



Faculty of Automotive and Construction Machinery Engineering

WARSAW UNIVERSITY OF TECHNOLOGY

Theory of Machines and Automatic Control Winter 2018/2019

Lecturer: Sebastian Korczak, PhD, Eng.

Lecture 10

Classification of basic automatic systems with examples.

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Classification of basic automatic systems

Element name	Equation	Transfer function
proportional	$y(t) = ku(t)$	k
first order (inertial)	$T \frac{dy(t)}{dt} + y(t) = ku(t)$	$\frac{k}{Ts + 1}$
integrator	$y(t) = k \int_0^t u(t) dt$ <p style="text-align: center;">or</p> $\frac{dy(t)}{dt} = ku(t)$	$\frac{k}{s}$

Classification of basic automatic systems

Element name	Equation	Transfer function
derivative	$y(t) = k \frac{du(t)}{dt}$	ks
derivative with inertia	$T \frac{dy(t)}{dt} + y(t) = k \frac{du(t)}{dt}$	$\frac{ks}{Ts + 1}$

Classification of basic automatic systems

Element name	Equation	Transfer function
delay	$y(t) = u(t - \tau)$	$e^{-\tau s}$
second order (oscillator)	$T_1^2 \frac{d^2 y(t)}{dt^2} + T_2 \frac{dy(t)}{dt} + y(t) = ku(t)$	$\frac{k}{T_1^2 s^2 + T_2 s + 1}$

Proportional element

1. Element equation: $y(t) = ku(t)$ $u(t)$ - input, $y(t)$ - output

Proportional element

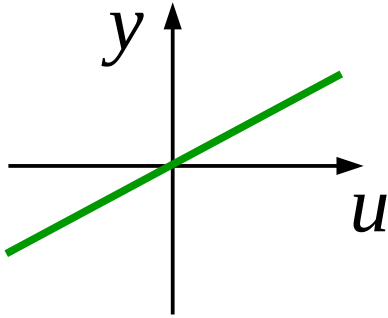
1. Element equation: $y(t) = ku(t)$ $u(t)$ - input, $y(t)$ - output

2. Static characteristic (steady state): for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

Proportional element

1. Element equation: $y(t) = ku(t)$ $u(t)$ - input, $y(t)$ - output

2. Static characteristic (steady state): $y = ku$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

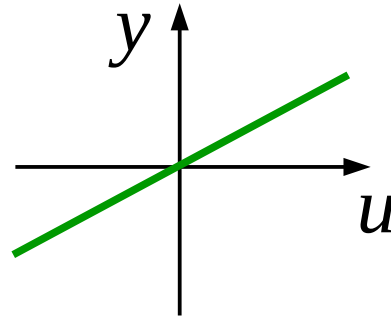


for $k > 0$

Proportional element

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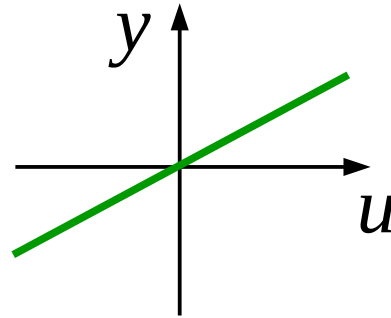
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3. Transfer function:

Proportional element

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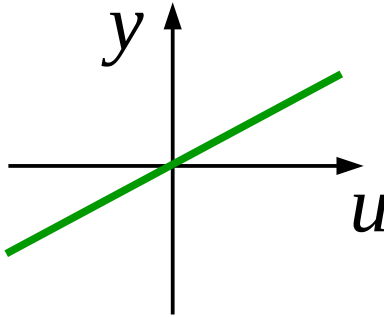
for $k > 0$

3. Transfer function: $H(s) = k$

Proportional element

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for $k > 0$

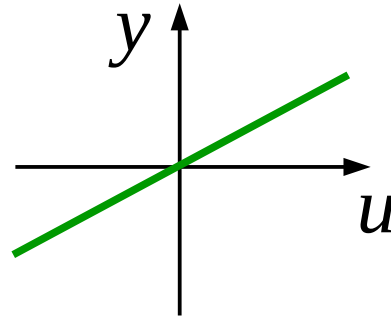
3. Transfer function: $H(s) = k$

4. Step response: for $u(t) = u_0 1(t)$

Proportional element

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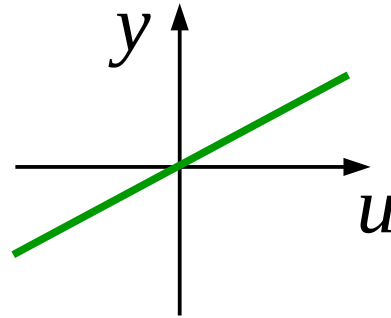
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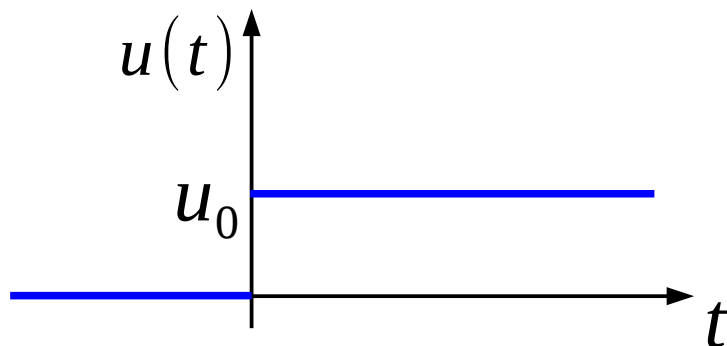
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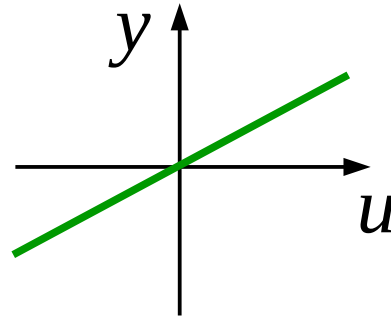
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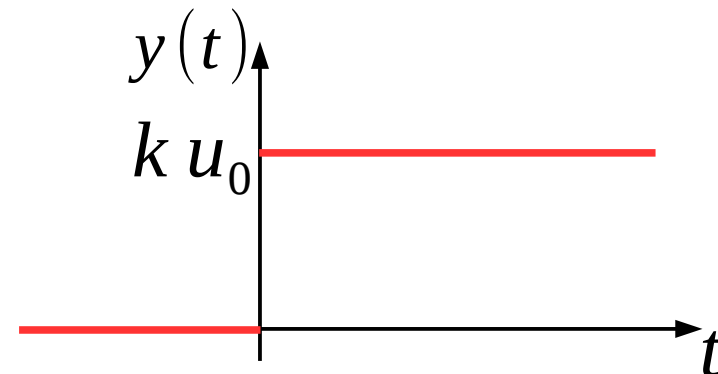
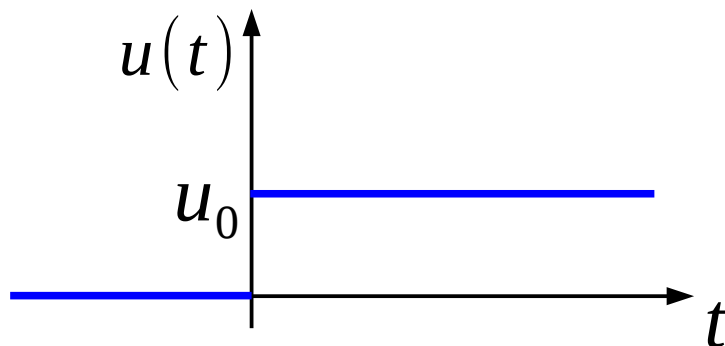
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Proportional element

5. Frequency response:

Proportional element

5. Frequency response: $H(j\omega) = k$

Proportional element

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Proportional element

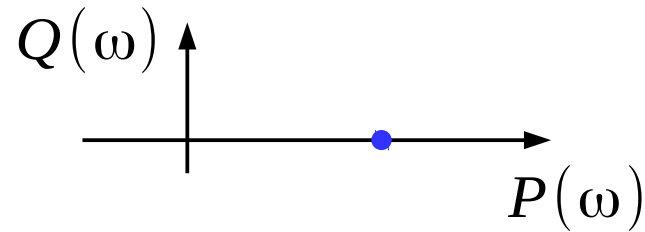
5. Frequency response: $H(j\omega) = k$ $P(\omega) = k$, $Q(\omega) = 0$

6. Nyquist plot:

Proportional element

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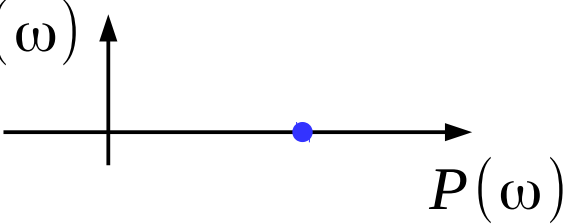
6. Nyquist plot:



for $k > 0$

Proportional element

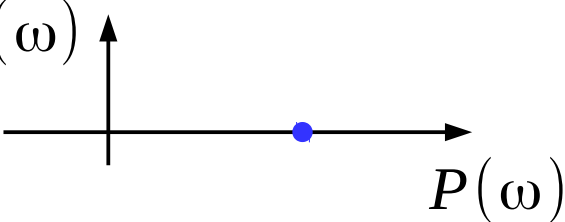
5. Frequency response: $H(j\omega) = k$ $P(\omega) = k, Q(\omega) = 0$

6. Nyquist plot:  $Q(\omega)$ for $k > 0$
 $P(\omega)$

7. Bode plot:

Proportional element

5. Frequency response: $H(j\omega) = k$ $P(\omega) = k, Q(\omega) = 0$

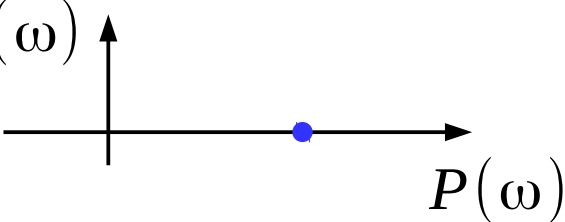
6. Nyquist plot:  for $k > 0$

The Nyquist plot shows a single point on the positive real axis, representing the frequency response of a proportional element. The horizontal axis is labeled $P(\omega)$ and the vertical axis is labeled $Q(\omega)$. A blue dot is placed on the positive $P(\omega)$ axis.

