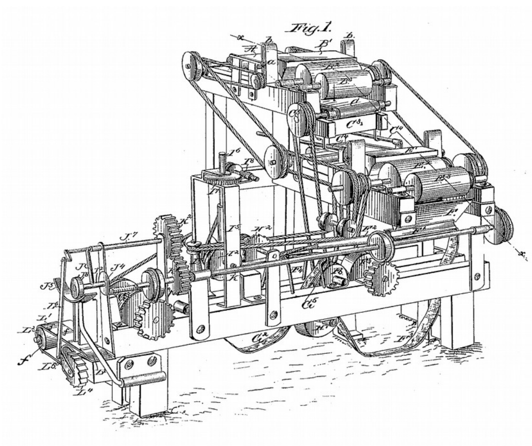
$$v_r = \frac{dx_r(t)}{dt}$$

Total system's power

$$P(F_i, M_i, \omega_i, v_i, \ldots)$$

Reduction of forces

System power



Total system's power

$$P\left(F_i, M_i, \omega_i, v_i, \ldots\right)$$

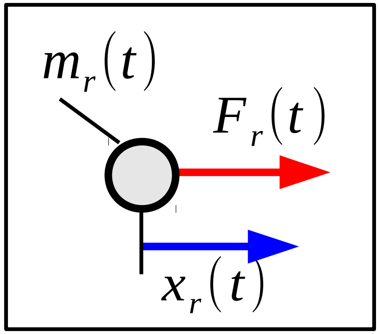
or

$$P = F_r v_r$$

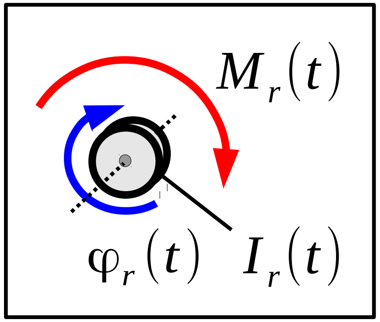
reduced force

$$P = M_r \omega_r$$

reduced torque

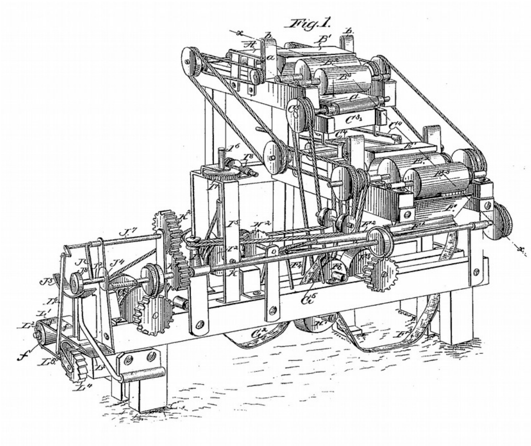


$$v_r = \frac{dx_r(t)}{dt}$$



$$\omega_r = \frac{d\varphi_r(t)}{dt}$$

Reduction of masses – details



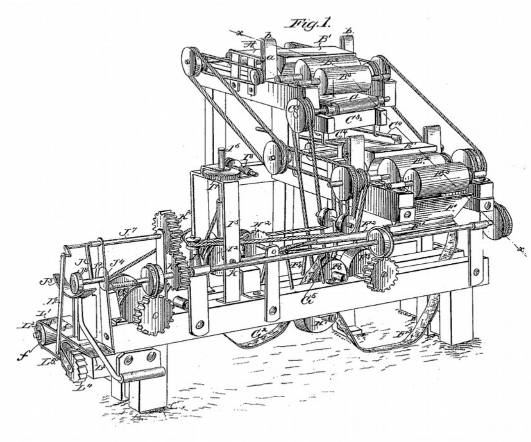
Kinetic energy

$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

Reduction of masses – details

Kinetic energy



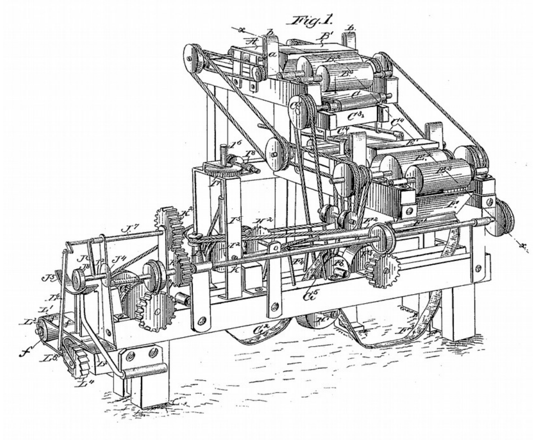
$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

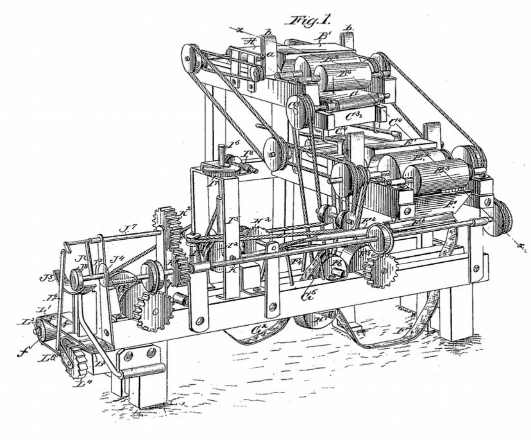
n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

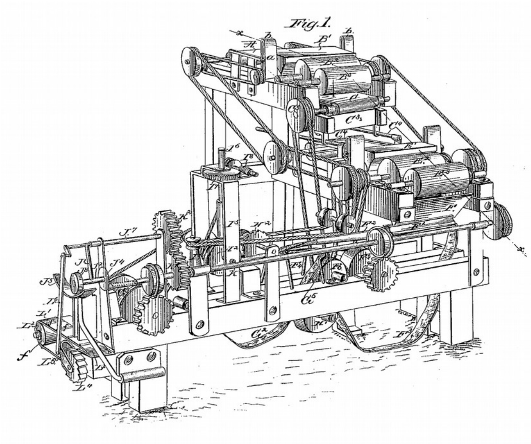
n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$\frac{1}{2}I_r\omega_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

Reduction of masses – details



Kinetic energy

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 + \sum_{j=1}^{k} \frac{1}{2} I_j \omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2} m_r v_r^2 = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 + \sum_{j=1}^{k} \frac{1}{2} I_j \omega_j^2$$

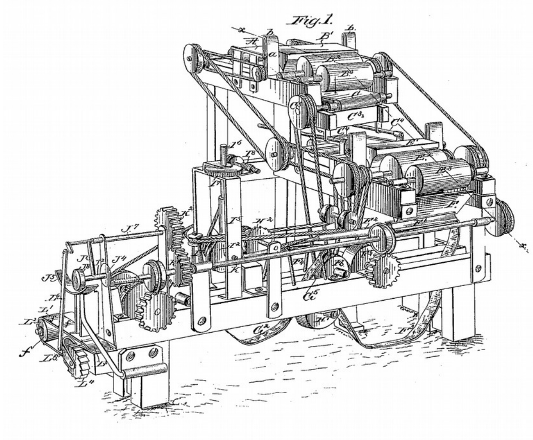
$$\frac{1}{2} I_r \omega_r^2 = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2 + \sum_{j=1}^{k} \frac{1}{2} I_j \omega_j^2$$

$$m_r = \sum_{i=1}^{n} m_i \frac{v_i^2}{v_r^2} + \sum_{j=1}^{k} I_j \frac{\omega_j^2}{v_r^2}$$

$$I_r = \sum_{i=1}^{n} m_i \frac{v_i^2}{\omega_r^2} + \sum_{j=1}^{k} I_j \frac{\omega_j^2}{\omega_r^2}$$

Reduction of masses – details

Kinetic energy



$$T=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

n – translating elements
k – rotating elements

$$\frac{1}{2}m_r v_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

$$\frac{1}{2}I_r \omega_r^2=\sum_{i=1}^{n}\frac{1}{2}m_i v_i^2+\sum_{j=1}^{k}\frac{1}{2}I_j\omega_j^2$$

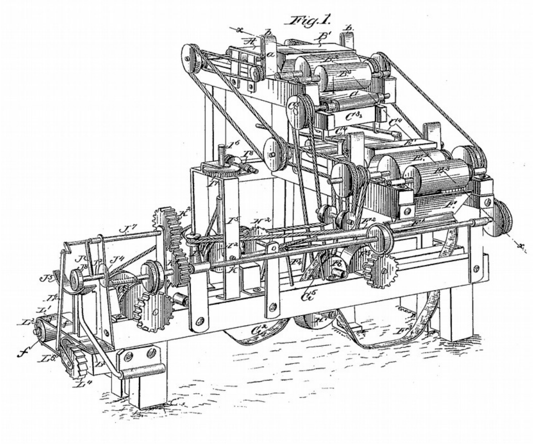
$$m_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{v_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{v_r^2}$$

$$I_r=\sum_{i=1}^{n}m_i\frac{v_i^2}{\omega_r^2}+\sum_{j=1}^{k}I_j\frac{\omega_j^2}{\omega_r^2}$$

$v_r,\ \omega_r$ – arbitrary chosen velocities

Reduction of forces – details

Work

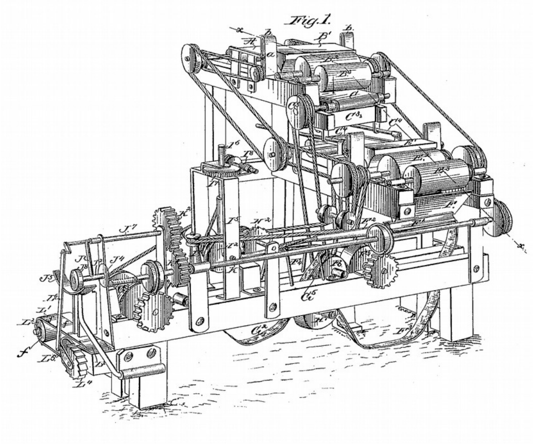


$$dW = \sum_{i=1}^{n} P_i ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

Reduction of forces – details

Work



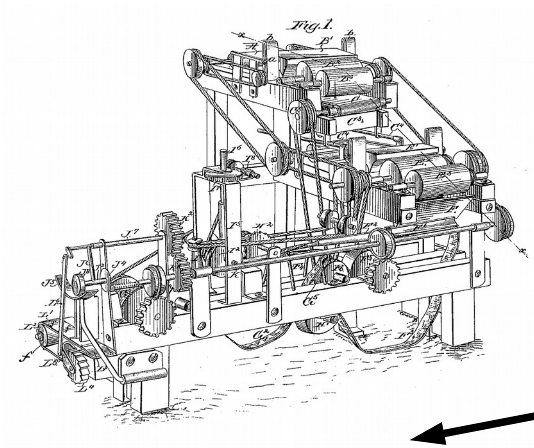
$$dW=\sum_{i=1}^{n} P_i\, ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j\, d\varphi_j$$

$$\alpha_i = \sphericalangle\left(P_i, ds_i\right)$$

n – translating elements
k – rotating elements

Reduction of forces – details

Work



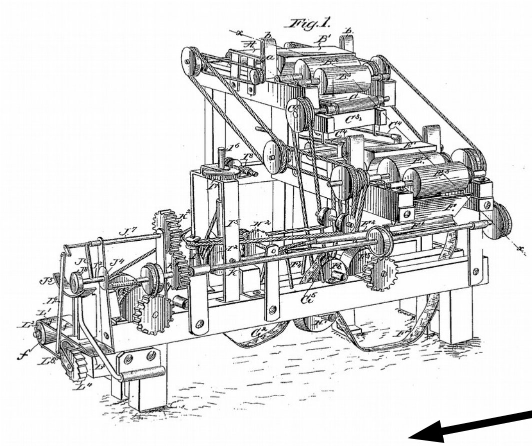
$$dW = \sum_{i=1}^{n} P_i \, ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j \, d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r \, ds_r = \sum_{i=1}^{n} P_i \, ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j \, d\varphi_j$$

Reduction of forces – details

Work



$$dW = \sum_{i=1}^{n} P_i \, ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j \, d\varphi_j$$

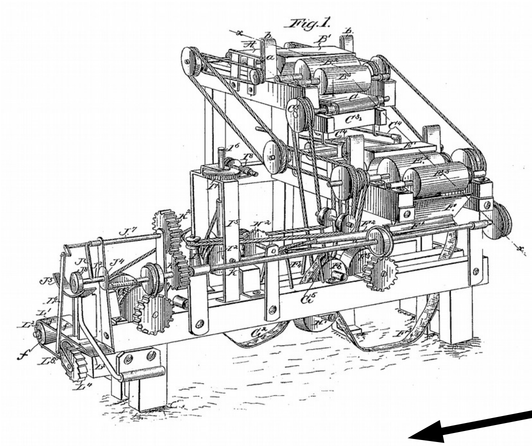
n – translating elements
k – rotating elements

$$P_r \, ds_r = \sum_{i=1}^{n} P_i \, ds_i \cos\alpha_i + \sum_{j=1}^{k} M_j \, d\varphi_j$$

$$P_r = \sum_{i=1}^{n} P_i \frac{ds_i}{ds_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

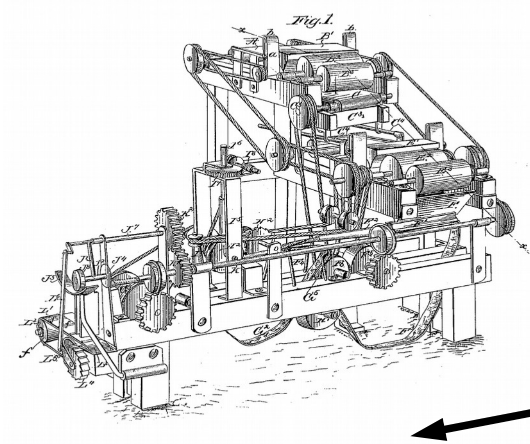
$$P_r ds_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i dt}{v_r dt}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j dt}{v_r dt}$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r ds_r=\sum_{i=1}^{n} P_i ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j d\varphi_j$$

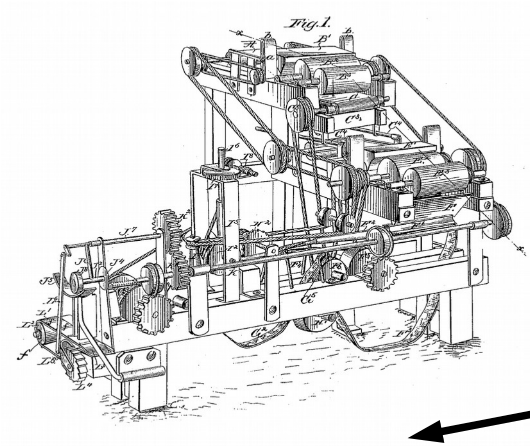
$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r} \cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i dt}{v_r dt} \cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j dt}{v_r dt}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i}{v_r} \cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{v_r}$$

Reduction of forces – details

Work



$$dW=\sum_{i=1}^{n} P_i\, ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j\, d\varphi_j$$

n – translating elements
k – rotating elements

$$P_r\, ds_r=\sum_{i=1}^{n} P_i\, ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j\, d\varphi_j$$

$$P_r=\sum_{i=1}^{n} P_i \frac{ds_i}{ds_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{ds_r}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i\, dt}{v_r\, dt}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j\, dt}{v_r\, dt}$$

$$P_r=\sum_{i=1}^{n} P_i \frac{v_i}{v_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{v_r}$$

$$M_r\, d\varphi_r=\sum_{i=1}^{n} P_i\, ds_i \cos\alpha_i+\sum_{j=1}^{k} M_j\, d\varphi_j$$

$$M_r=\sum_{i=1}^{n} P_i \frac{ds_i}{d\varphi_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{d\varphi_j}{d\varphi_r}$$

$$M_r=\sum_{i=1}^{n} P_i \frac{v_i\, dt}{\omega_r\, dt}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j\, dt}{\omega_r\, dt}$$

$$M_r=\sum_{i=1}^{n} P_i \frac{v_i}{\omega_r}\cos\alpha_i+\sum_{j=1}^{k} M_j \frac{\omega_j}{\omega_r}$$