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k|$
 $L(\omega) = 20 \log A(\omega)$

Proportional element

5. Frequency response: $H(j\omega) = k$ $P(\omega) = k, Q(\omega) = 0$

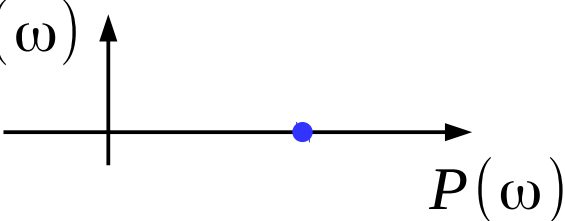
6. Nyquist plot:  for $k > 0$

The Nyquist plot shows a horizontal axis labeled $P(\omega)$ and a vertical axis labeled $Q(\omega)$. A blue dot is plotted on the positive $P(\omega)$ axis, representing the frequency response for $k > 0$.

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k|$ $\varphi(\omega) = \arctan \frac{Q}{P} = \begin{cases} 0, & \text{dla } k \geq 0 \\ \pi, & \text{dla } k < 0 \end{cases}$
 $L(\omega) = 20 \log A(\omega)$

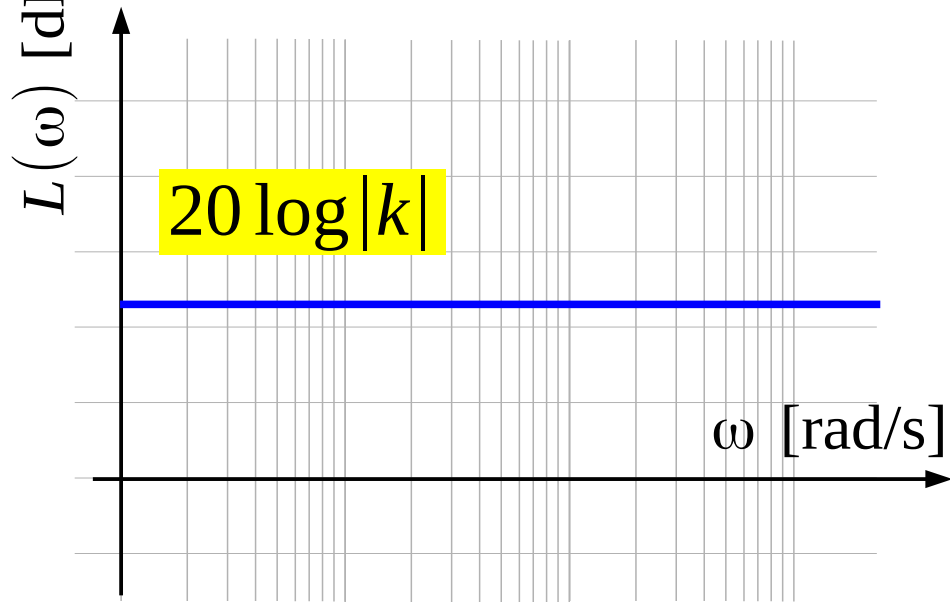
Proportional element

5. Frequency response: $H(j\omega) = k$ $P(\omega) = k, Q(\omega) = 0$

6. Nyquist plot:  for $k > 0$

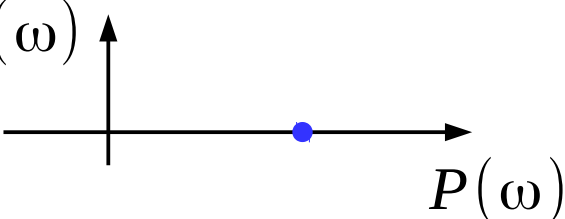
The Nyquist plot shows a horizontal axis labeled $P(\omega)$ and a vertical axis labeled $Q(\omega)$. A single blue dot is plotted on the positive $P(\omega)$ axis, representing the frequency response for $k > 0$.

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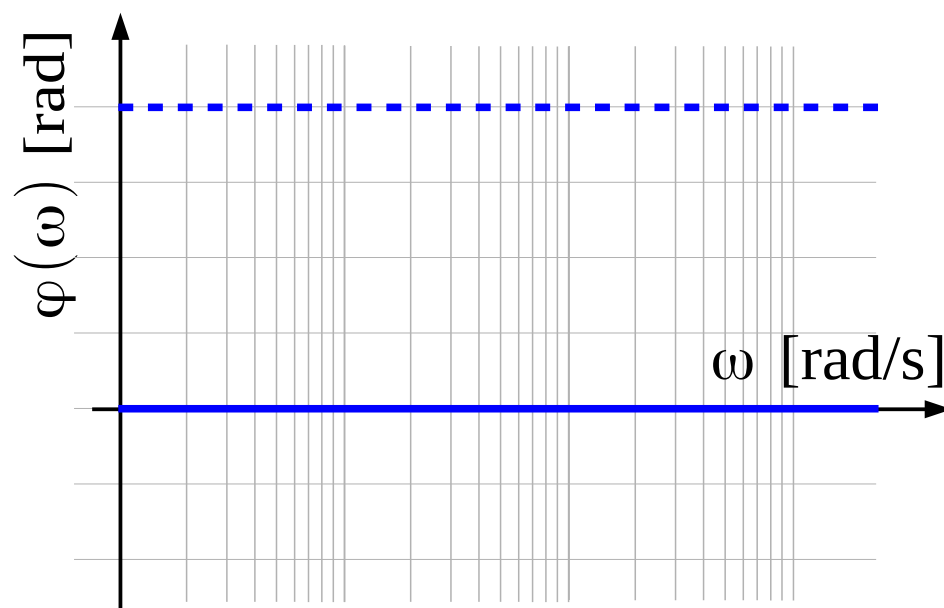
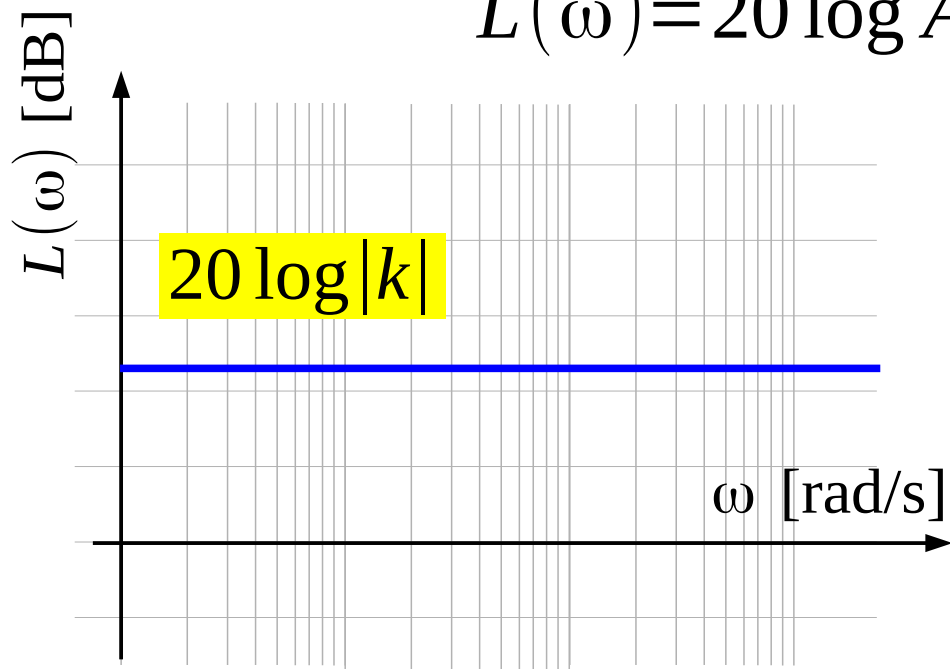


Proportional element

5. Frequency response: $H(j\omega) = k$ $P(\omega) = k$, $Q(\omega) = 0$

6. Nyquist plot:  $Q(\omega)$ vs $P(\omega)$ for $k > 0$

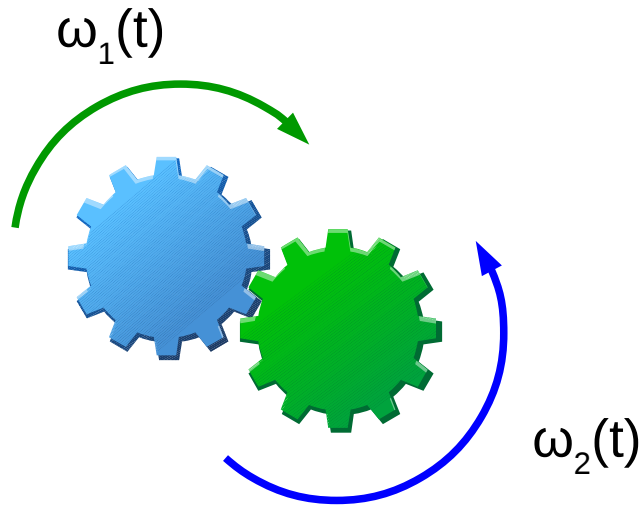
7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k|$ $\varphi(\omega) = \arctan \frac{Q}{P} = \begin{cases} 0, & \text{dla } k \geq 0 \\ \pi, & \text{dla } k < 0 \end{cases}$
 $L(\omega) = 20 \log A(\omega)$



Proportional element

Examples

①



GEARBOX:

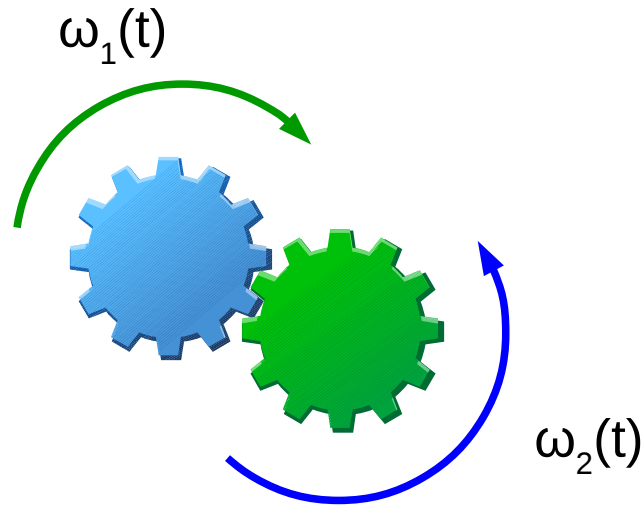
input – angular velocity $\omega_1(t)$

output – angular velocity $\omega_2(t)$

Proportional element

Examples

1

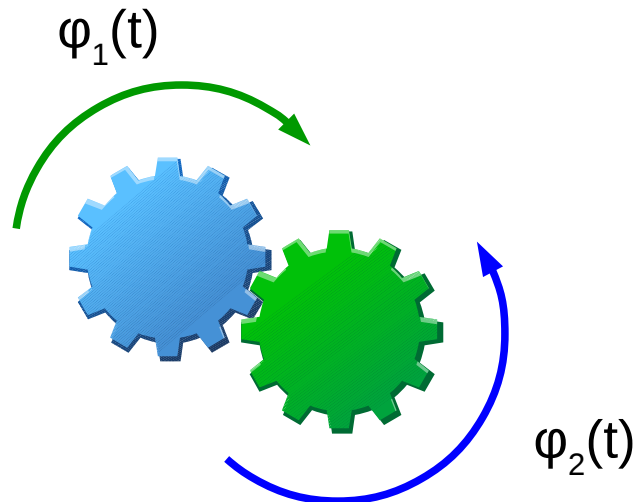


GEARBOX:

input – angular velocity $\omega_1(t)$

output – angular velocity $\omega_2(t)$

2



GEARBOX:

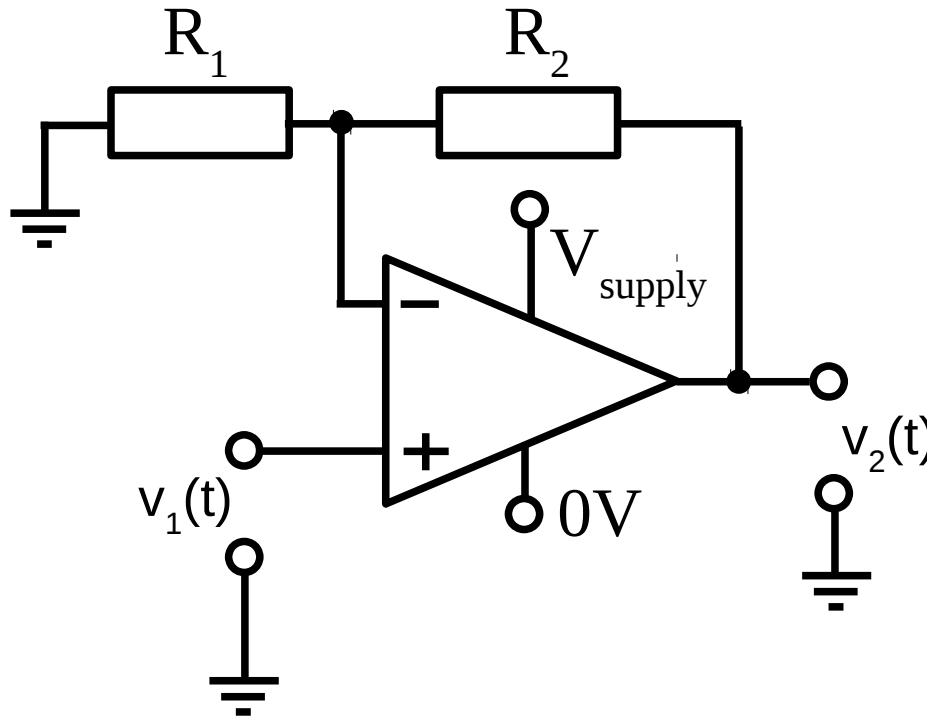
input – rotation angle $\varphi_1(t)$

output – rotation angle $\varphi_2(t)$

Proportional element

Examples

3



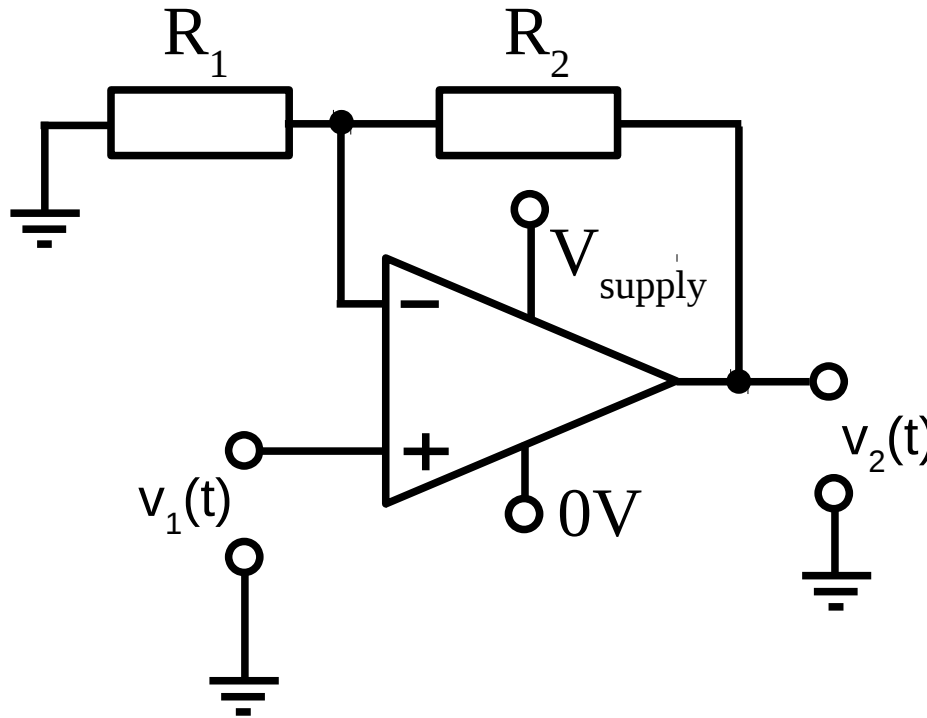
OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = v_1(t) \left(1 + \frac{R_2}{R_1} \right)$$

Proportional element

Examples

3



OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = v_1(t) \left(1 + \frac{R_2}{R_1} \right)$$

4

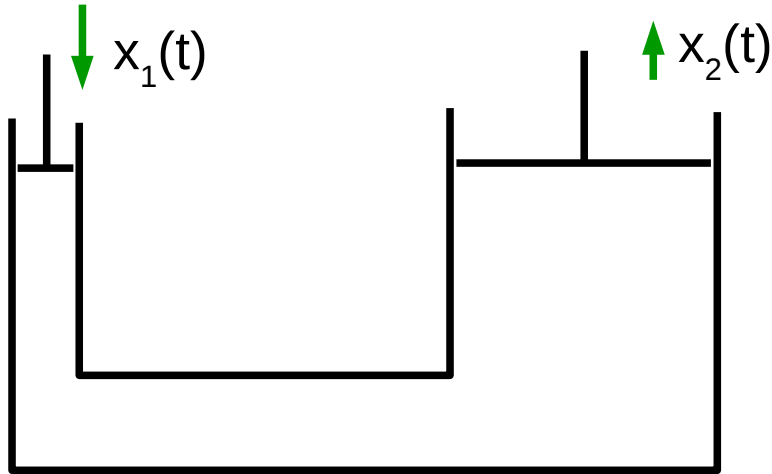


BEAM in steady state:
input – force F_1
output – force F_2

Proportional element

Examples

5



HYDRAULIC LEVER:

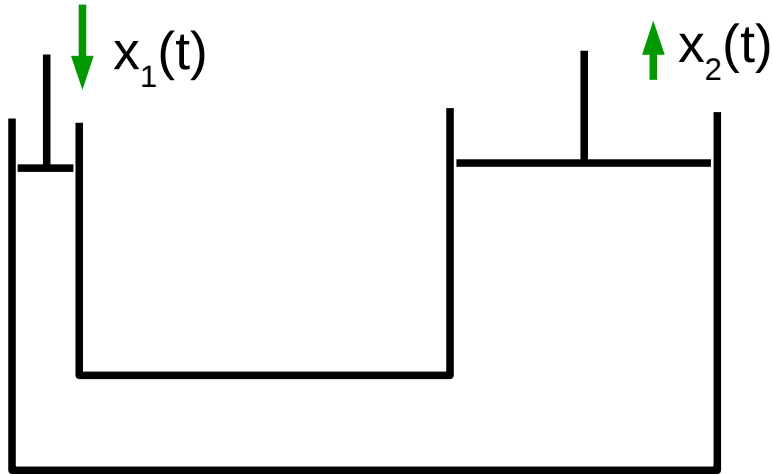
input – displacement $x_1(t)$

output – displacement $x_2(t)$

Proportional element

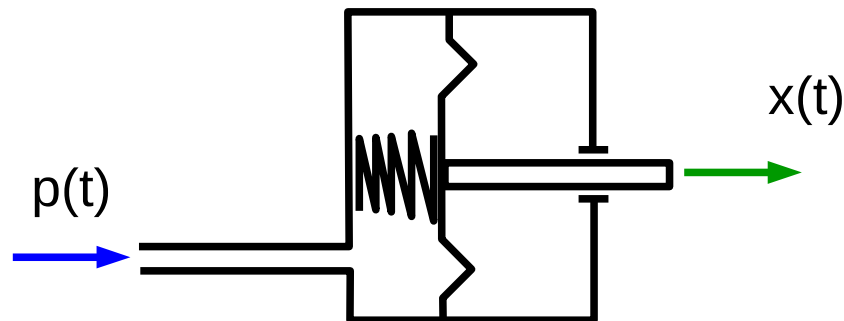
Examples

5



HYDRAULIC LEVER:
input – displacement $x_1(t)$
output – displacement $x_2(t)$

6



PRESSURE ACTUATOR:
input – pressure $p_1(t)$
output – displacement $x(t)$

First-order inertial element

1. Element equation: $T \frac{dy(t)}{dt} + y(t) = ku(t)$

$u(t)$ - input
 $y(t)$ - output

First-order inertial element

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2. Static characteristic (steady state):

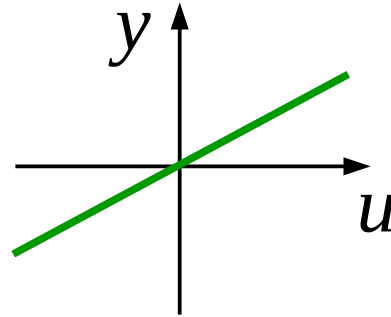
for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

First-order inertial element

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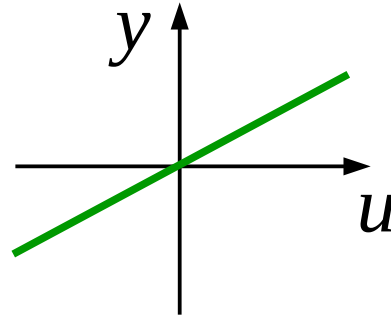
for $k > 0$

First-order inertial element

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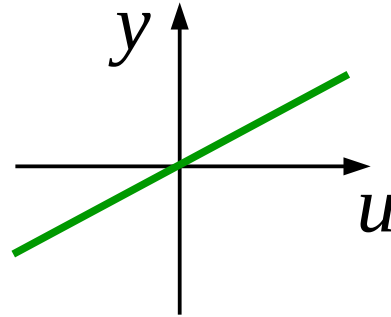
3. Transfer function:

First-order inertial element

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for $k > 0$

3. Transfer function: $H(s) = \frac{k}{Ts + 1}$

First-order inertial element

4. Step response:

First-order inertial element

4. Step response:

input: $u(t) = u_0 \mathbf{1}(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

First-order inertial element

4. Step response:

input: $u(t) = u_0 \mathbf{1}(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Laplace of output: $Y(s) = H(s) U(s) = \frac{k u_0}{s(Ts + 1)}$

First-order inertial element

4. Step response:

$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

$$\text{Laplace of output: } Y(s) = H(s) U(s) = \frac{k u_0}{s(Ts + 1)}$$

$$\text{output: } y(t) = L^{-1}\{Y(s)\} = k u_0 (1 - e^{-t/T})$$

First-order inertial element

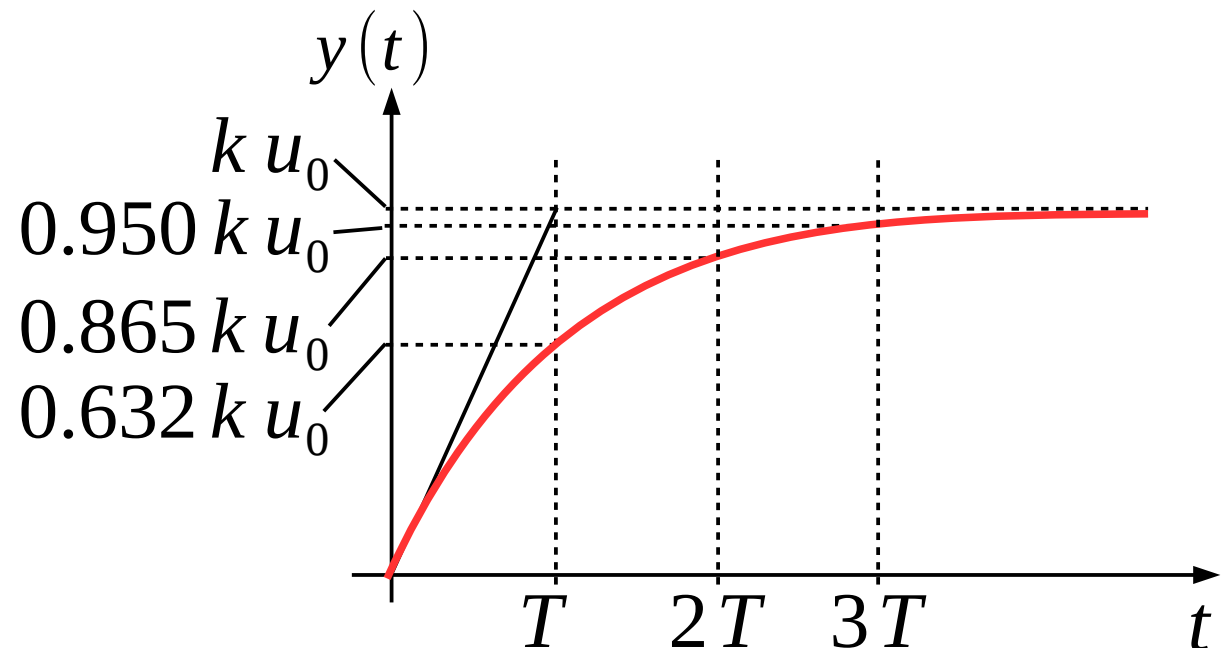
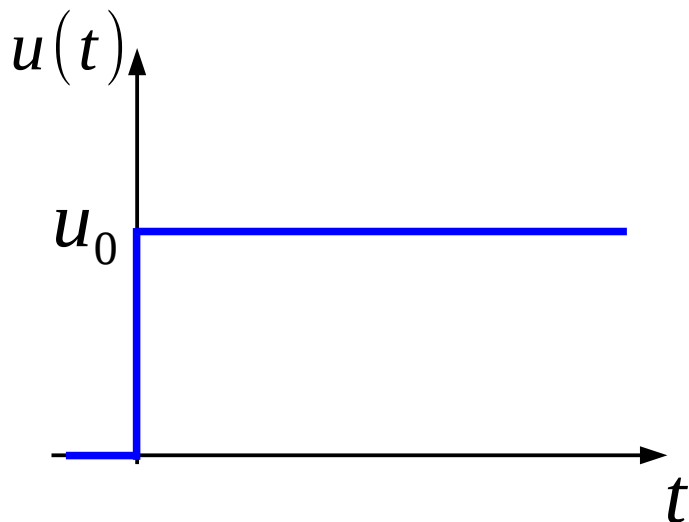
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First-order inertial element

5. Frequency response:

First-order inertial element

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First-order inertial element

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$$P(\omega) = \frac{k}{T^2\omega^2 + 1}, \quad Q(\omega) = \frac{-kT\omega}{T^2\omega^2 + 1}$$

First-order inertial element

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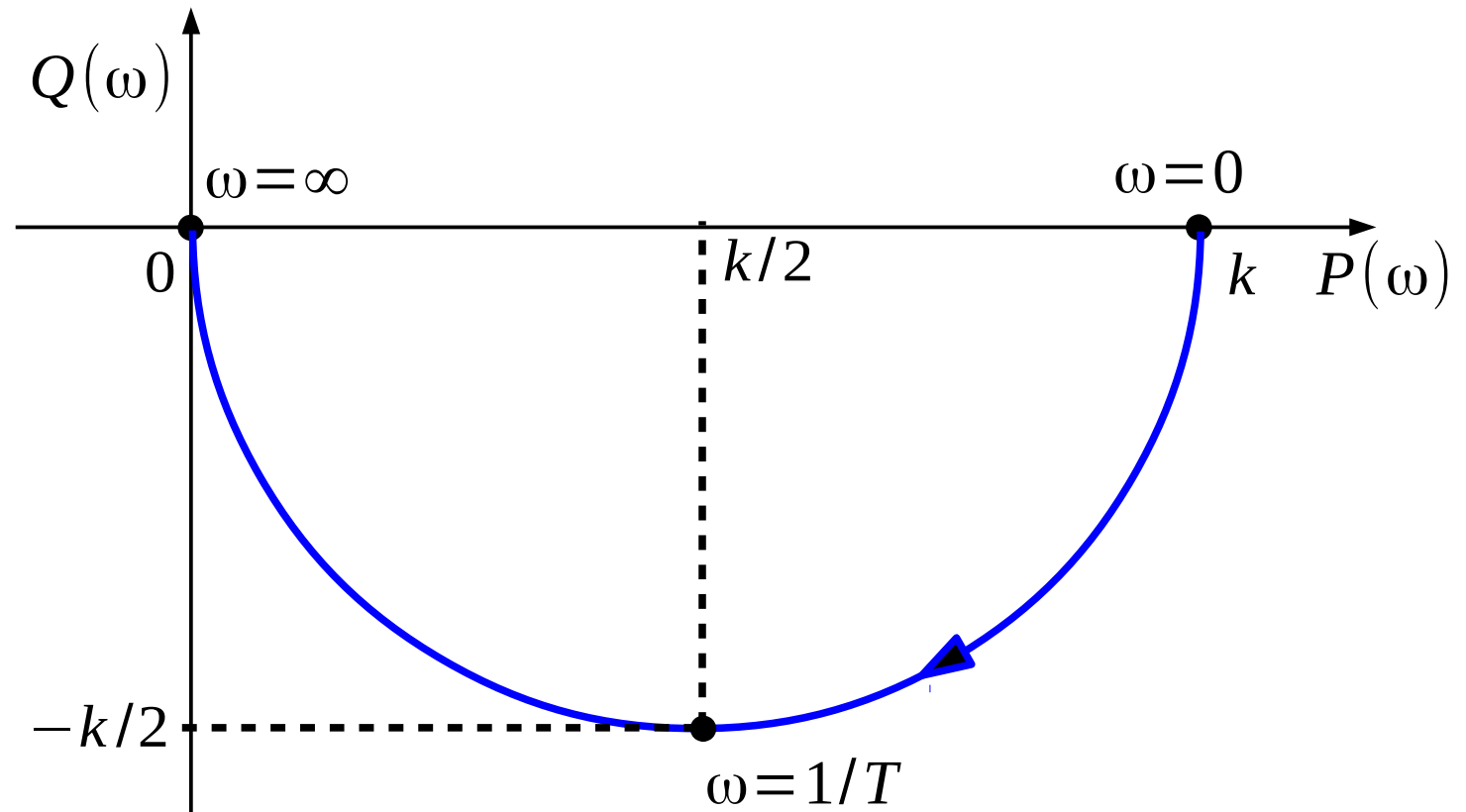
6. Nyquist plot:

First-order inertial element

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6. Nyquist plot:
for $k > 0$



First-order inertial element

7. Bode plot:

First-order inertial element

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First-order inertial element

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$$L(\omega) = 20 \log A(\omega) = 20 \log |k| - 20 \log \sqrt{T^2 \omega^2 + 1}$$

First-order inertial element

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$$\varphi(\omega) = \arctan \frac{Q}{P} = \arctan(-T \omega)$$