Reduction of masses/moments of inertia

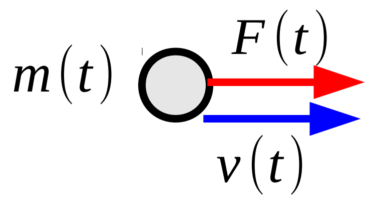
$$m_r = \sum_{i=1}^{n} m_i \frac{v_i^2}{v_r^2} + \sum_{j=1}^{k} I_j \frac{\omega_j^2}{v_r^2} \qquad I_r = \sum_{i=1}^{n} m_i \frac{v_i^2}{\omega_r^2} + \sum_{j=1}^{k} I_j \frac{\omega_j^2}{\omega_r^2}$$

Reduction of forces/torques

$$P_r = \sum_{i=1}^{n} P_i \frac{v_i}{v_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{\omega_j}{v_r} \qquad M_r = \sum_{i=1}^{n} P_i \frac{v_i}{\omega_r} \cos\alpha_i + \sum_{j=1}^{k} M_j \frac{\omega_j}{\omega_r}$$

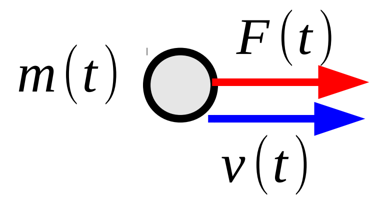
Machine equation of motion

Linear motion



Machine equation of motion

Linear motion



elementary change of kinetic energy

elementary work

$$dT = dW$$

complete differential of kinetic energy

$$d\left(\frac{1}{2}m(t)v(t)^2\right) = F(t)\,dx$$

$$\frac{1}{2}dm(t)v(t)^2 + m(t)v(t)dv(t) = F(t)\,dx$$

$$\frac{1}{2}dm(t)v(t)^2 + m(t)\frac{dx(t)}{dt}dv(t) = F(t)\,dx$$

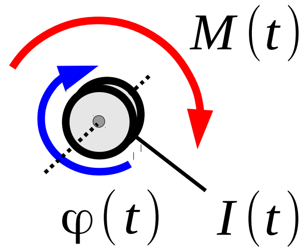
$$\frac{dm(t)}{dx}\frac{v(t)^2}{2} + m\frac{dv(t)}{dt} = F(t)$$

$$\boxed{\frac{dm(t)}{dt}\frac{v(t)}{2} + m\frac{dv(t)}{dt} = F(t)}$$

$$if\;\; m = const. \Rightarrow m\frac{dv(t)}{dt} = P(t) \quad or \quad m\ddot{x}(t) = F(t)$$

Machine equation of motion

Angular motion



$$dT = dW$$

$$d\left(I\frac{\omega(t)^2}{2}\right) = M(t)\,d\varphi$$

$$\ldots$$

$$\ldots$$

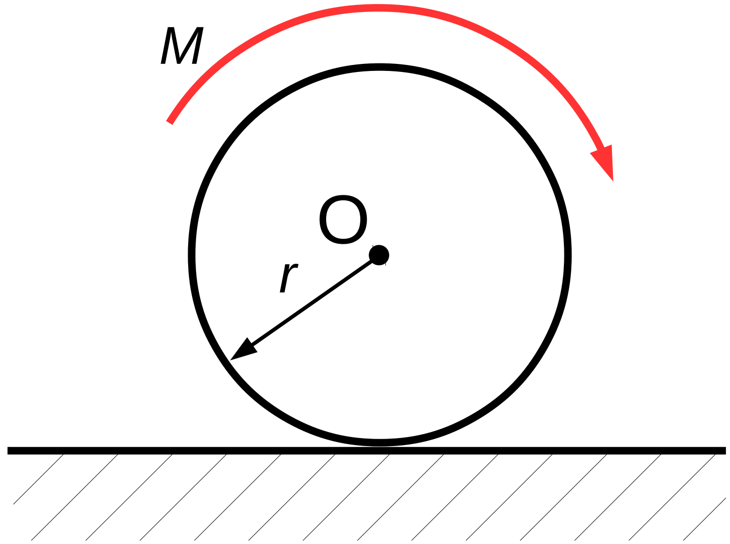
$$\frac{dI(t)}{d\varphi}\frac{\omega(t)^2}{2} + I(t)\frac{d\omega(t)}{dt} = M(t)$$

$$\boxed{\frac{dI(t)}{dt}\frac{\omega(t)}{2} + I(t)\frac{d\omega(t)}{dt} = M(t)}$$

$$if\quad I = const. \Rightarrow\; I\frac{d\omega(t)}{dt} = M(t)\quad or\quad I\ddot{\varphi}(t) = M(t)$$

Reduction of masses and forces

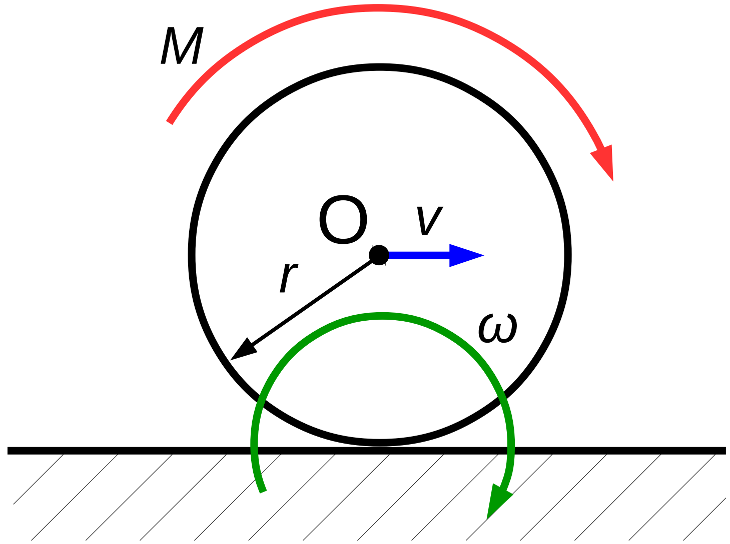
Rolling wheel (without a slip)



Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Reduction of masses and forces

Rolling wheel (without a slip)



Given: m – wheel's mass,
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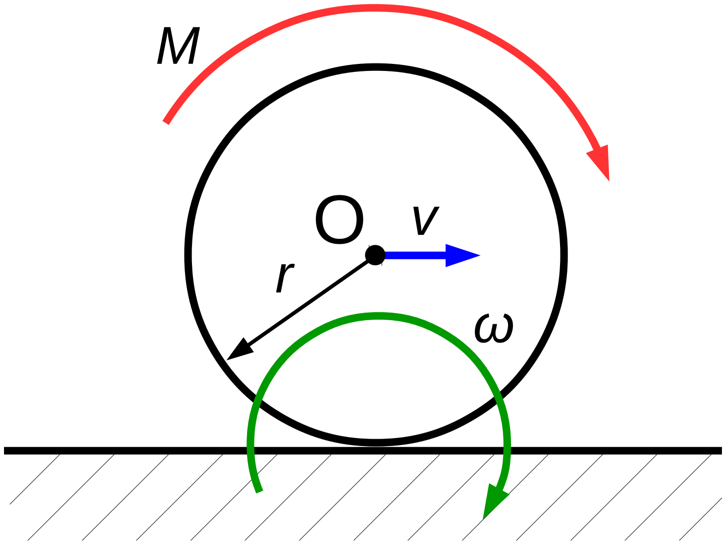
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

Reduction of masses and forces

Rolling wheel (without a slip)



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Reduction of masses and forces

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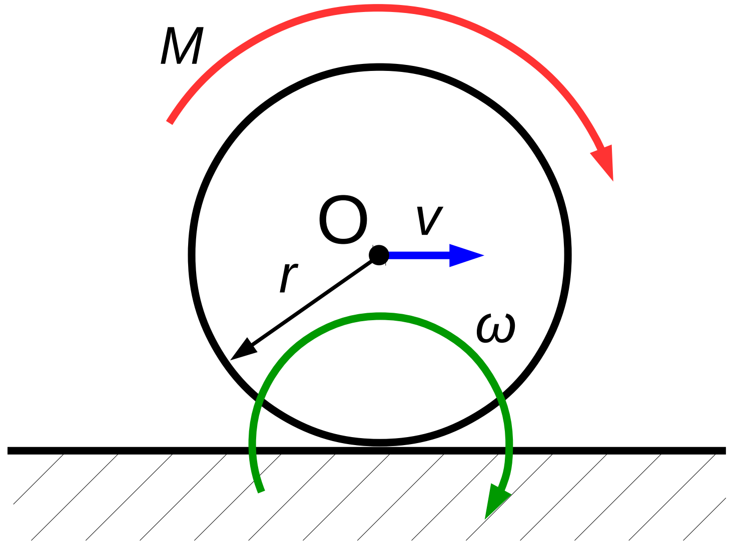

Given: m – wheel's mass,
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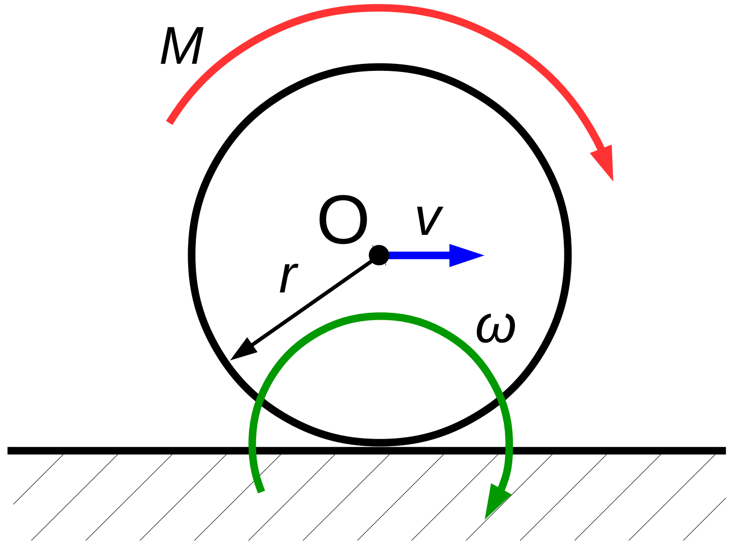
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\omega^2$$

Reduction of masses and forces

Rolling wheel (without a slip)



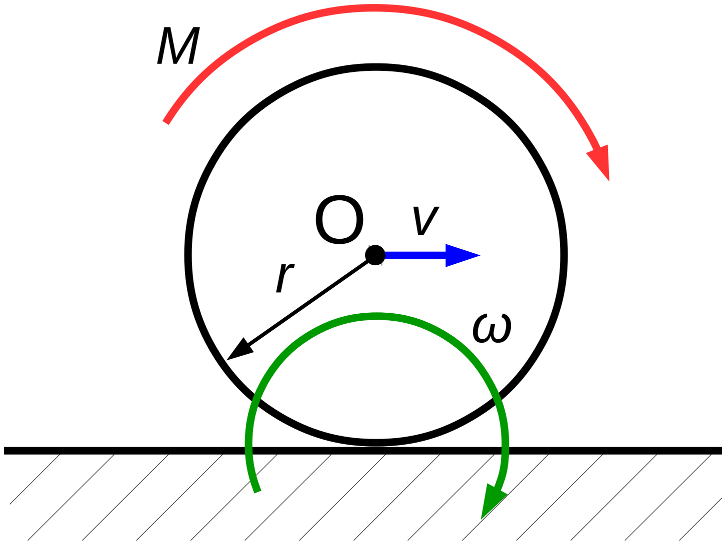
Given: m – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\omega^2$$ but $v=\omega r$

Reduction of masses and forces

Rolling wheel (without a slip)



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r – wheel's radius,
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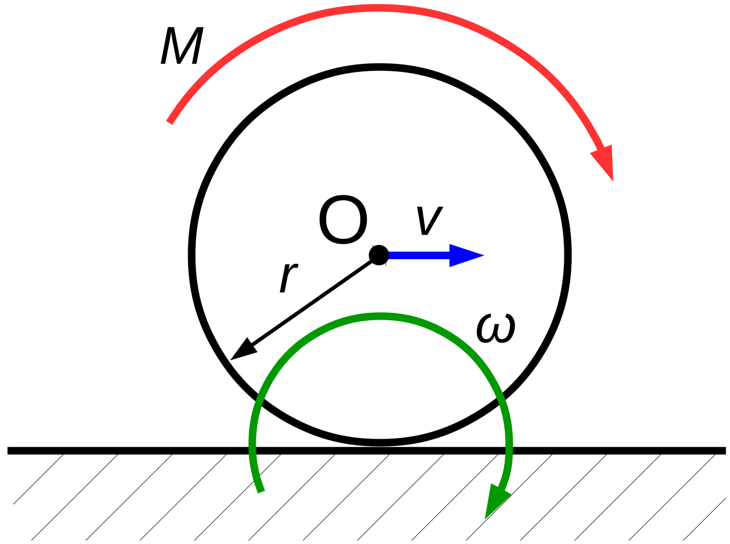
Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}$$

Reduction of masses and forces

Rolling wheel (without a slip)



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r – wheel's radius,
M – torque.

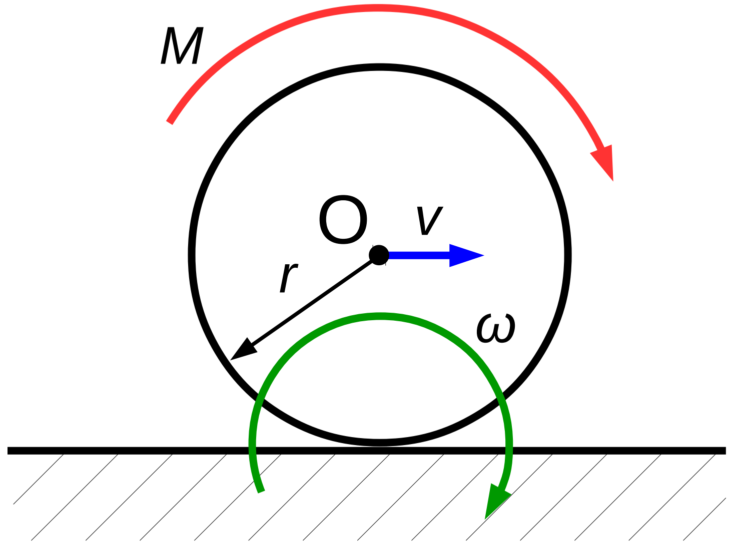
Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r$$

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2$$

Reduction of masses and forces

Rolling wheel (without a slip)



Given: *m* – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

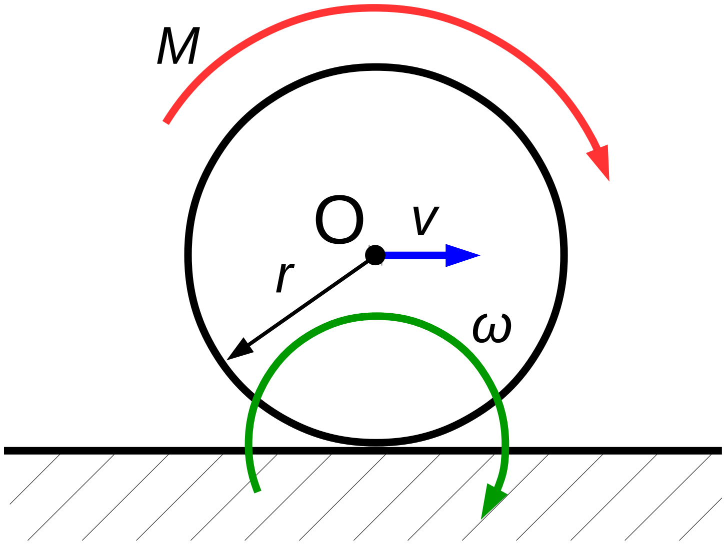
$$T = \frac{1}{2} m v^2 + \frac{1}{2} I_O \omega^2 \qquad \text{but } v = \omega r$$

$$T = \frac{1}{2} m v^2 + \frac{1}{2} I_O \frac{v^2}{r^2} = \frac{1}{2}\left(m + \frac{I_O}{r^2}\right) v^2 = \frac{1}{2} m_r v^2$$

$$m_r = m + \frac{I_O}{r^2} = \text{const.}$$

Reduction of masses and forces

Rolling wheel (without a slip)



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I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

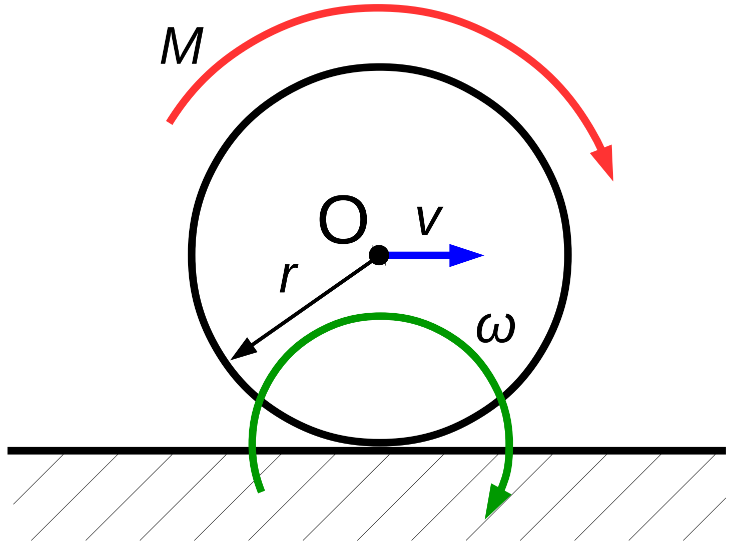
$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r \qquad P=M\omega$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

Reduction of masses and forces

Rolling wheel (without a slip)



Given: *m* – wheel's mass,
I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

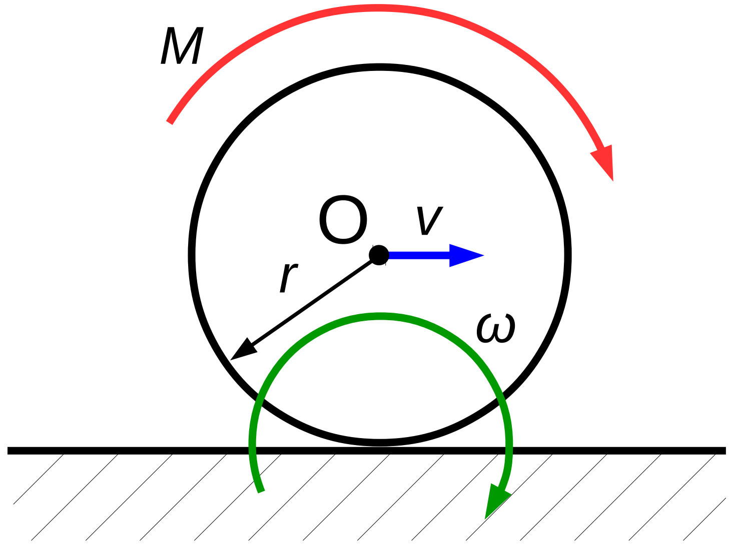
$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r \qquad P=M\omega$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2 \qquad P=M\frac{v}{r}$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

Reduction of masses and forces

Rolling wheel (without a slip)



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I_O – wheel's mass moment of inertia in point O,
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M – torque.

Assume:
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ω – angular velocity of the wheel.

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r$$

$$T=\frac{1}{2}m v^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

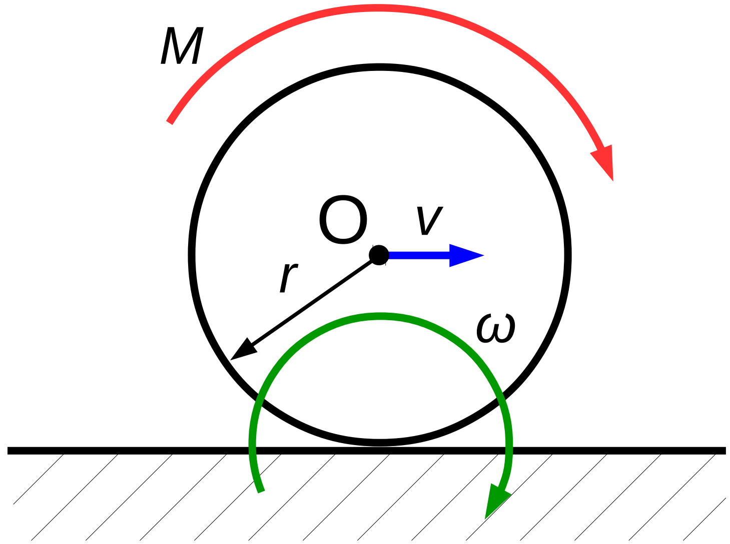
$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

$$P=M\omega$$

$$P=M\frac{v}{r}=\frac{M}{r}v$$

Reduction of masses and forces

Rolling wheel (without a slip)



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I_O – wheel's mass moment of inertia in point O,
r – wheel's radius,
M – torque.

Assume:
v – velocity of the wheel's center,
ω – angular velocity of the wheel.

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\omega^2 \qquad \text{but } v=\omega r$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

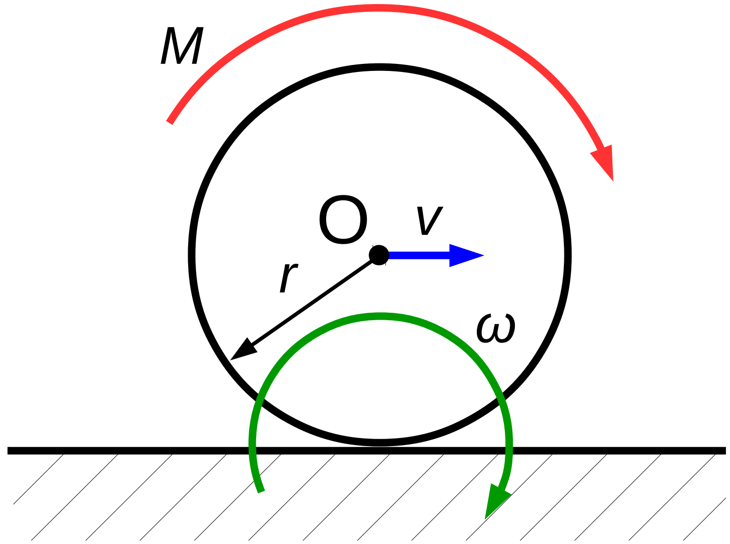
$$P=M\omega$$

$$P=M\frac{v}{r}=\frac{M}{r}v=F_r v$$

$$F_r=\frac{M}{r}$$

Reduction of masses and forces

Rolling wheel (without a slip)



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Assume:
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ω – angular velocity of the wheel.

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$$P=M\omega$$

$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

$$P=M\frac{v}{r}=\frac{M}{r}v=F_r v$$

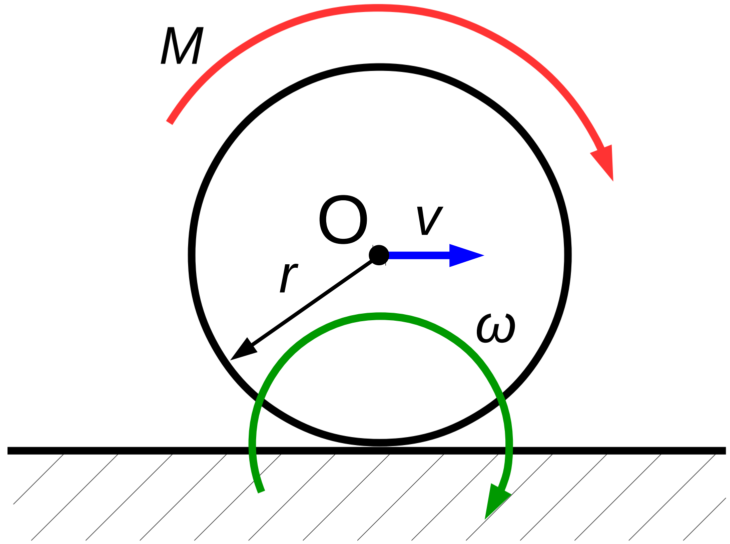
$$F_r=\frac{M}{r}$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

$$m_r\frac{dv}{dt}=F_r$$

Reduction of masses and forces

Rolling wheel (without a slip)



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ω – angular velocity of the wheel.

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$$T=\frac{1}{2}mv^2+\frac{1}{2}I_O\frac{v^2}{r^2}=\frac{1}{2}\left(m+\frac{I_O}{r^2}\right)v^2=\frac{1}{2}m_r v^2$$

$$m_r=m+\frac{I_O}{r^2}=\text{const.}$$

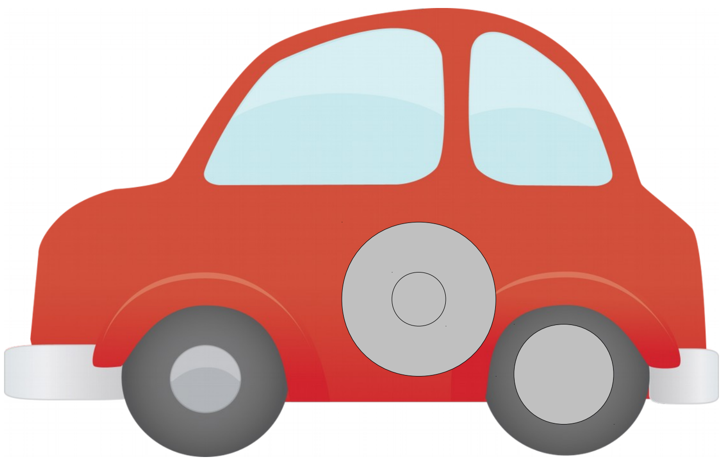
$$P=M\omega$$

$$P=M\frac{v}{r}=\frac{M}{r}v=F_r v$$

$$F_r=\frac{M}{r}$$

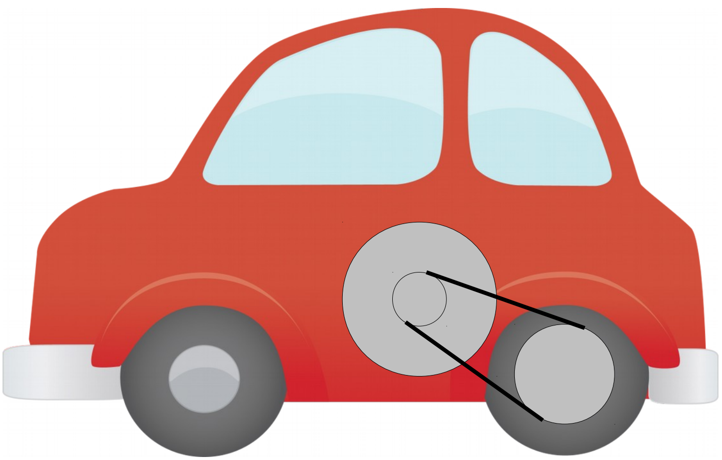
$$\boxed{\left(m+\frac{I_0}{r^2}\right)\frac{dv}{dt}=\frac{M}{r}}$$

Reduction of masses and forces – quiz



m_1 – total mass

m_{r1} – reduced mass



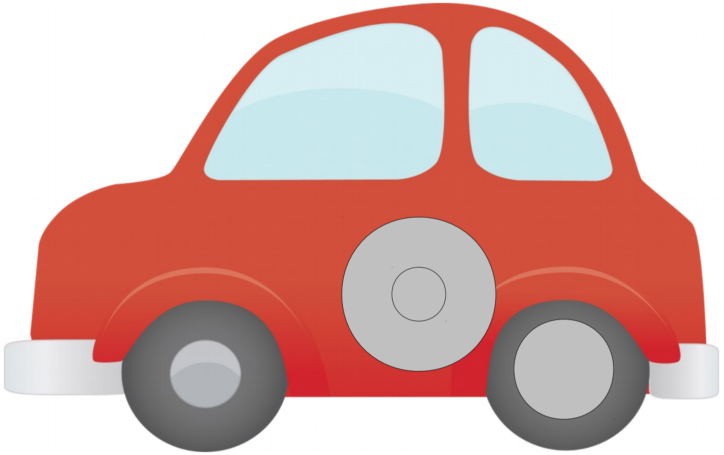
m_2 – total mass

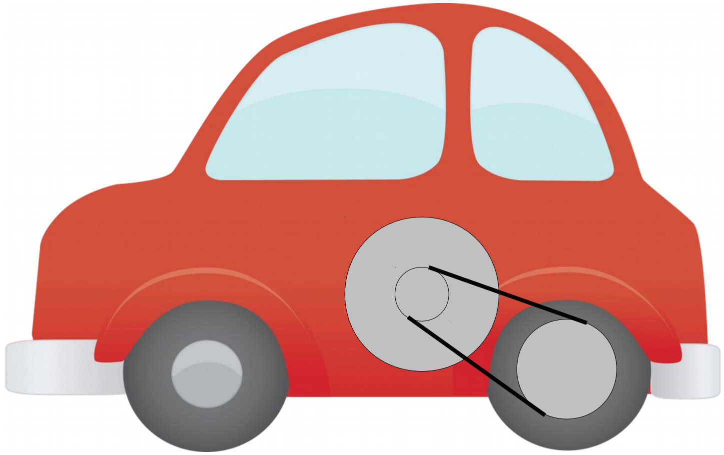
m_{r2} – reduced mass

$$m_1 = m_2$$

$$m_{r1} \; ? \; m_{r2}$$

Reduction of masses and forces – quiz





m_1 – total mass
m_{r1} – reduced mass

m_2 – total mass
m_{r2} – reduced mass

$$m_1 = m_2$$

$$m_{r1} < m_{r2}$$

Reduction of masses and forces

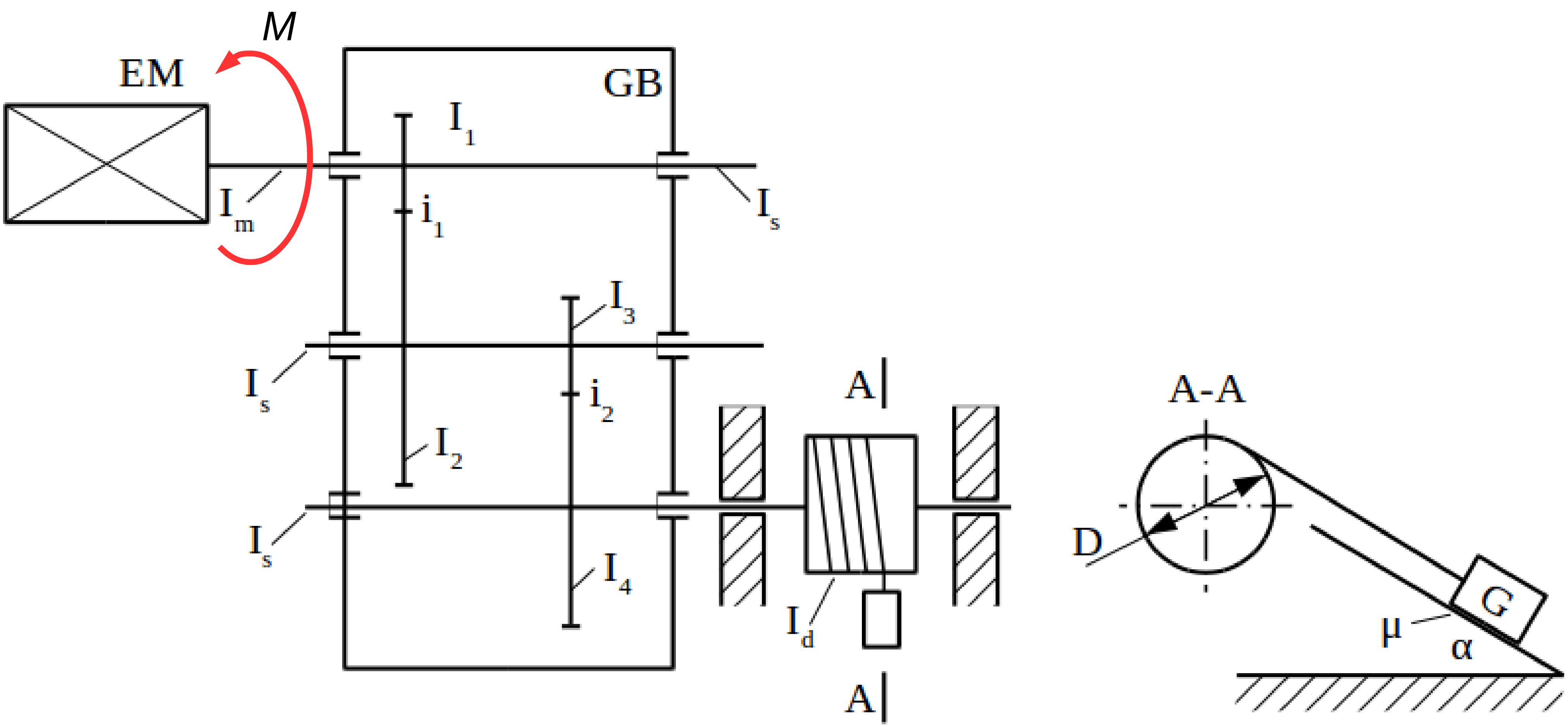
Example 1

In this example a drum winch is analyzed. It consist of:

- electric motor EM, which generates torque as a function of angular velocity ω: $M=A-B\omega$ where A and B are given constants; rotor moment of inertia is equal to I_m;
- two stage gearbox GB (reducer) with given gears moments of inertia I_1, I_2, I_3, I_4; shafts moments of inertia are equal to I_s; gears ratio are given as $i_1=\omega_2/\omega_1$ and $i_2=\omega_4/\omega_3$
- winch's drum has diameter D and moment of inertia I_d; drum is set on two ball bearing that generates resistance M_f assumed to be constant;
- inclined plane with angle α related to the horizon line;
- box of weight G pulled up with winch. Friction between object and inclined plane is represented as dry friction with μ coefficient.