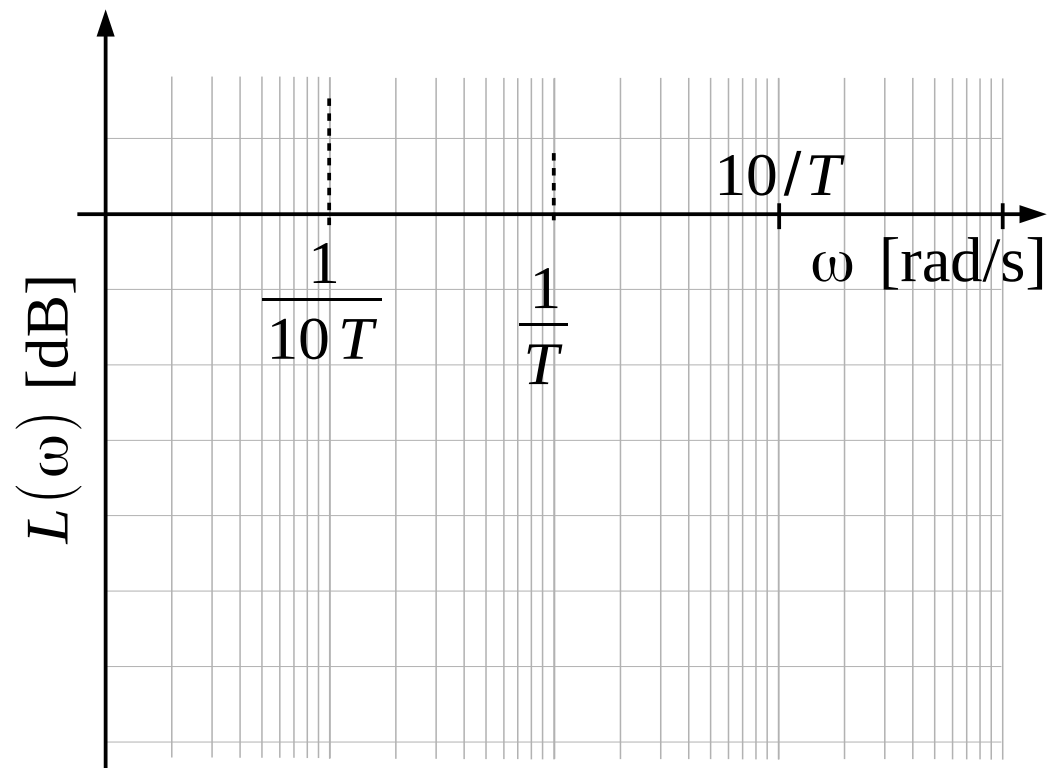
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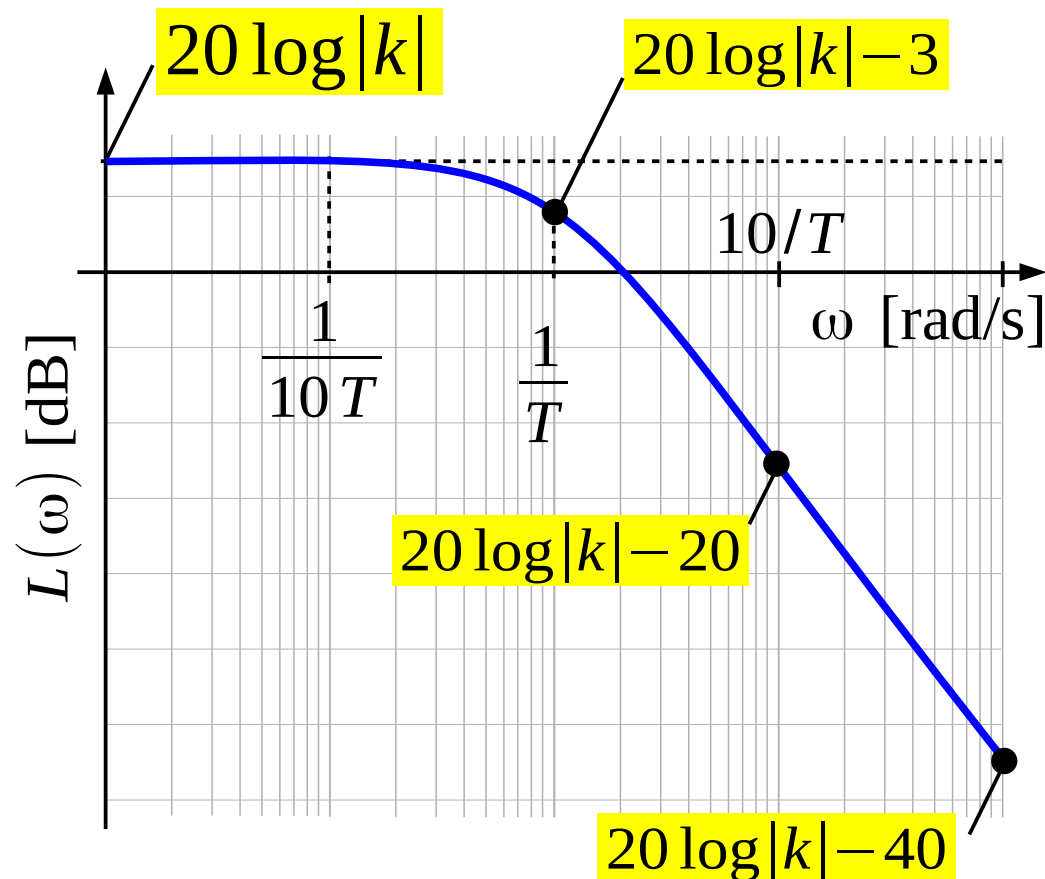
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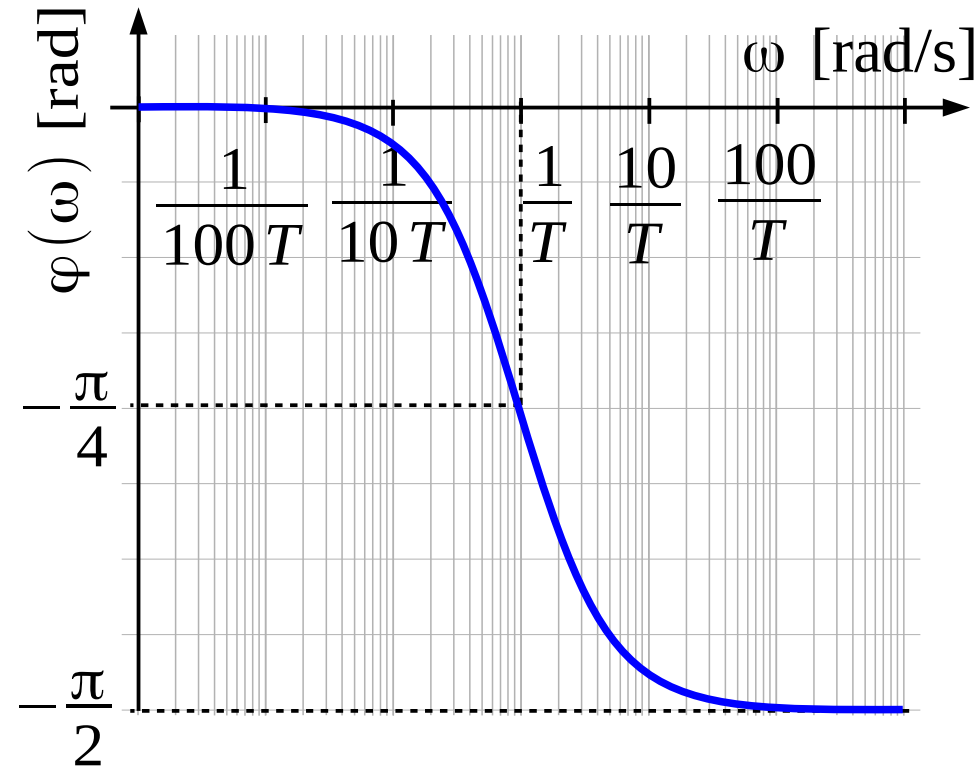
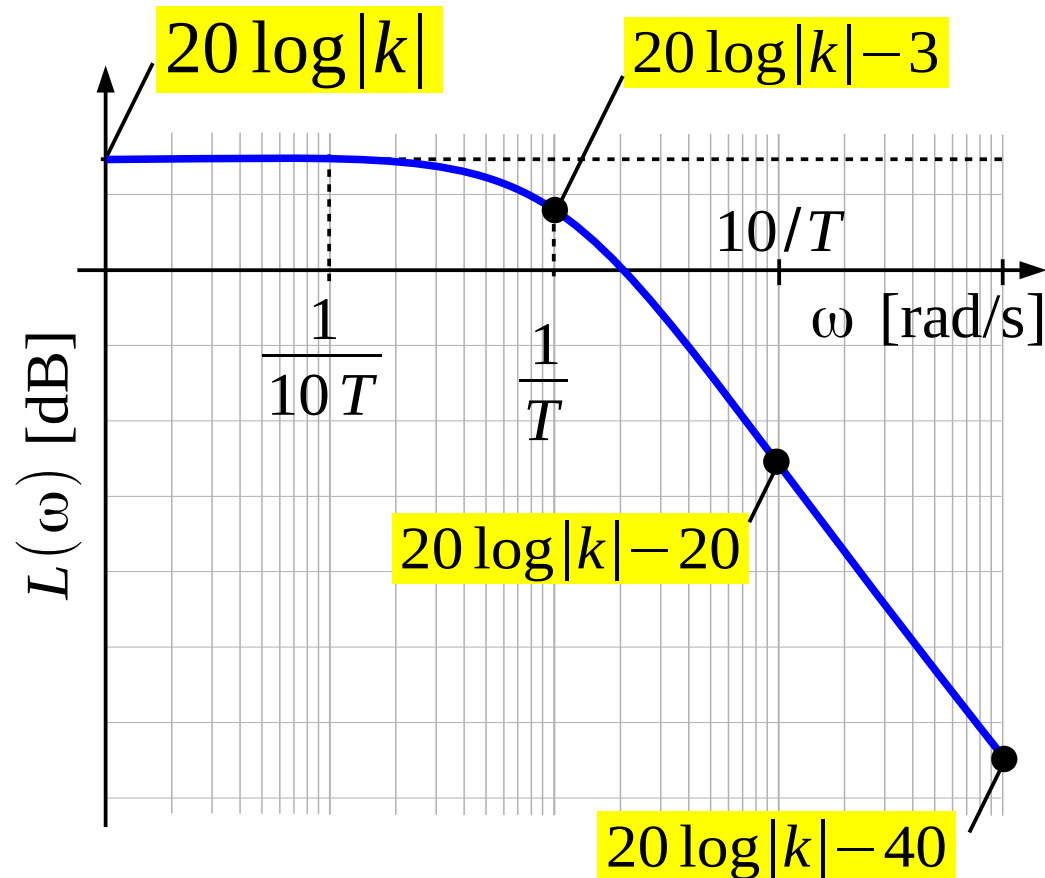


First-order inertial element

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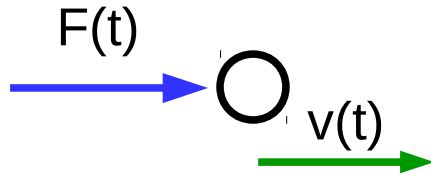
$$\varphi(\omega) = \arctan \frac{Q}{P} = \arctan(-T \omega) \quad \text{for } k > 0$$



First-order inertial element

Examples

①



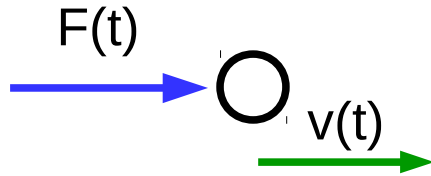
LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
input – force $F(t)$
output – velocity $v(t)$

example: car is driving on a flat surface with air resistance proportional to its velocity, described using machine equation of motion, with assumption of constant reduced mass.