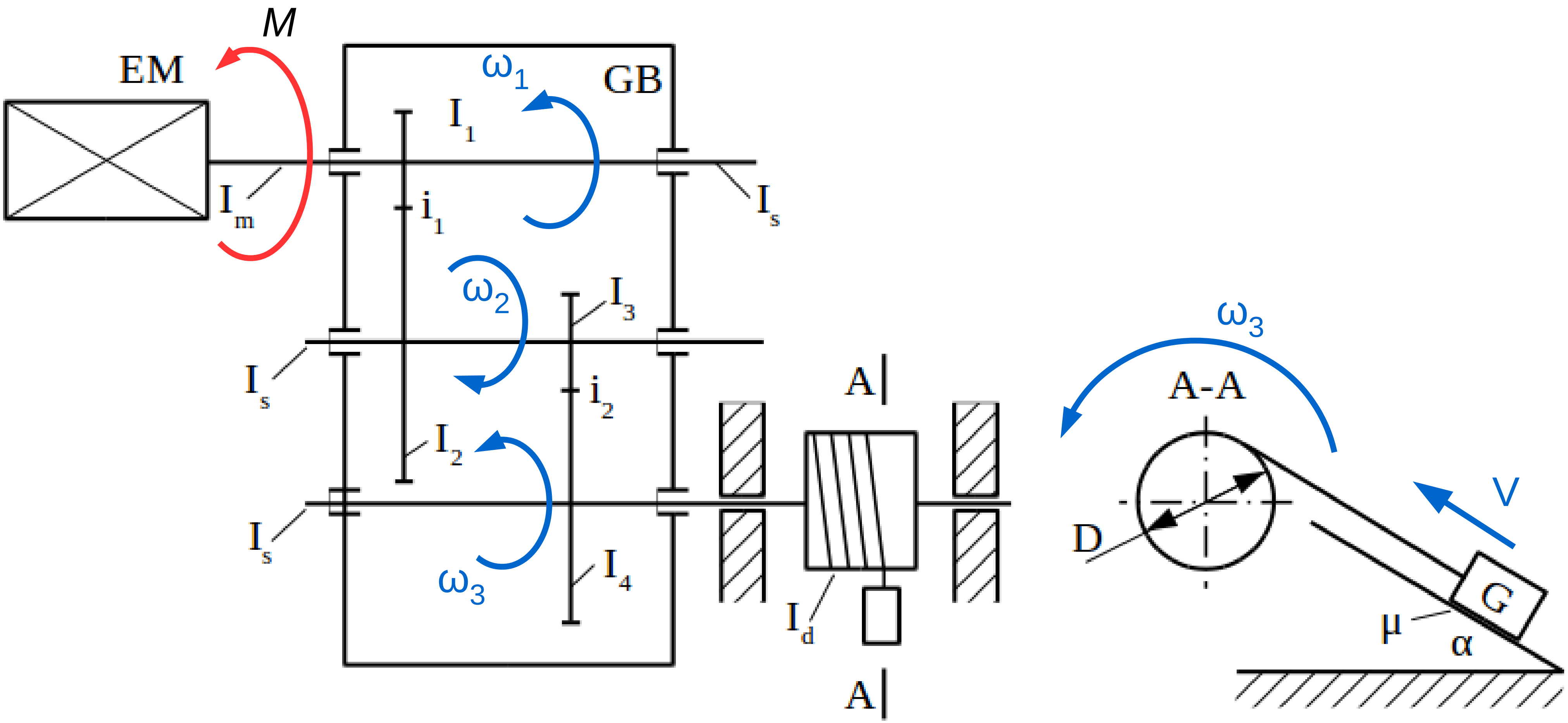
Reduction of masses and forces

Example 1



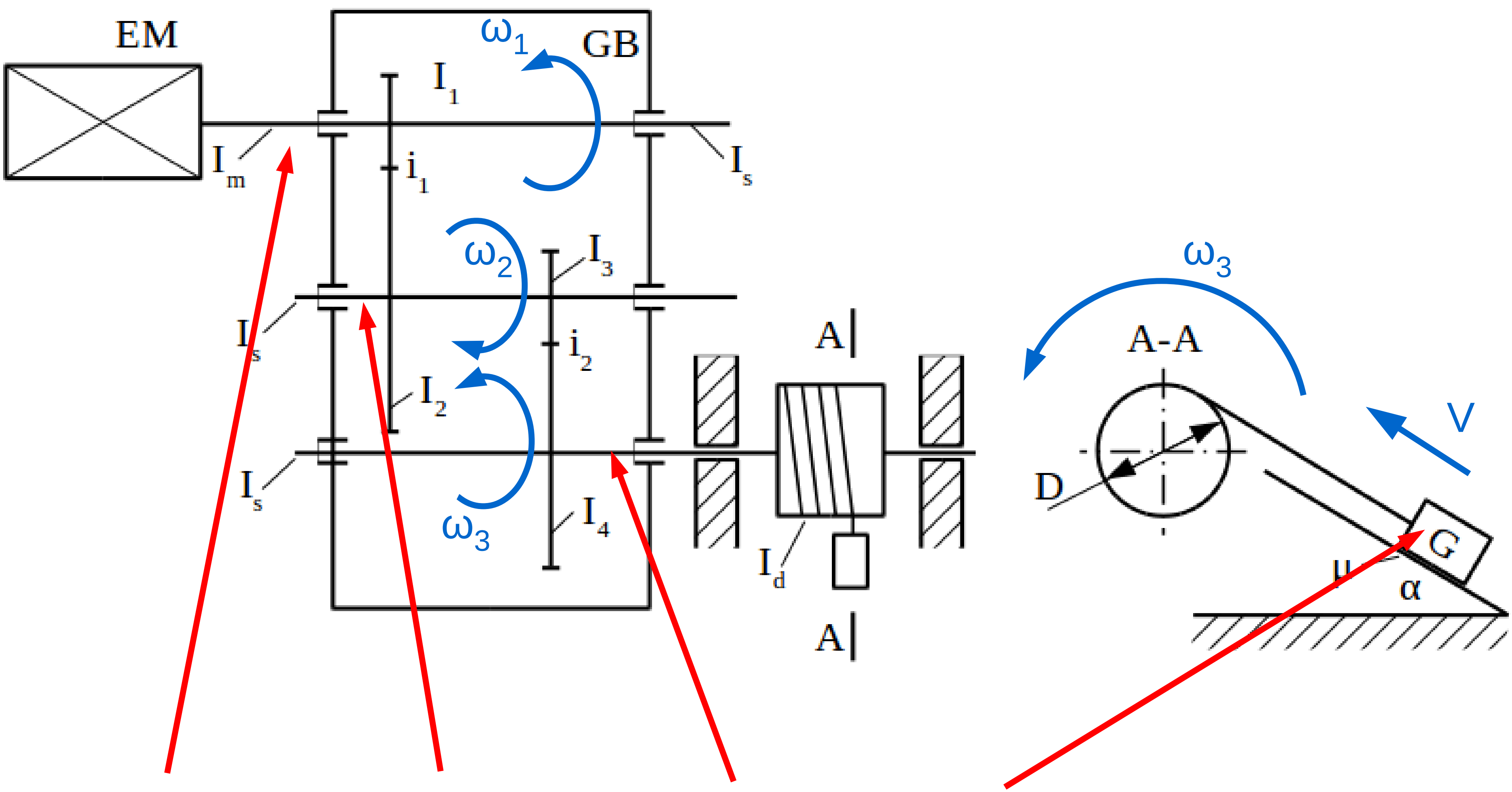
Reduction of masses and forces

Example 1



Reduction of masses and forces

Example 1



$$T=\frac{1}{2}\left(I_m+I_1+I_s\right)\omega_1^2+\frac{1}{2}\left(I_2+I_3+I_s\right)\omega_2^2+\frac{1}{2}\left(I_4+I_d+I_s\right)\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

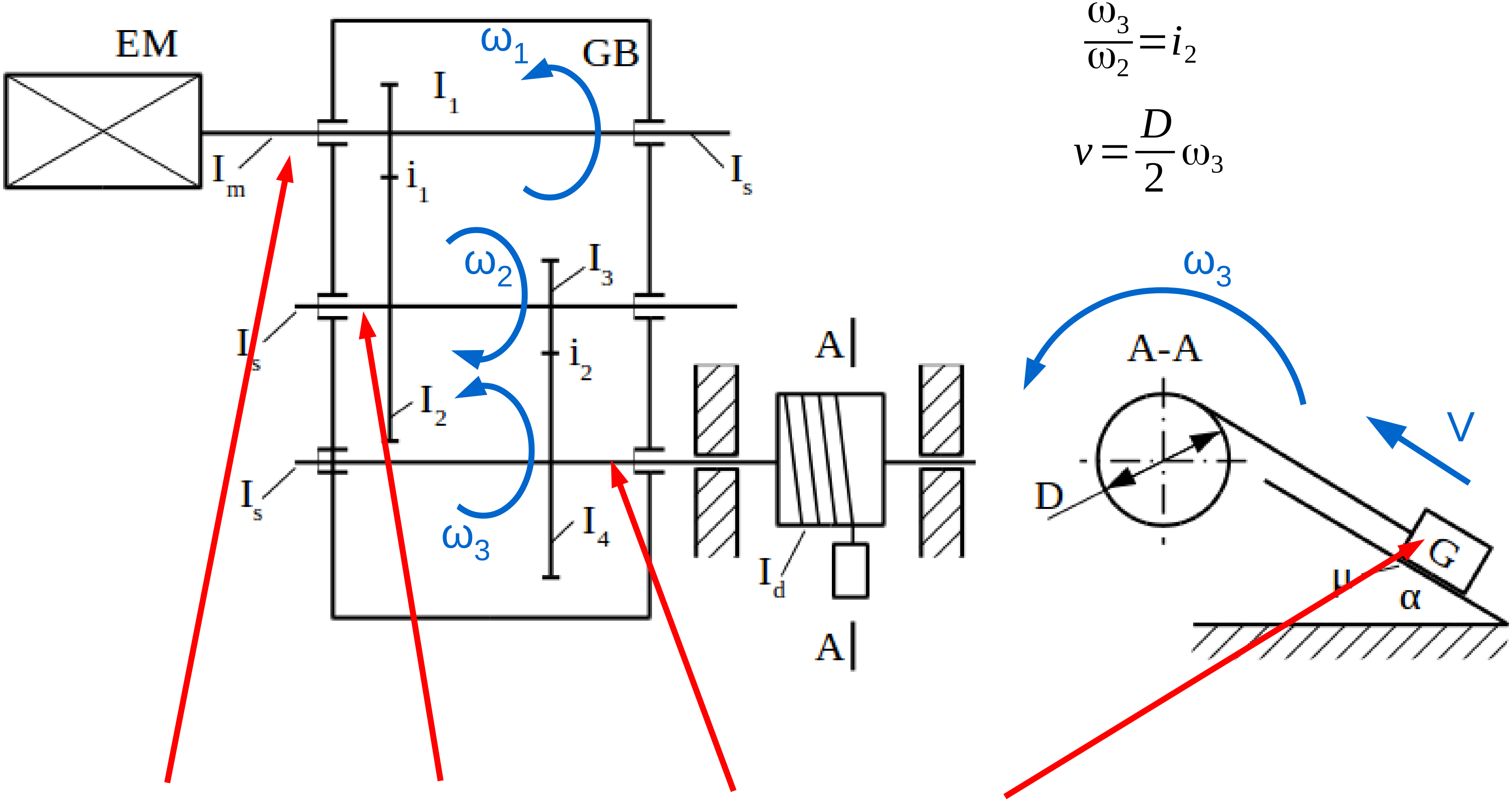
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1}=i_1$$

$$\frac{\omega_3}{\omega_2}=i_2$$

$$v=\frac{D}{2}\omega_3$$



$$T=\frac{1}{2}(I_m+I_1+I_s)\omega_1^2+\frac{1}{2}(I_2+I_3+I_s)\omega_2^2+\frac{1}{2}(I_4+I_d+I_s)\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

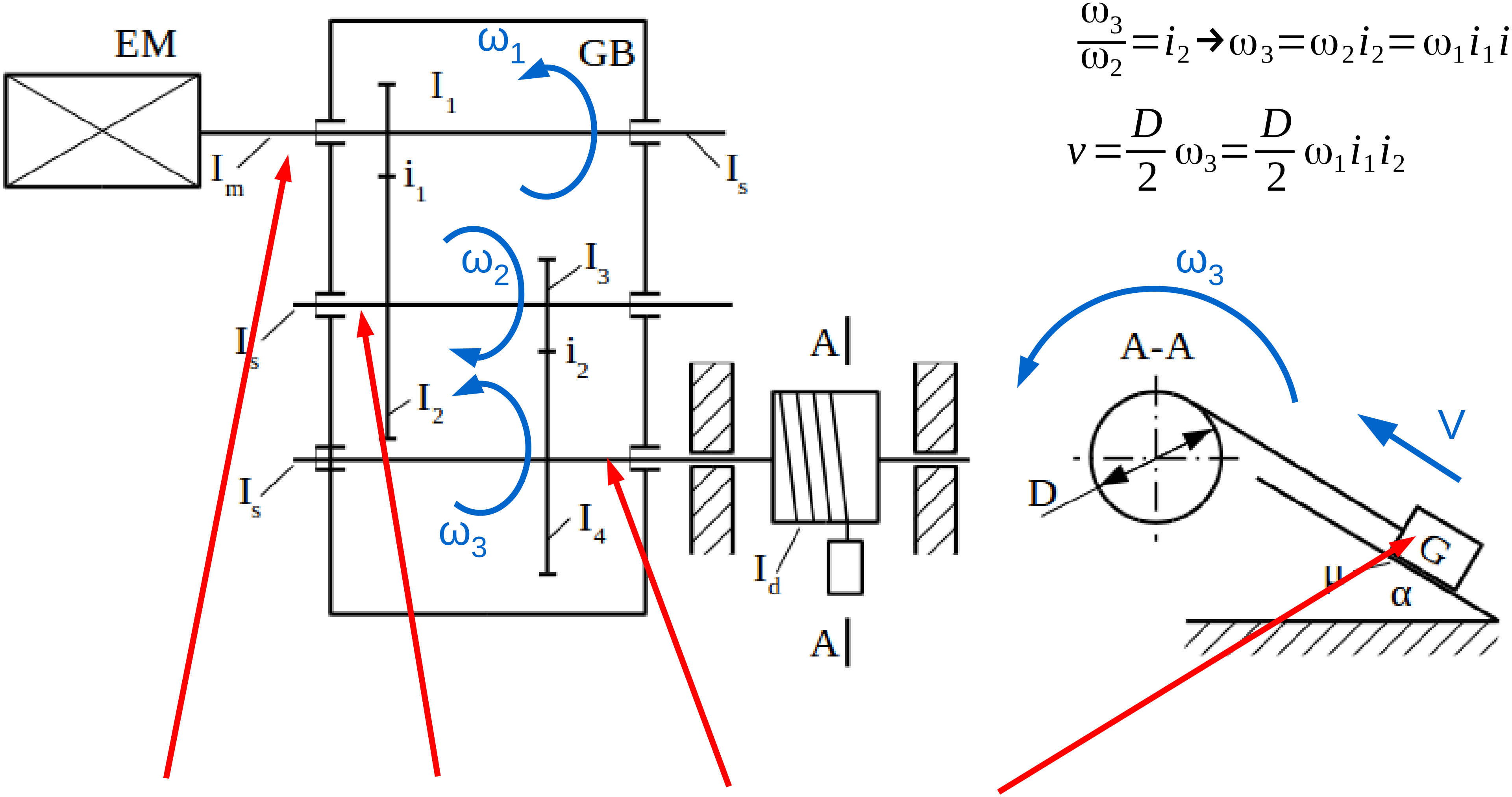
Reduction of masses and forces

Example 1

$$\frac{\omega_2}{\omega_1}=i_1 \rightarrow \omega_2=\omega_1 i_1$$

$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$



$$T=\frac{1}{2}\left(I_m+I_1+I_s\right)\omega_1^2+\frac{1}{2}\left(I_2+I_3+I_s\right)\omega_2^2+\frac{1}{2}\left(I_4+I_d+I_s\right)\omega_3^2+\frac{1}{2}\frac{G}{g}v^2$$

Reduction of masses and forces

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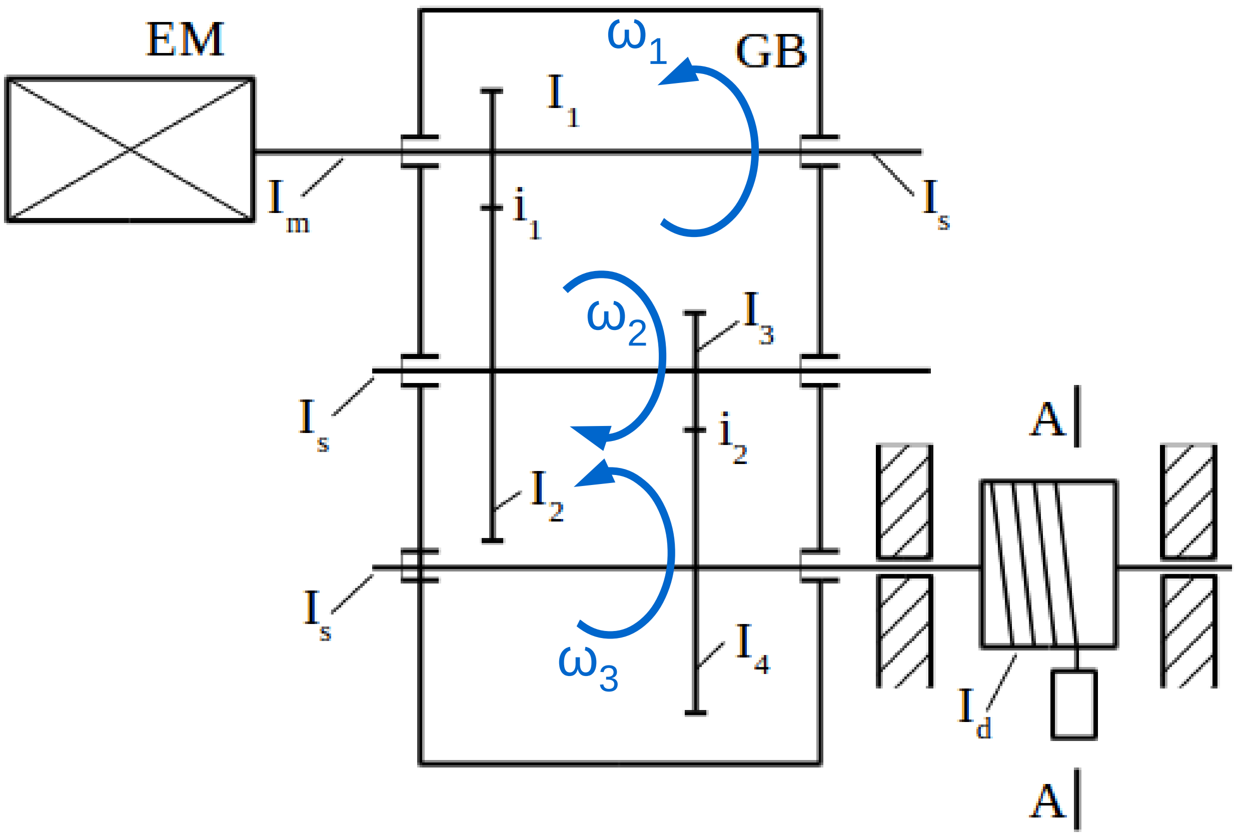
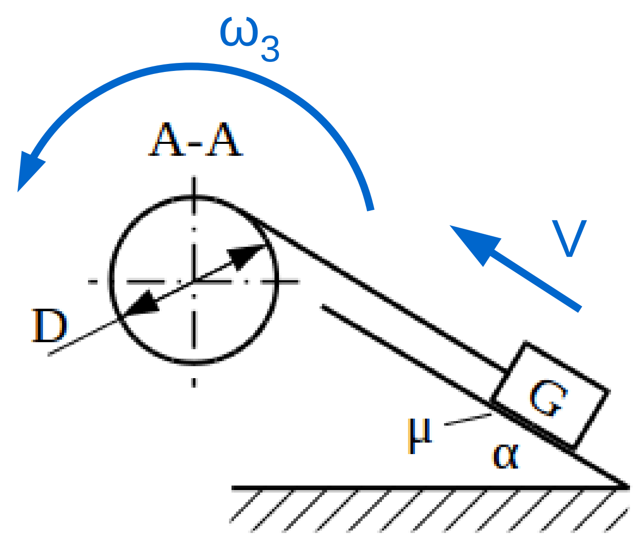
$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$





$$T=\frac{1}{2}(I_m+I_1+I_s)\omega_1^2+\frac{1}{2}(I_2+I_3+I_s)\omega_1^2 i_1^2+\frac{1}{2}(I_4+I_d+I_s)\omega_1^2 i_1^2 i_2^2+\frac{1}{2}\frac{G}{g}\frac{D^2}{4}\omega_1^2 i_1^2 i_2^2$$

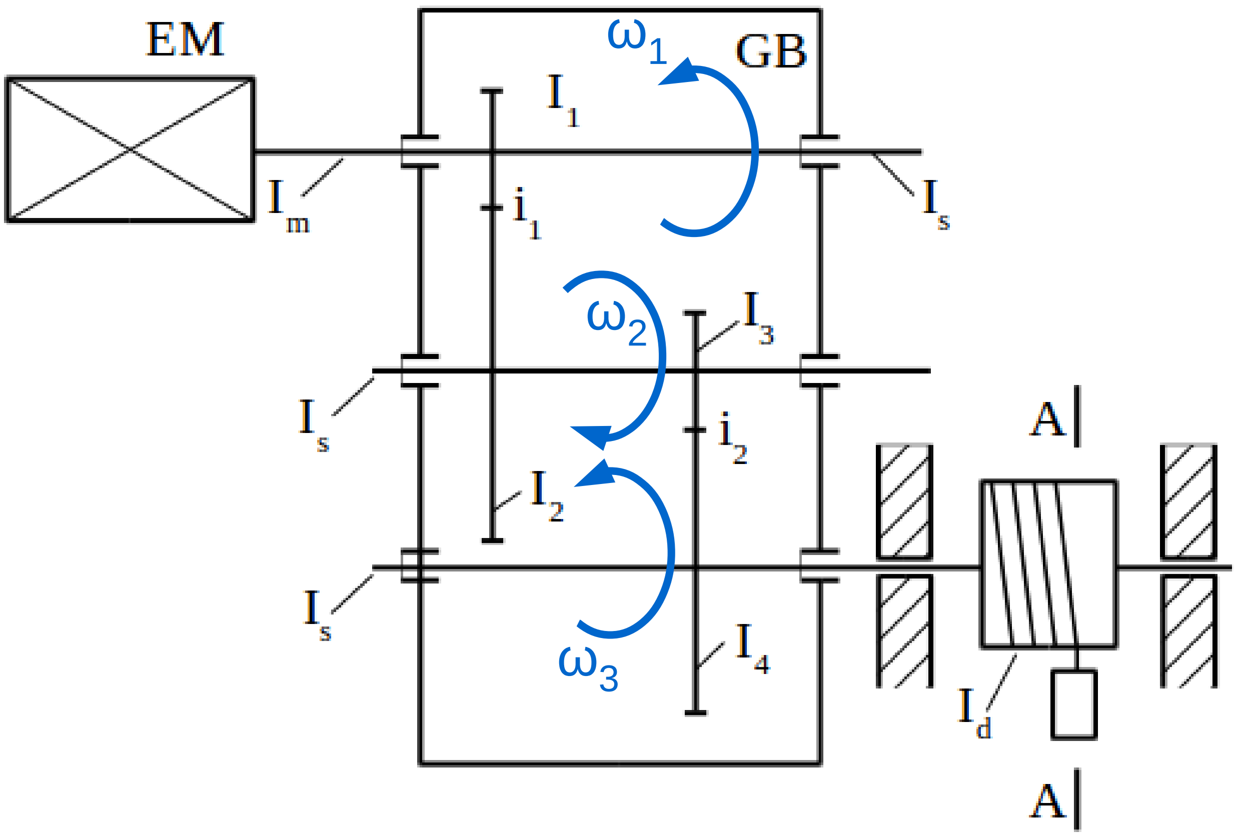
Reduction of masses and forces

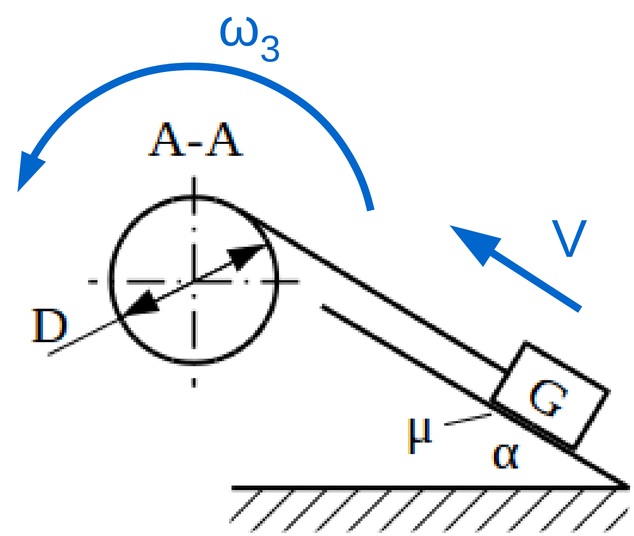
Example 1

$$\frac{\omega_2}{\omega_1}=i_1 \rightarrow \omega_2=\omega_1 i_1$$

$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$





$$T=\frac{1}{2}\left[\left(I_m+I_1+I_s\right)+\left(I_2+I_3+I_s\right)i_1^2+\left(I_4+I_d+I_s\right)i_1^2 i_2^2+\frac{G}{g}\frac{D^2}{4}i_1^2 i_2^2\right]\omega_1^2$$

Reduction of masses and forces

Example 1

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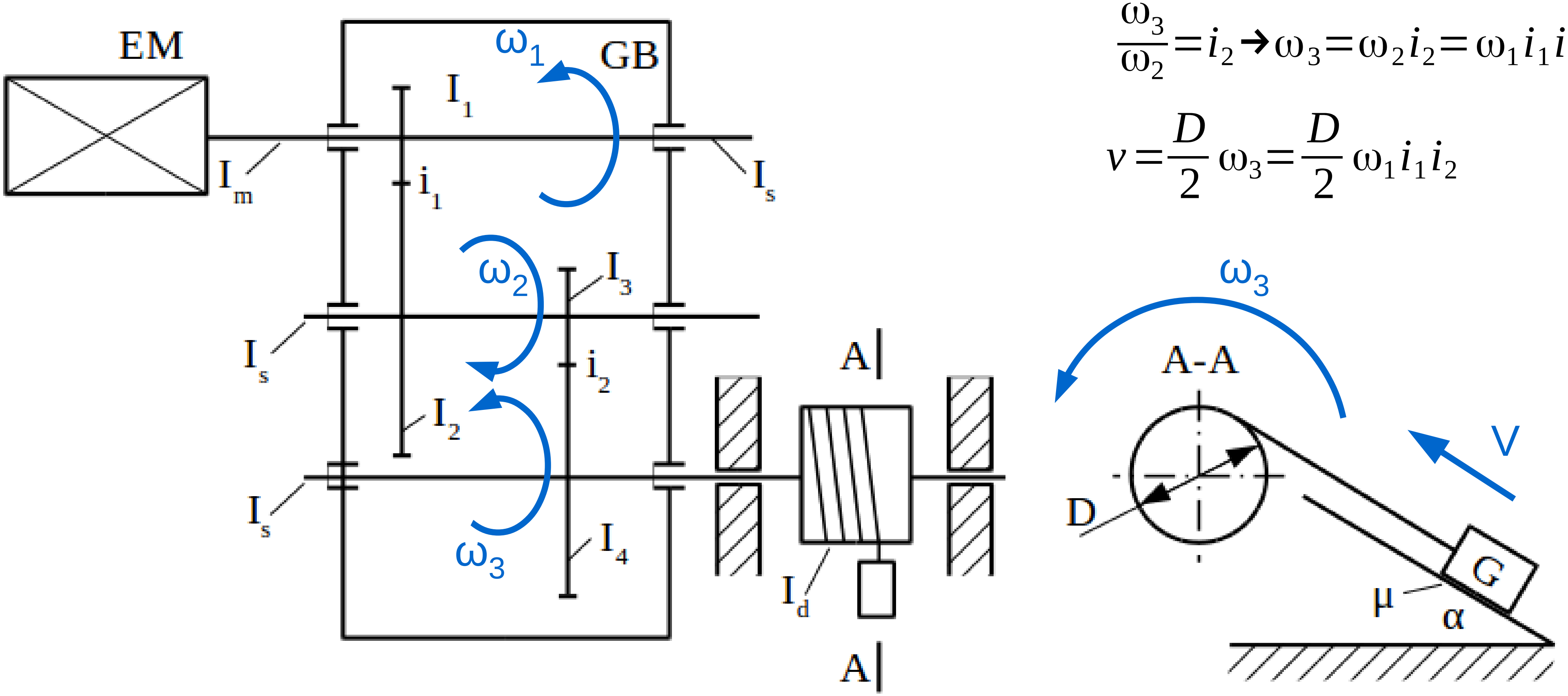
$$\frac{\omega_3}{\omega_2}=i_2 \rightarrow \omega_3=\omega_2 i_2=\omega_1 i_1 i_2$$

$$v=\frac{D}{2}\omega_3=\frac{D}{2}\omega_1 i_1 i_2$$

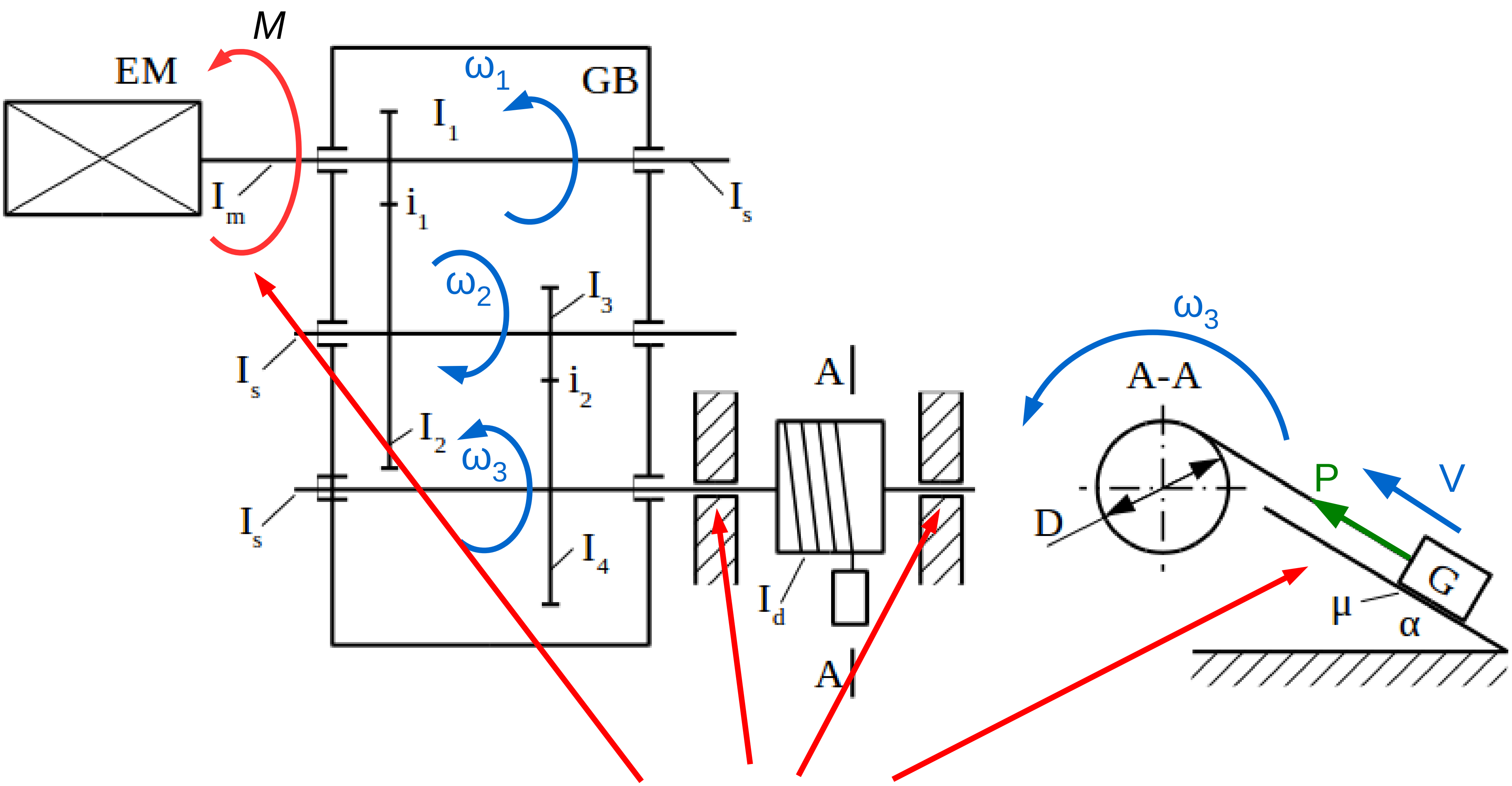


reduced moment of inertia

$$I_R=I_m+I_1+I_s+\left(I_2+I_3+I_s\right)i_1^2+\left(I_4+I_d+I_s\right)i_1^2 i_2^2+\frac{G}{g}\frac{D^2}{4}i_1^2 i_2^2$$