First-order inertial element

Examples

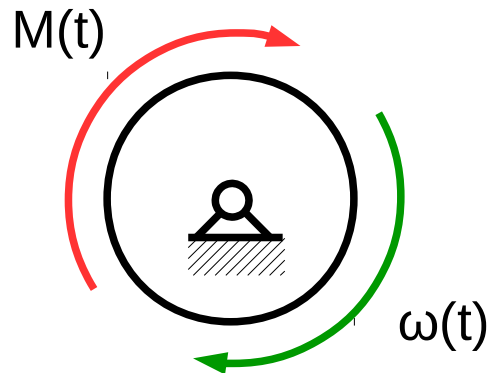
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②

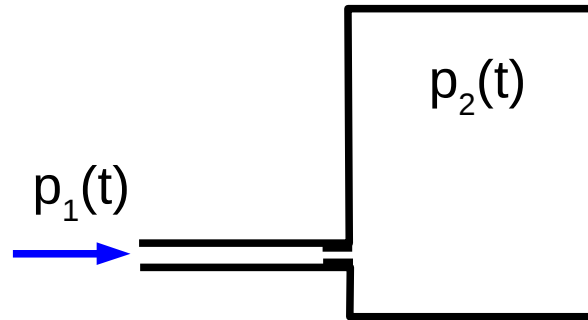


ANGULAR MOTION OF A RIGID BODY WITH LINEAR DAMPING:
input – torque $M(t)$
output – angular velocity $\omega(t)$

First-order inertial element

Examples

3

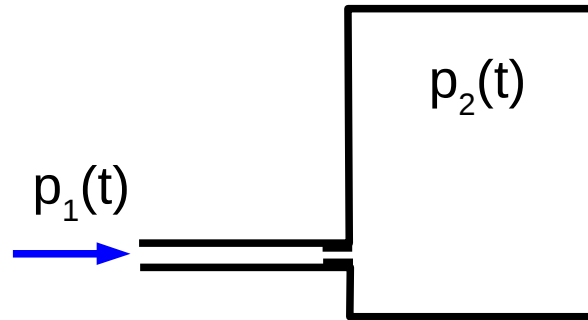


AIR CONTAINER:
input – pressure $p_1(t)$
output – pressure $p_2(t)$

First-order inertial element

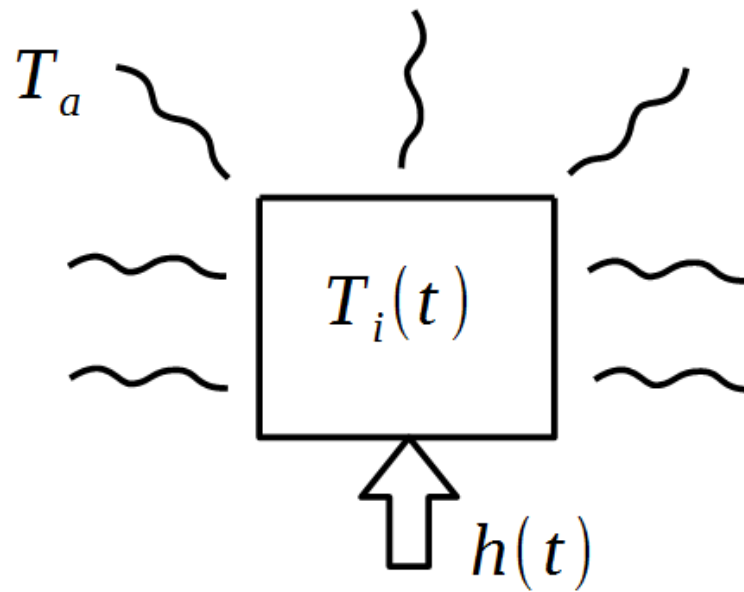
Examples

3



AIR CONTAINER:
input – pressure $p_1(t)$
output – pressure $p_2(t)$

4



HEATED OBJECT WITH SMALL
INERTIA:
input – heater power $h(t)$
output – object temperature $T_i(t)$

Integrator

1. Element equation: $\frac{dy(t)}{dt} = k u(t)$

$u(t)$ - input
 $y(t)$ - output

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2. Static characteristic (steady state):

for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

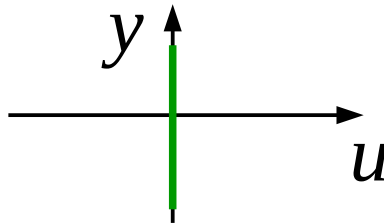
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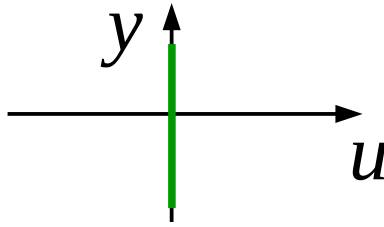
Integrator

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3. Transfer function:

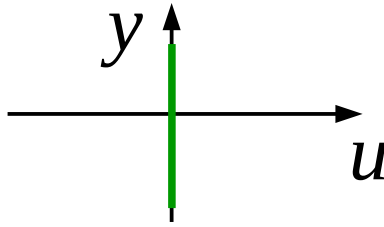
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3. Transfer function: $H(s) = \frac{k}{s}$

Integrator

4. Step response:

Integrator

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$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

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Integrator

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$$\text{output: } y(t) = L^{-1}\{Y(s)\} = k u_0 t$$

Integrator

4. Step response:

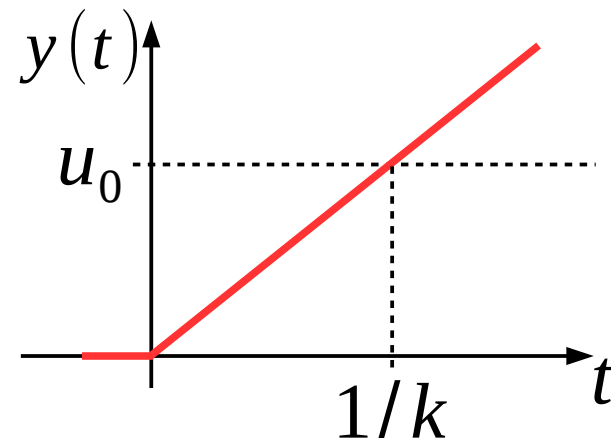
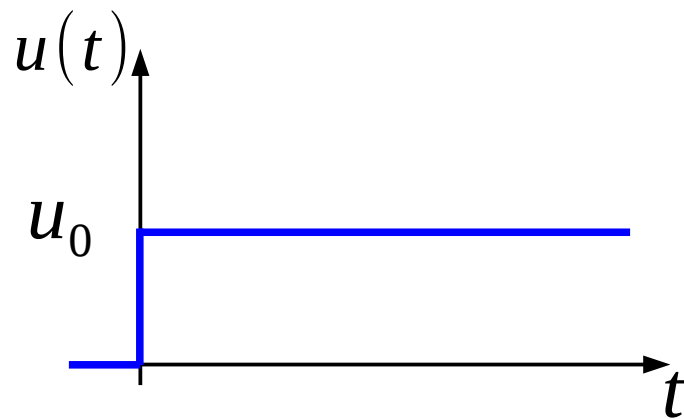
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for $k > 0$



Integrator

5. Frequency response:

Integrator

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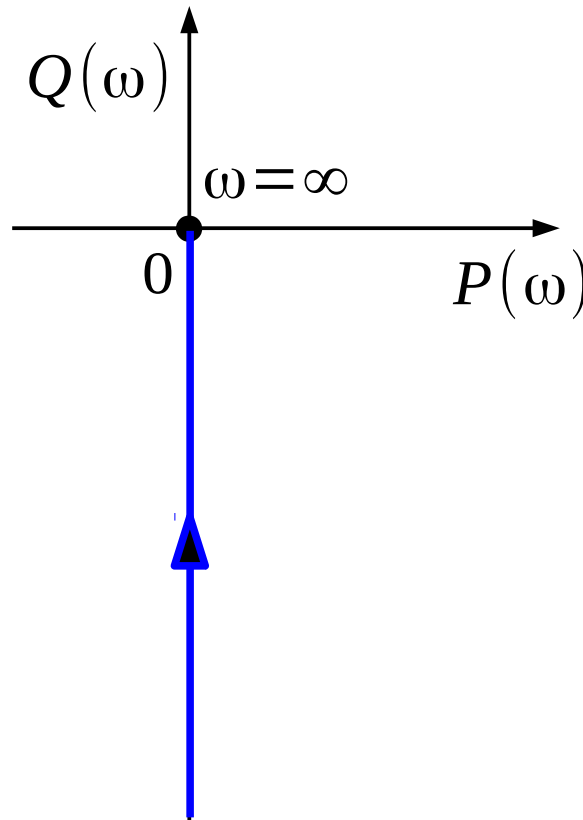
6. Nyquist plot:

Integrator

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$$P(\omega) = 0, \quad Q(\omega) = -\frac{k}{\omega}$$

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Integrator

7. Bode plot:

Integrator

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = \left| \frac{k}{\omega} \right|$

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Integrator

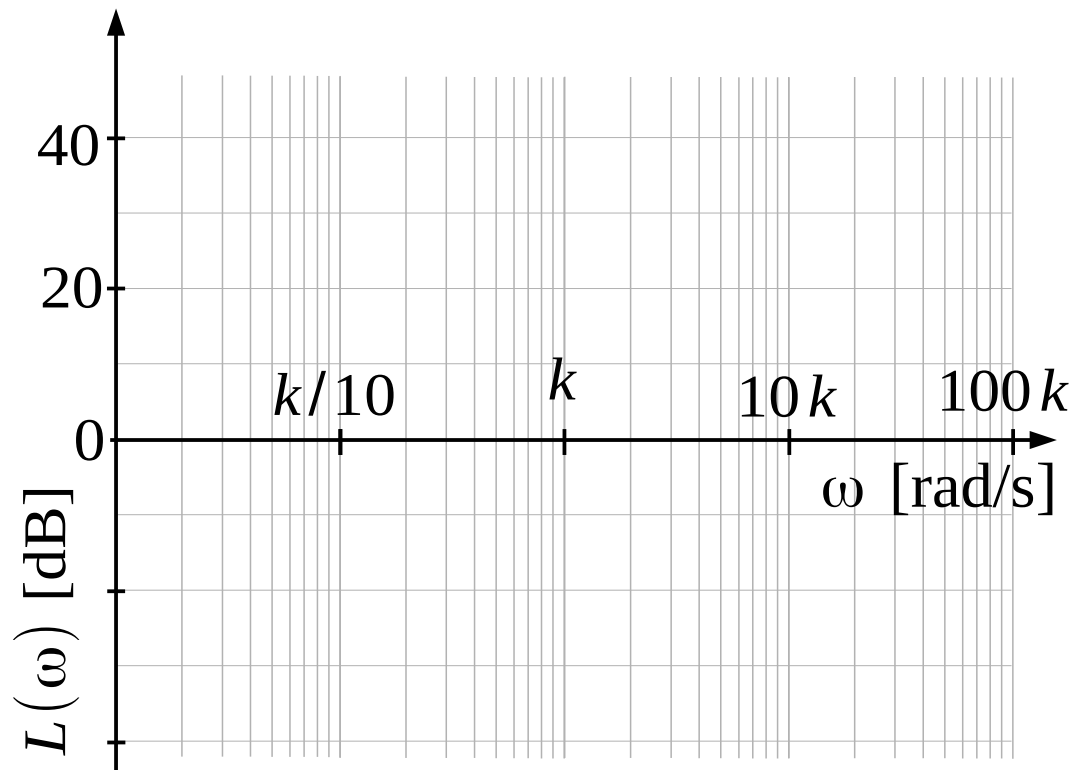
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$$L(\omega) = 20 \log A(\omega) = 20 \log \left| \frac{k}{\omega} \right| \quad \varphi(\omega) = \arctan \frac{Q}{P} = \arctan(-\infty)$$