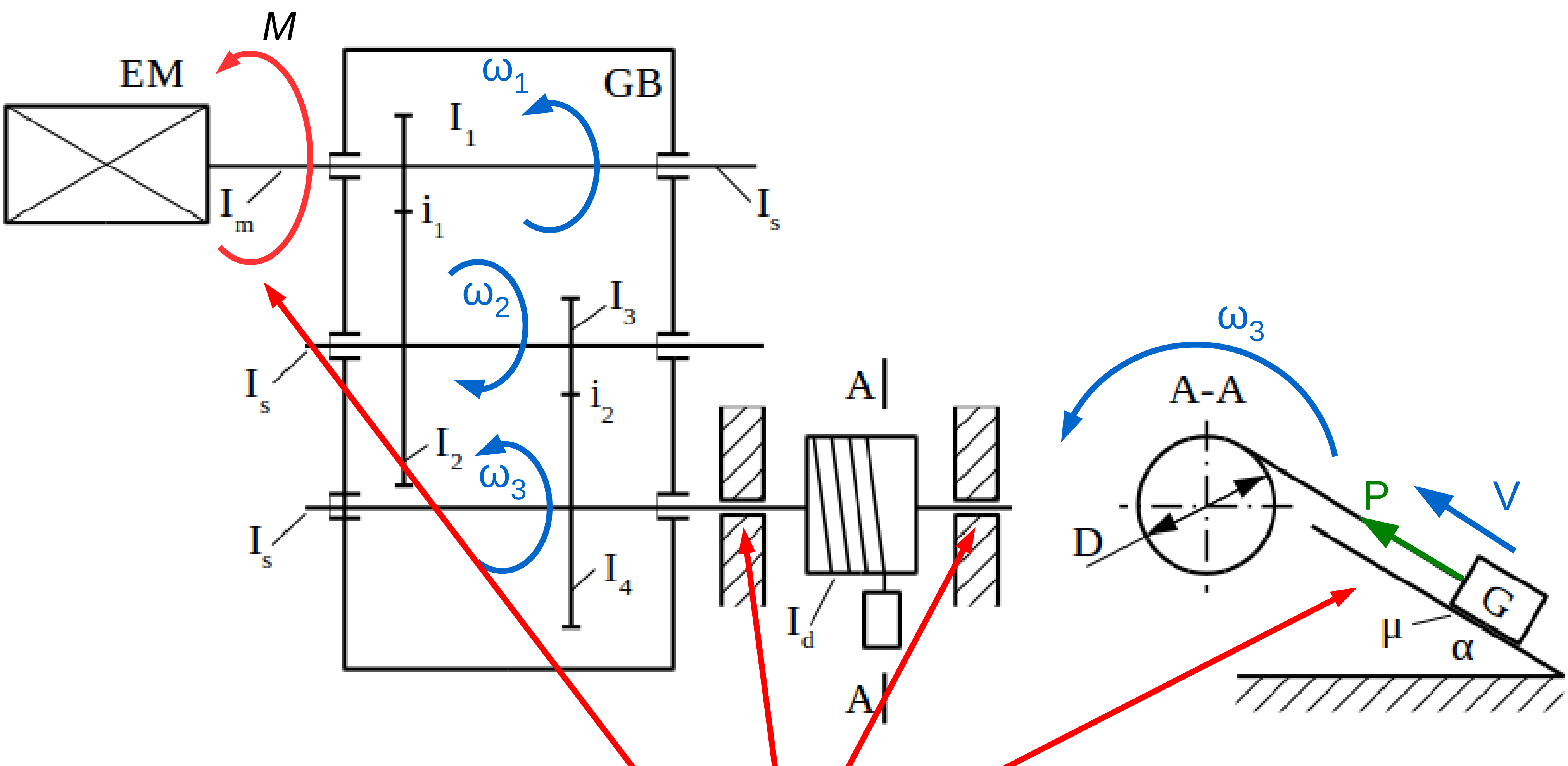
Reduction of masses and forces

Example 1



Reduction of masses and forces

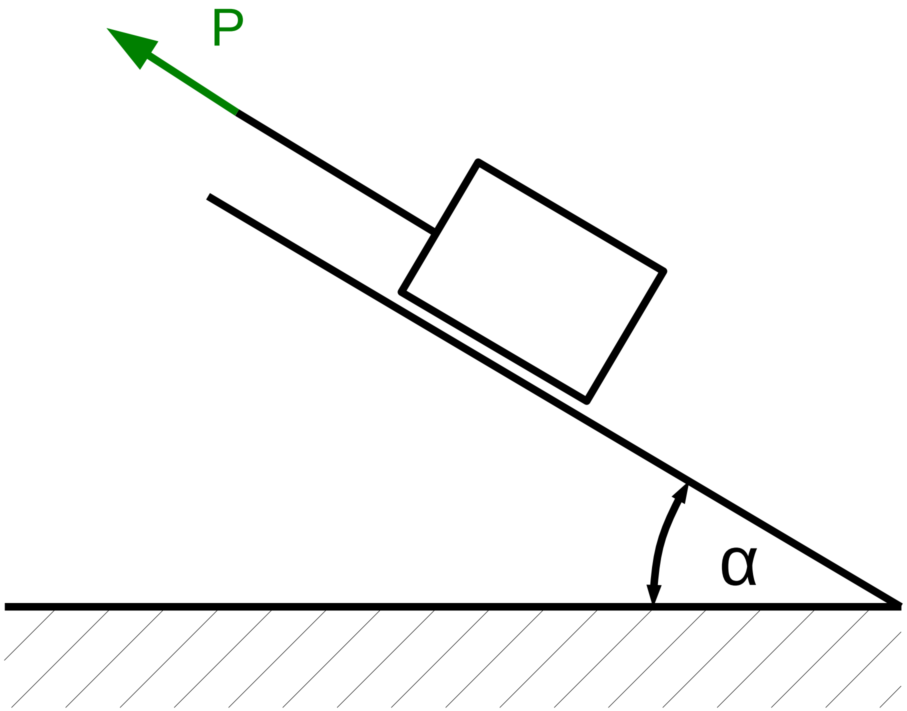
Example 1



$$N = M_s \omega_1 - M_f \omega_3 - P v$$

Reduction of masses and forces

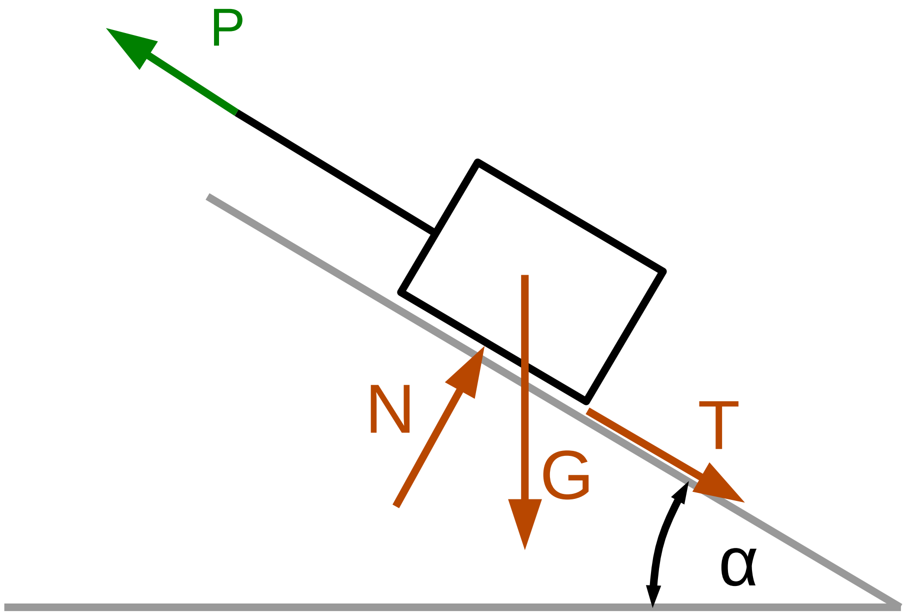
Example 1



P=…

Reduction of masses and forces

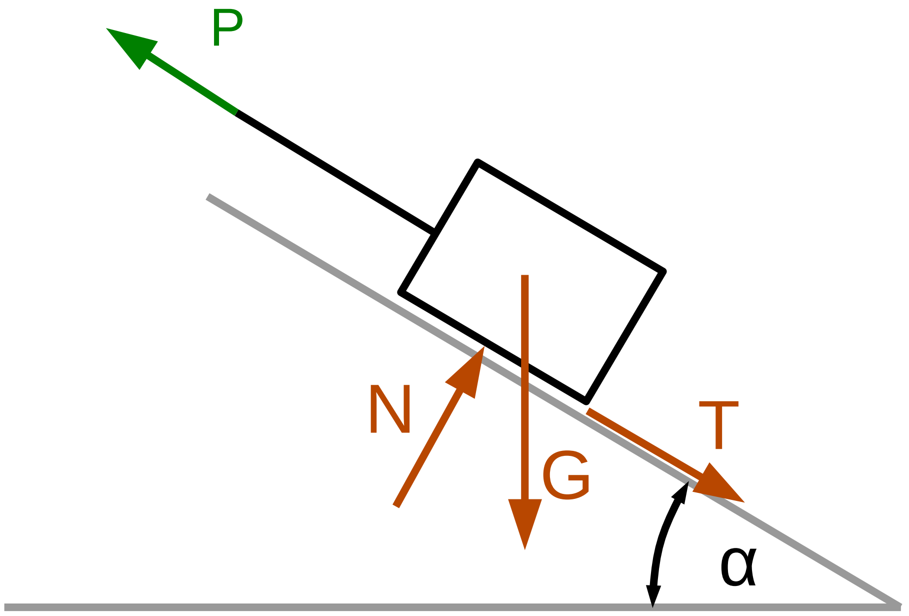
Example 1



P=…

Reduction of masses and forces

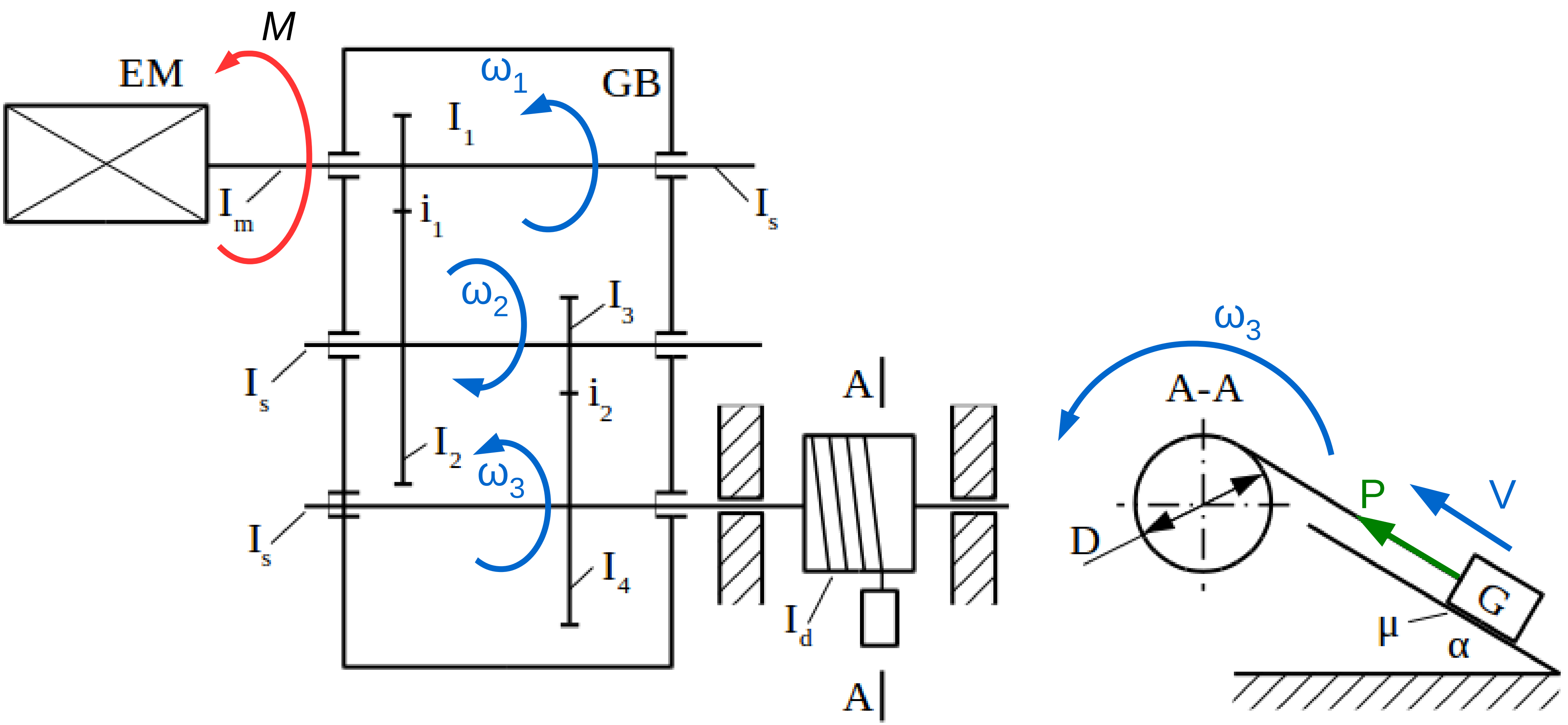
Example 1



$$P = T + G\sin\alpha = \mu N + G\sin\alpha = \mu G\cos\alpha + G\sin\alpha$$

Reduction of masses and forces

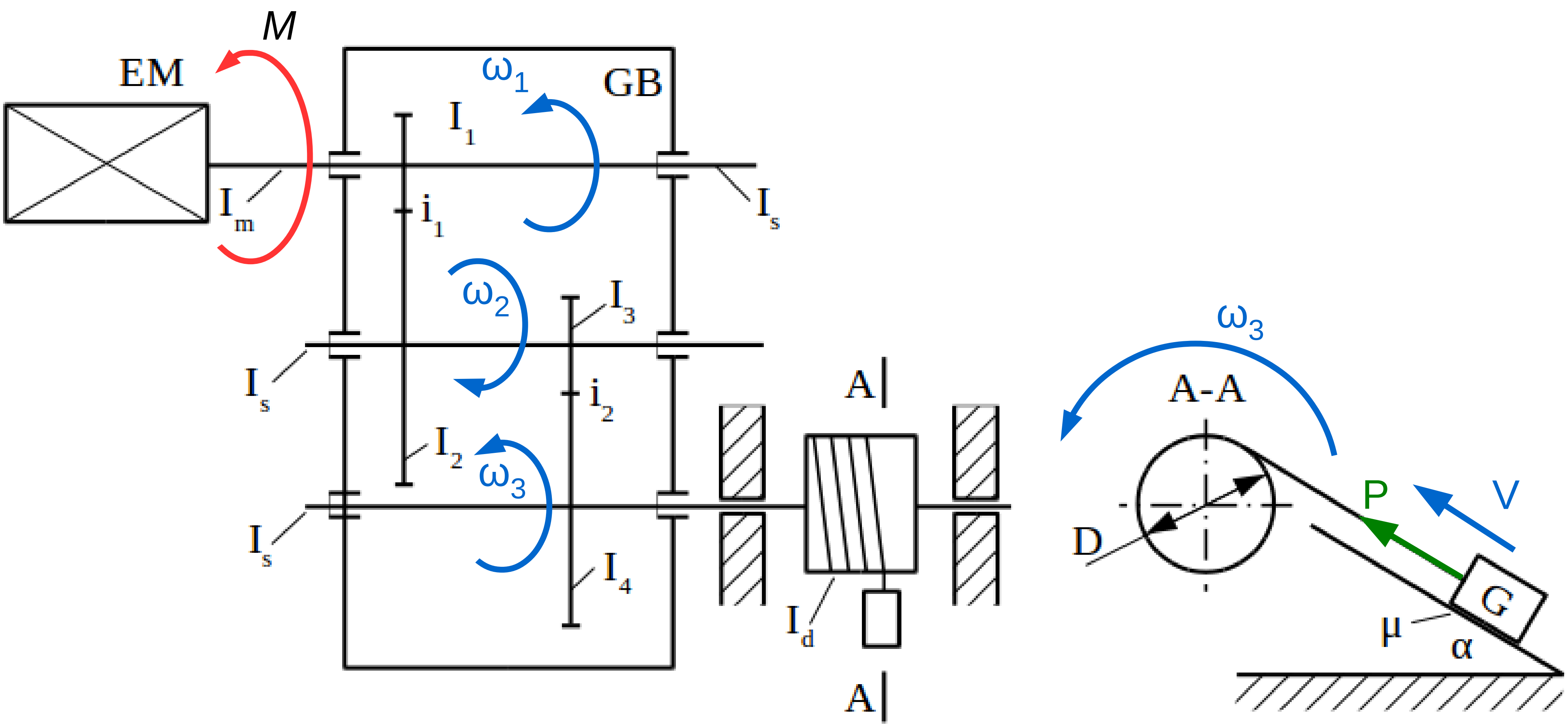
Example 1



$$N = M_s \omega_1 - M_f \omega_3 - (\mu G \cos\alpha + G \sin\alpha) v$$

Reduction of masses and forces

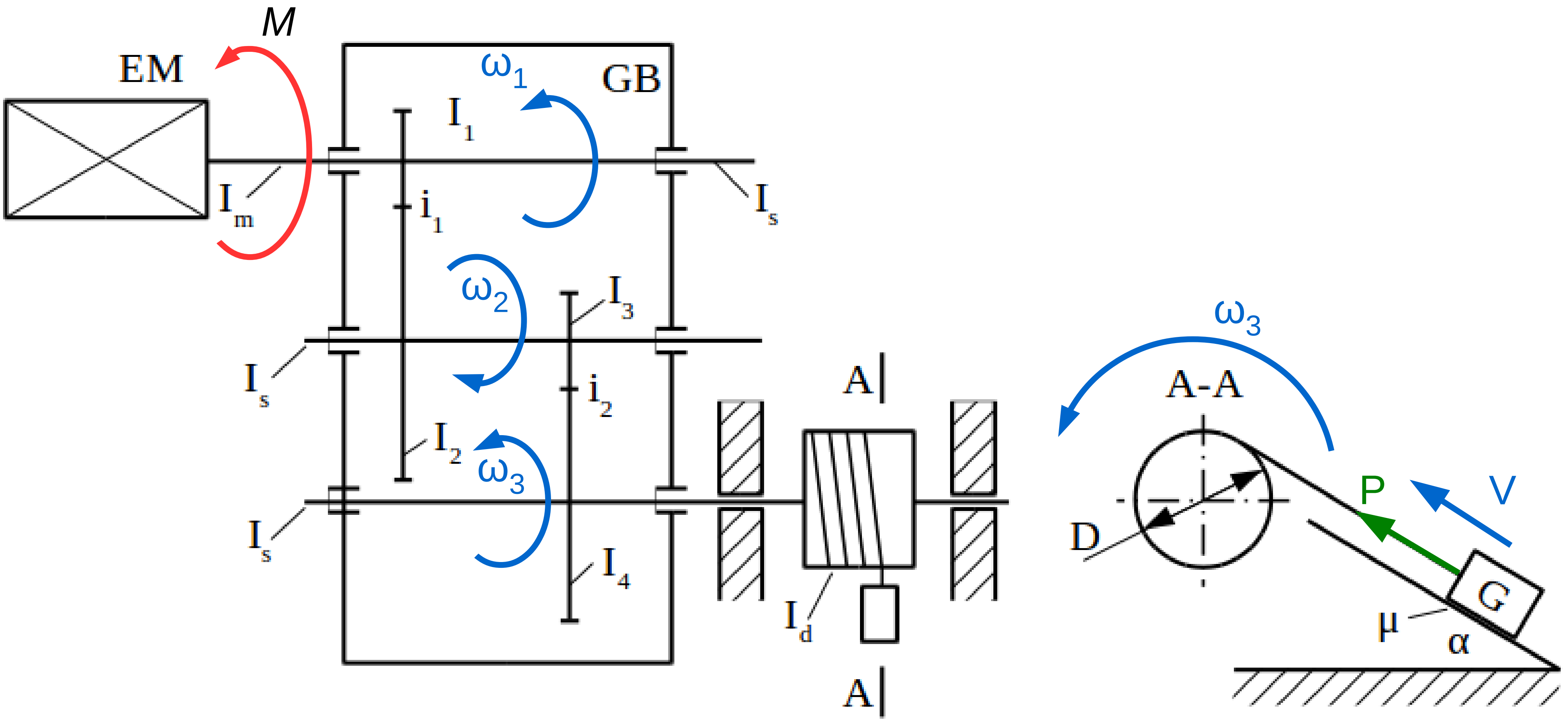
Example 1



$$N = M_s \omega_1 - M_f \omega_1 i_1 i_2 - (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} \omega_1 i_1 i_2$$

Reduction of masses and forces

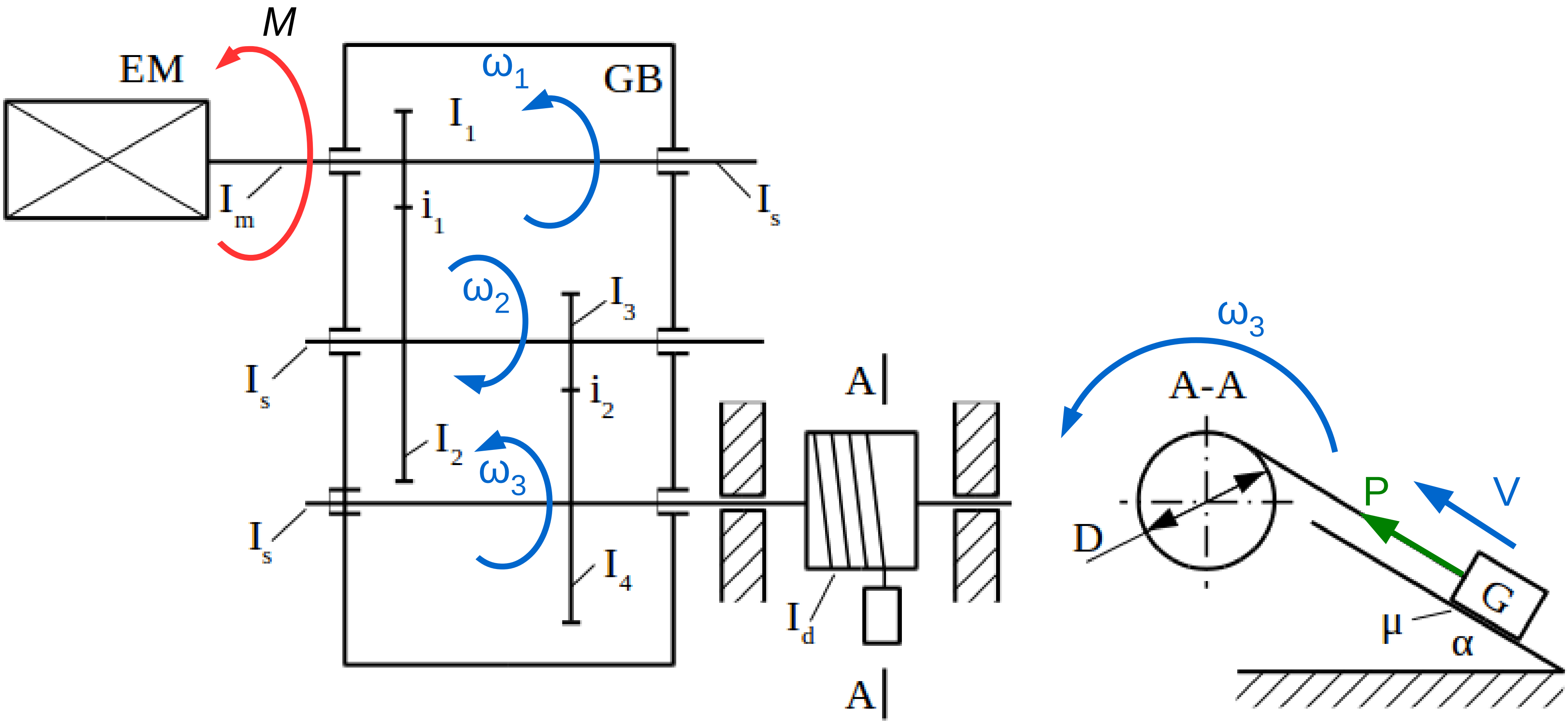
Example 1



$$N=\left[M_s-M_f i_1 i_2-\left(\mu G\cos\alpha+G\sin\alpha\right)\frac{D}{2}i_1 i_2\right]\omega_1$$

Reduction of masses and forces

Example 1



reduced torque

$$M_R = M_s - M_f i_1 i_2 - (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2$$

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = M_s - \left(M_f i_1 i_2 + (\mu G \cos\alpha + G \sin\alpha) \frac{D}{2} i_1 i_2 \right)$$

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = \underbrace{M_s}_{\substack{\text{driving torque } M_D \\ \text{(electric motor torque)}}} - \underbrace{\left(M_f i_1 i_2 + \left(\mu G \cos\alpha + G \sin\alpha \right) \frac{D}{2} i_1 i_2 \right)}_{\text{passive torque } M_P}$$

Reduction of masses and forces

Example 1

Reduced torque

$$M_R = M_s - \underbrace{\left(M_f\, i_1\, i_2 + \left(\mu\, G \cos\alpha + G \sin\alpha \right) \frac{D}{2}\, i_1\, i_2 \right)}_{\text{passive torque } M_P}$$

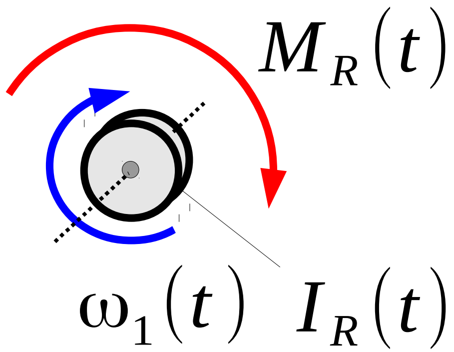
driving torque M_D
(electric motor torque)

$$M_R = M_D - M_P = \left(A - B\,\omega_1 \right) - M_P$$