Integrator

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = \left| \frac{k}{\omega} \right|$ for $k > 0$

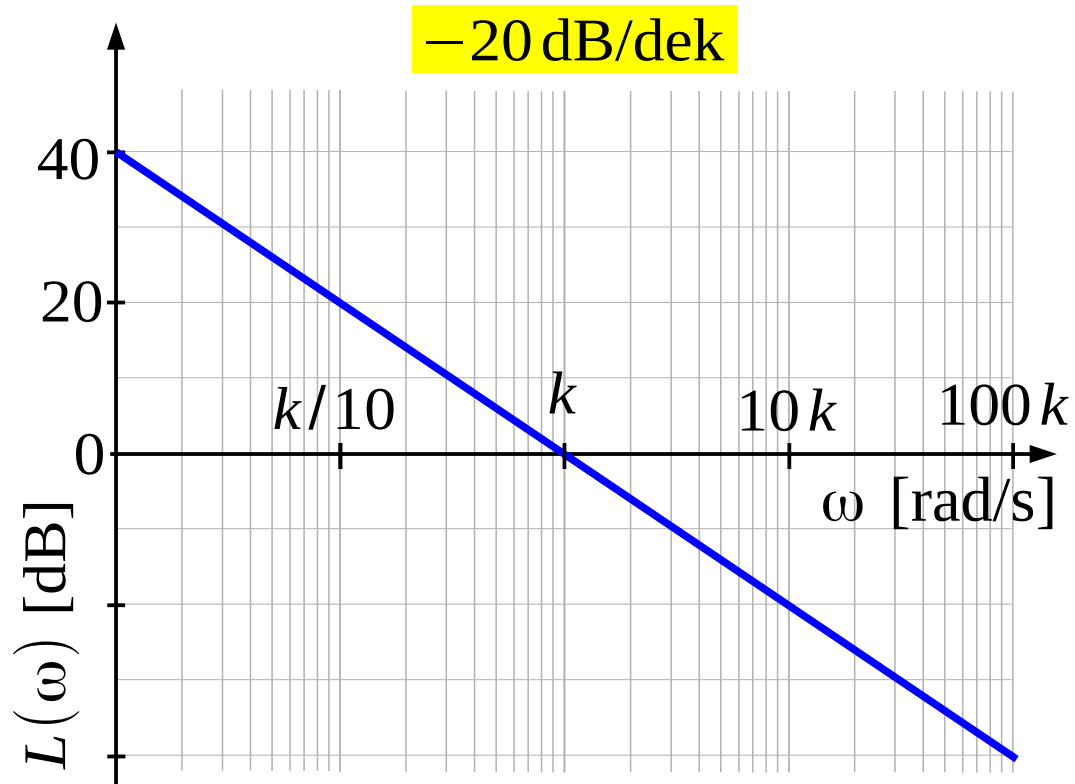
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Integrator

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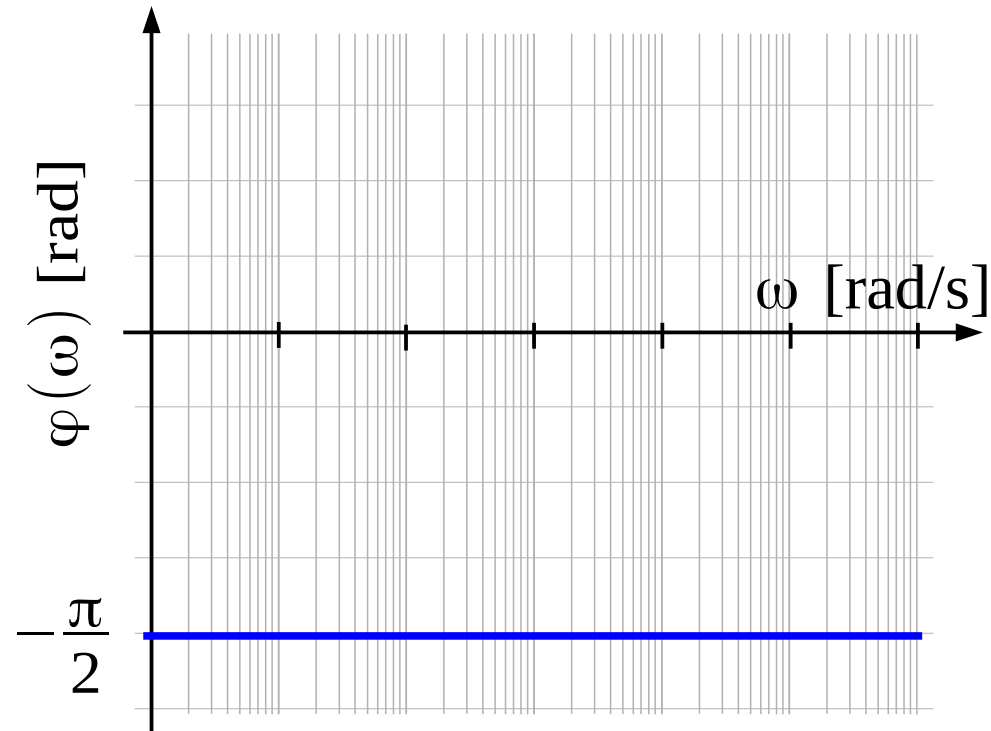
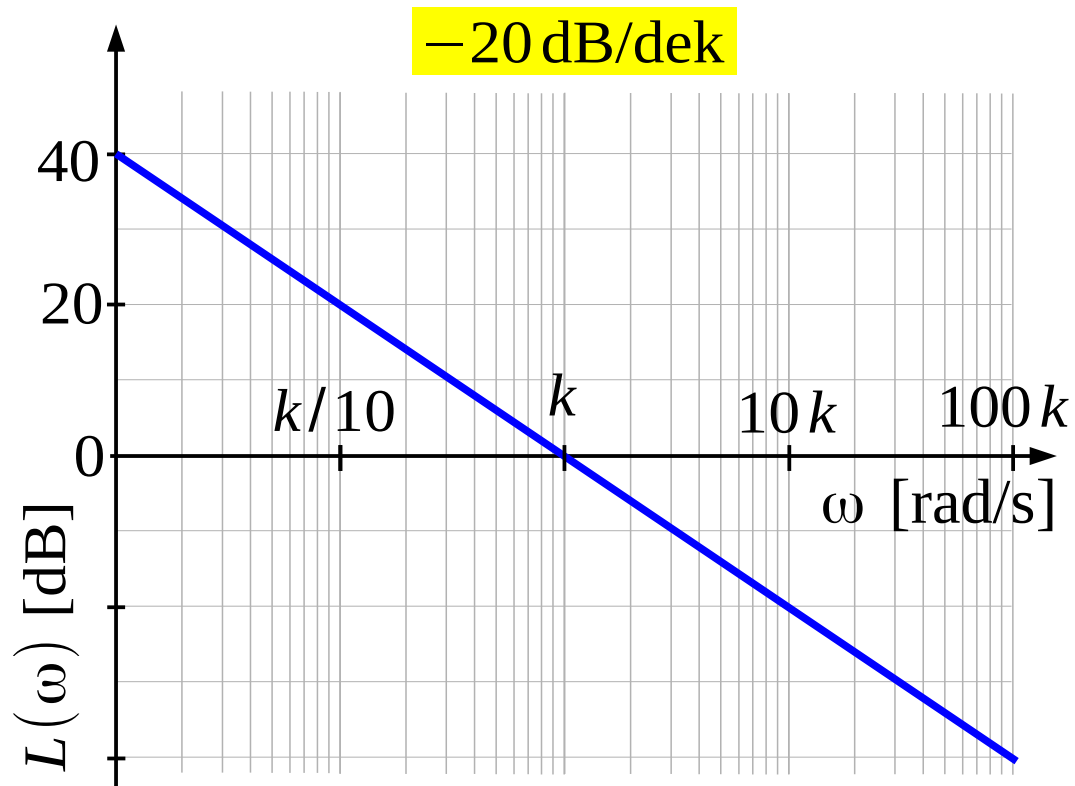
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Integrator

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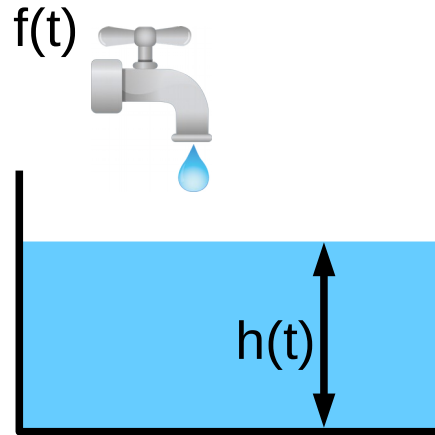
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Integrator