Reduction of masses and forces

Example 1

Start-up process

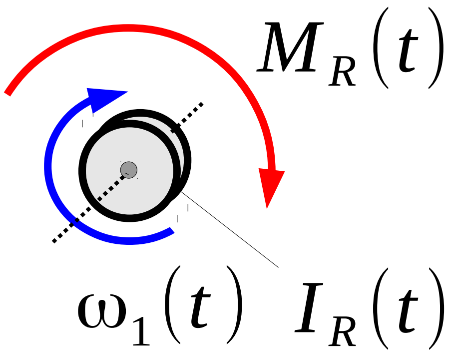


$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

Reduction of masses and forces

Example 1

Start-up process



$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

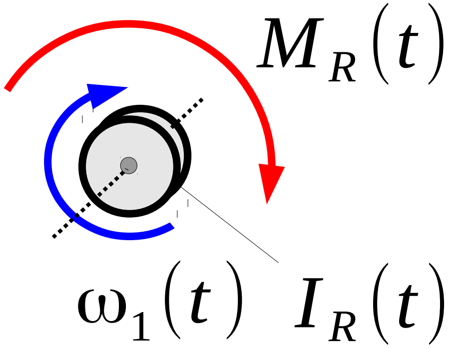
$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

A, M_P, B, I_R - constants

Reduction of masses and forces

Example 1

Start-up process



$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

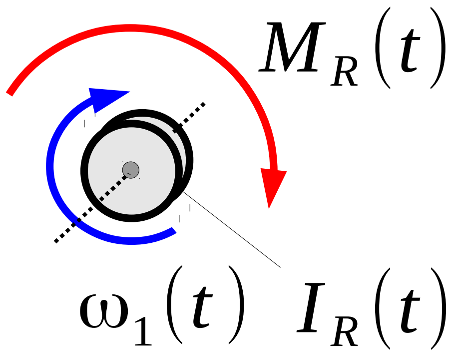
$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

non-homogeneous
1st order ODE
with const. coef.

general solution particular solution

Reduction of masses and forces

Example 1



Start-up process

$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

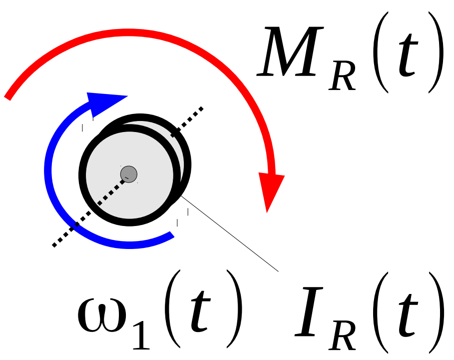
$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

general solution particular solution

$$\omega_{1g}(t) = E e^{-\frac{B}{I_R}t}$$

Reduction of masses and forces

Example 1



Start-up process

$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

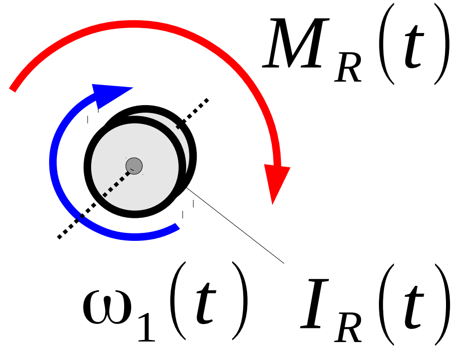
$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

general solution particular solution

$$\omega_{1g}(t) = E e^{-\frac{B}{I_R}t} \qquad \omega_{1p}(t) = F$$

Reduction of masses and forces

Example 1



Start-up process

$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

general solution particular solution

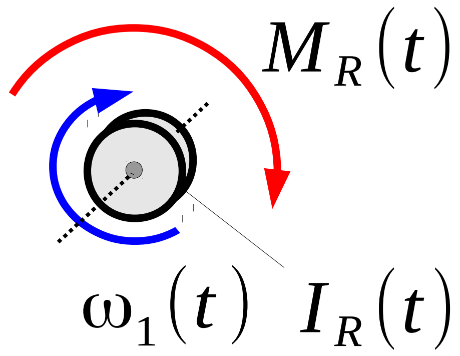
$$\omega_{1g}(t) = E e^{-\frac{B}{I_R}t} \qquad \omega_{1p}(t) = F$$

initial condition

$$\omega_1(t=0) = 0$$

Reduction of masses and forces

<u>Example 1</u>



Start-up process

$$I_R \frac{d\omega_1}{dt} = M_R \qquad M_R = A - B\omega_1 - M_P$$

$$\frac{d\omega_1}{dt} + \frac{B}{I_R}\omega_1 = \frac{A - M_P}{I_R}$$

general solution

$$\omega_{1g}(t) = E e^{-\frac{B}{I_R}t}$$

particular solution

$$\omega_{1p}(t) = F$$

initial condition

$$\omega_1(t=0) = 0$$

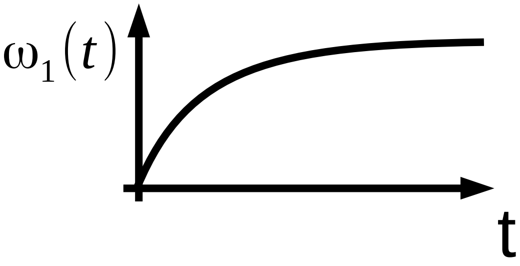
$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$

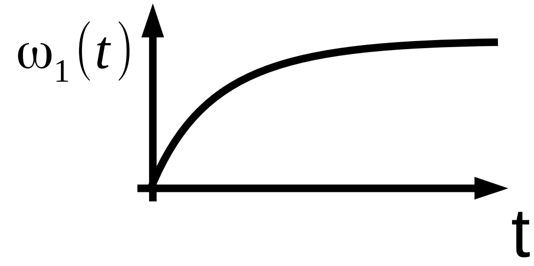


Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t) = \frac{A - M_P}{B}\left(1 - e^{-\frac{B}{I_R}t}\right)$$



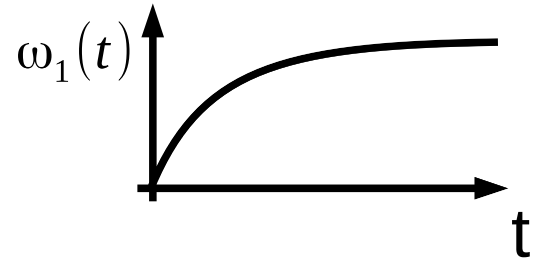
steady state value (maximum) $\omega_{max} = \frac{A - M_P}{B}$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$



steady state value (maximum) $\omega_{max}=\frac{A-M_P}{B}$

START-UP TIME
(95% of maximum value)

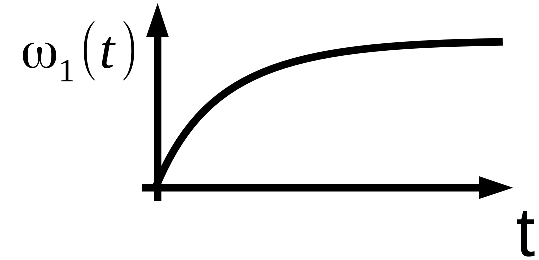
$$0{,}95\,\frac{A-M_P}{B}=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t_{95}}\right)$$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$



steady state value (maximum) $\omega_{max}=\frac{A-M_P}{B}$

START-UP TIME
(95% of maximum value)

$$0{,}95\frac{A-M_P}{B}=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t_{95}}\right)$$

$$t_{95}\approx 3\frac{I_R}{B}$$

Reduction of masses and forces

Example 1

Start-up process

$$\omega_1(t)=\frac{A-M_P}{B}\left(1-e^{-\frac{B}{I_R}t}\right)$$

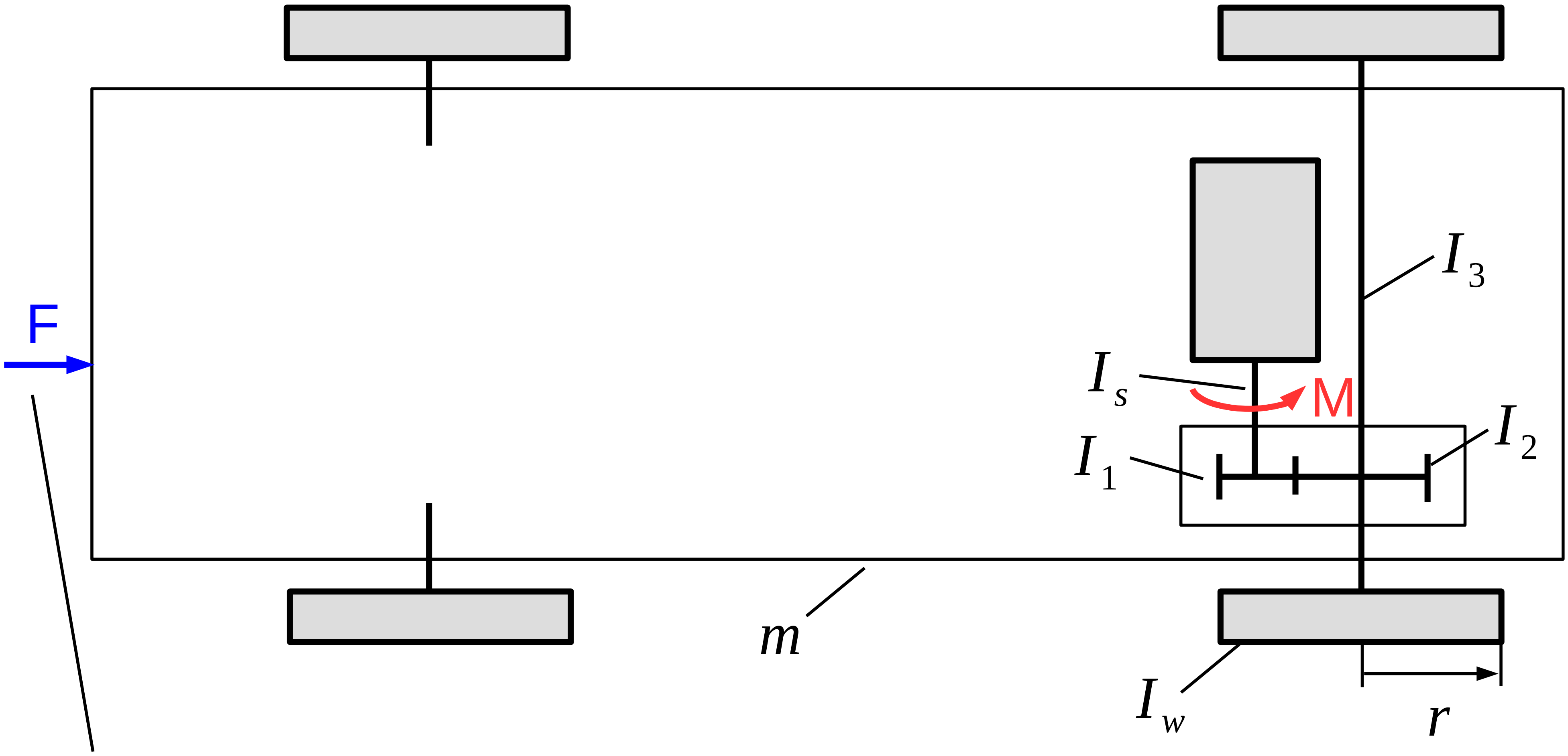
electric motor angular velocity

$$v(t)=\frac{D}{2}\omega_1(t)\,i_1\,i_2$$

box linear velocity

Reduction of masses and forces

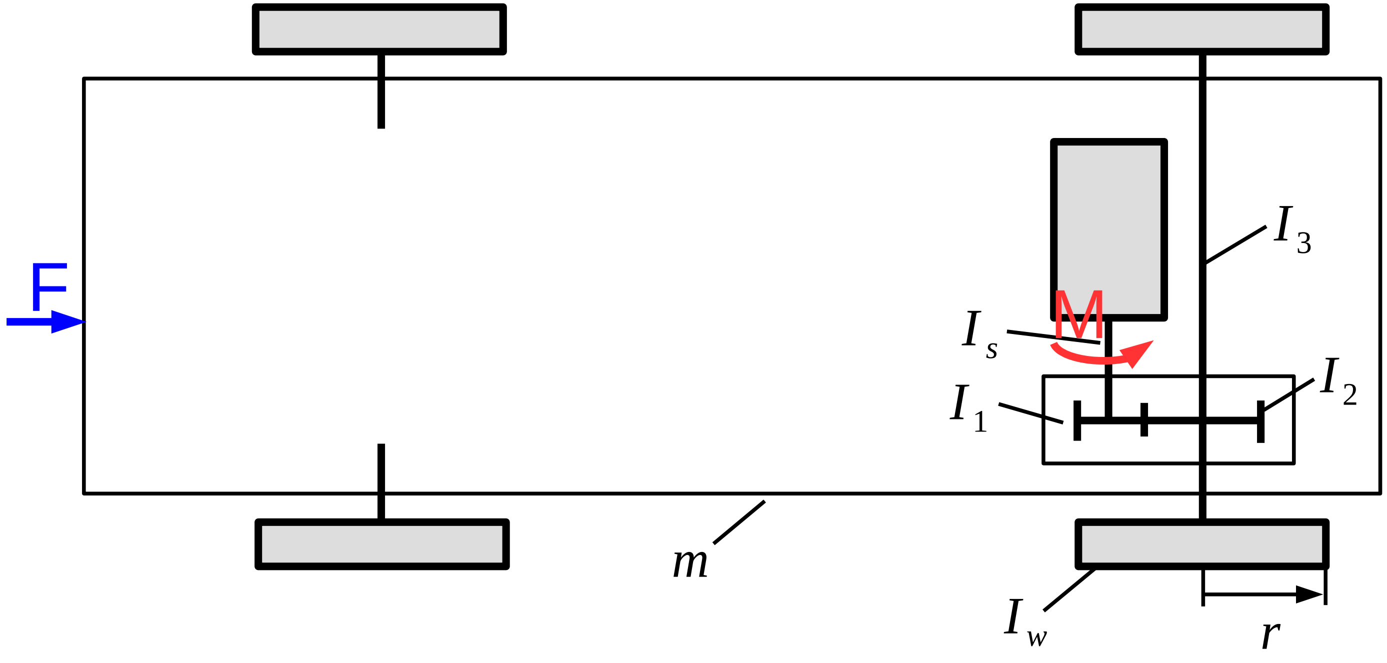
Example 2



air resistance proportional to velocity

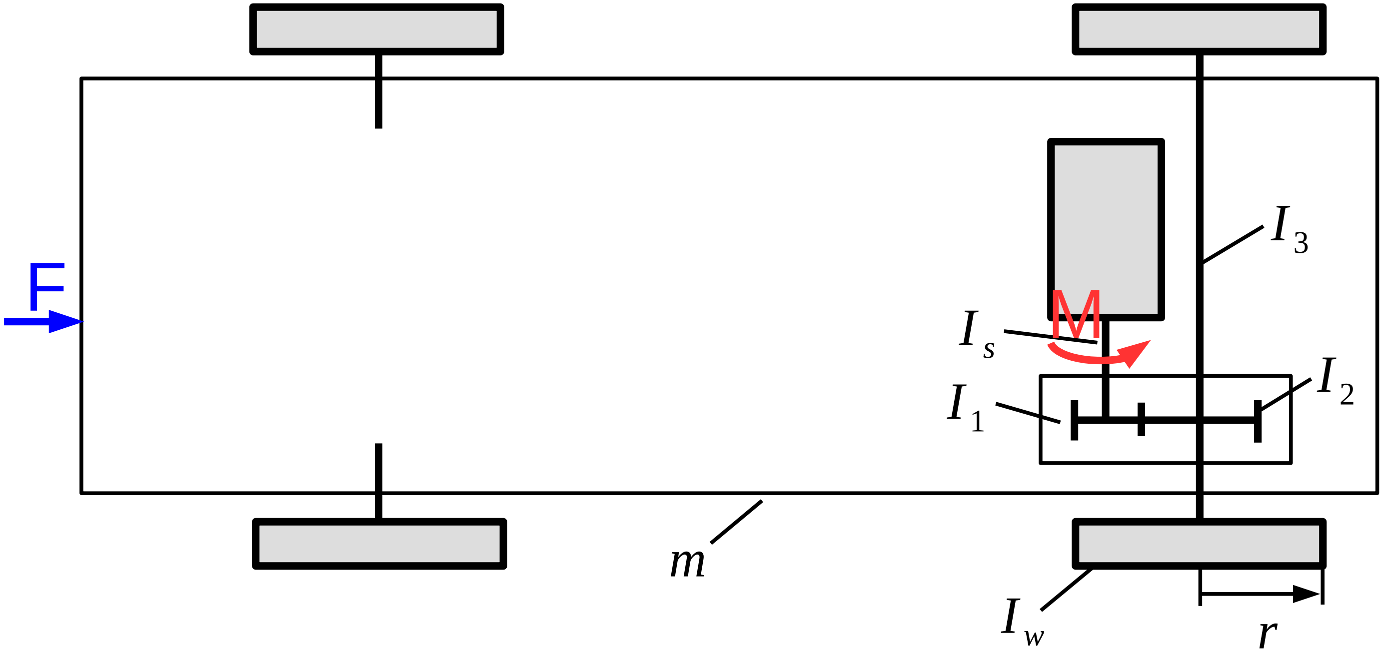
Reduction of masses and forces

Example 2



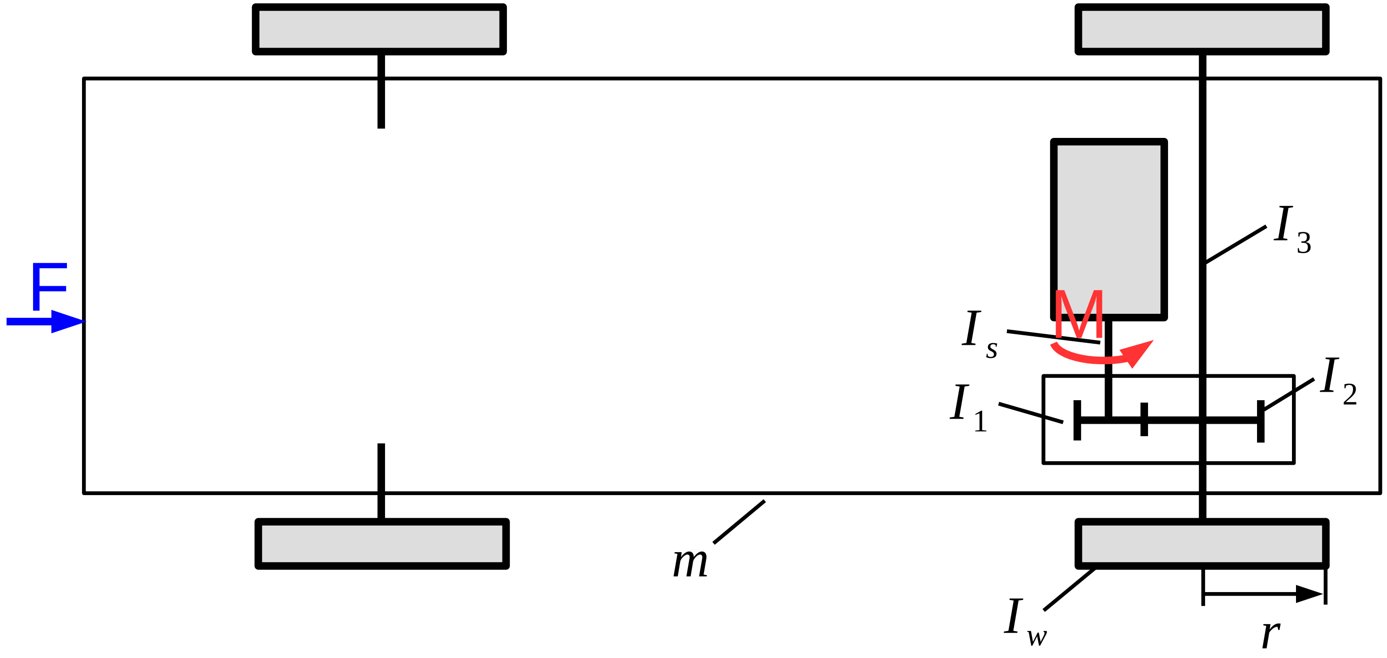
Reduction of masses and forces

Example 2



Reduction of masses and forces

Example 2



Reduction of masses and forces

Example 2