Examples

1

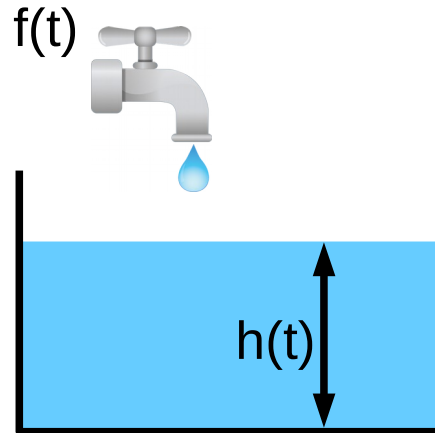


PRISM LIQUID TANK:
input – liquid inflow $f(t)$
output – liquid level $h(t)$

Integrator

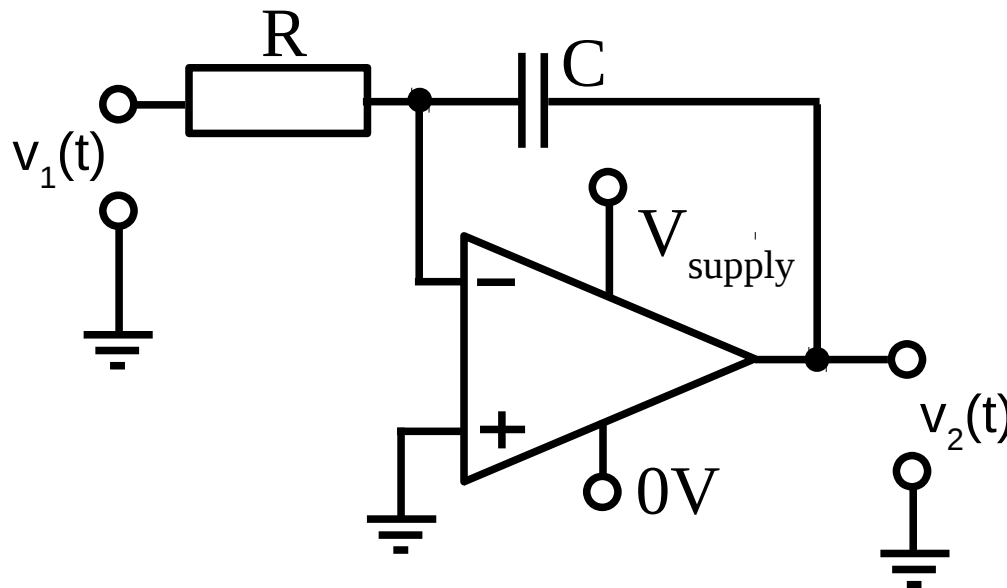
Examples

①



PRISM LIQUID TANK:
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output – liquid level $h(t)$

②



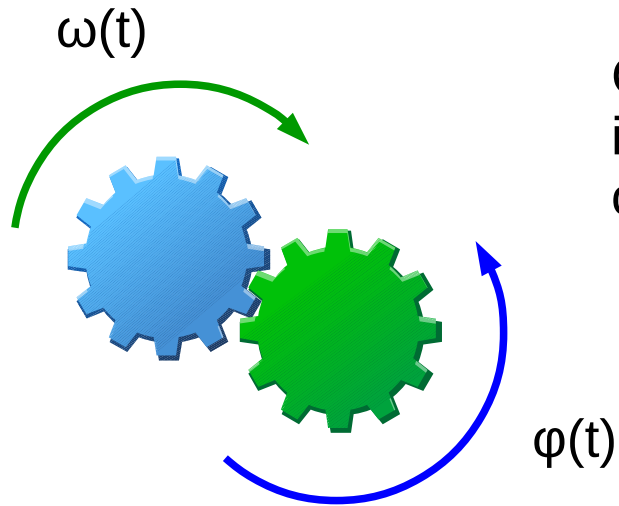
OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = \frac{1}{RC} \int_0^t v_1(t) dt$$

Integrator

Examples

3



GEARBOX:

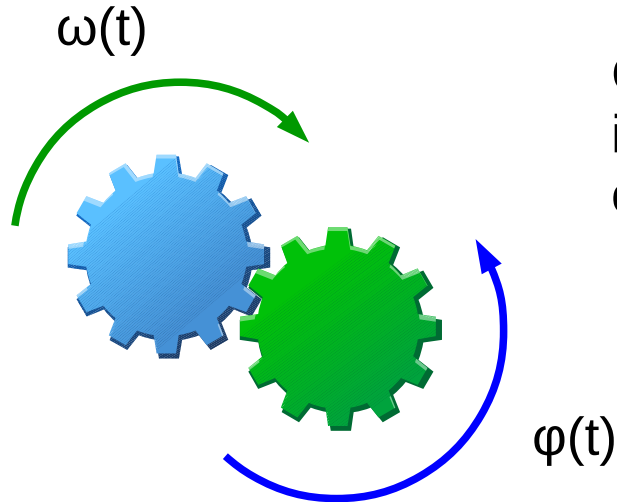
input – angular velocity $\omega(t)$

output – rotation angle $\phi(t)$

Integrator

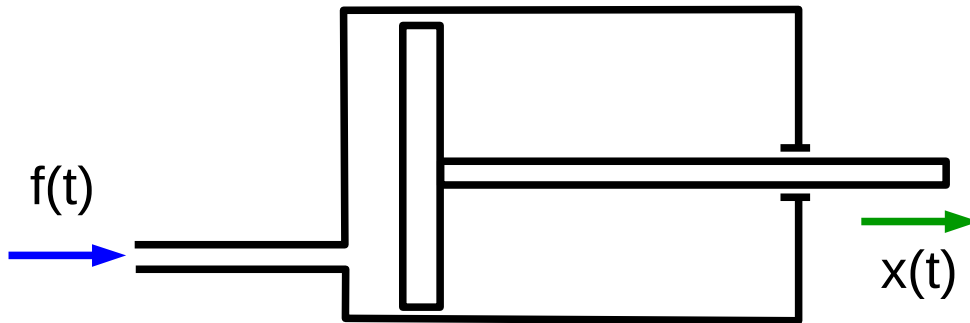
Examples

3



GEARBOX:
input – angular velocity $\omega(t)$
output – rotation angle $\phi(t)$

4



HYDRAULIC CYLINDER:
input – volume inflow $f(t)$
output – displacement $x(t)$

Differentiator

1. Element equation: $y(t) = k \frac{du(t)}{dt}$

$u(t)$ - input
 $y(t)$ - output

Differentiator

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2. Static characteristic (steady state):

for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

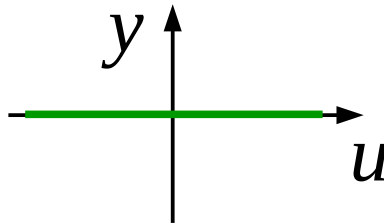
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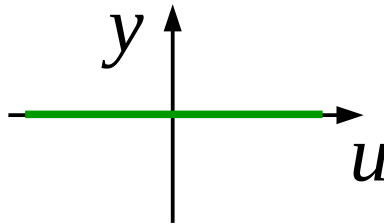
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3. Transfer function:

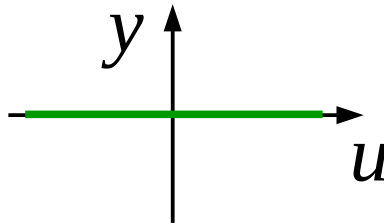
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3. Transfer function: $H(s) = k s$

Differentiator

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Differentiator

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$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

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Differentiator

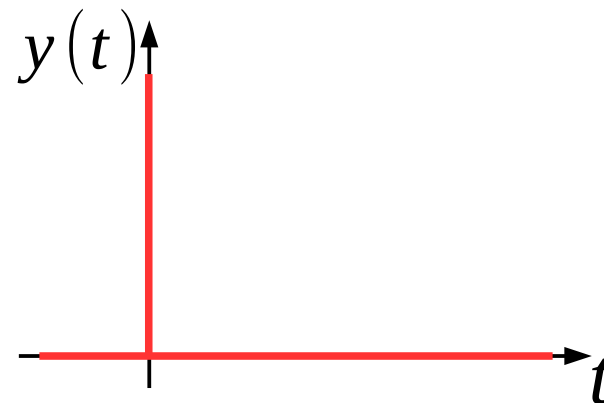
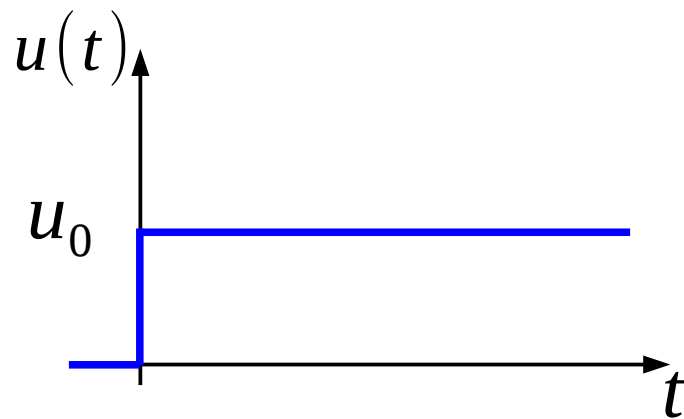
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Differentiator

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Differentiator

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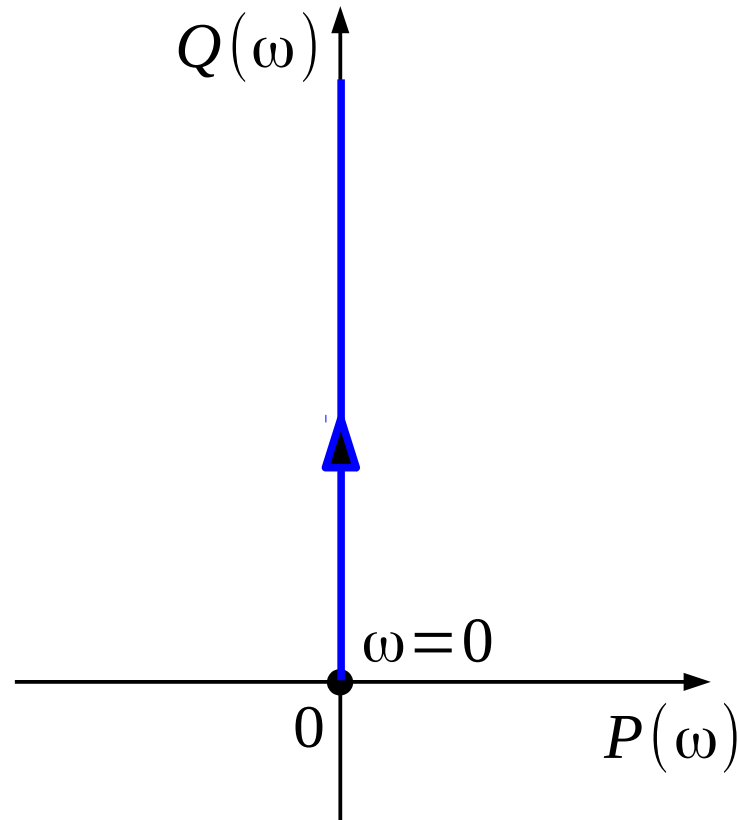
6. Nyquist plot:

Differentiator

5. Frequency response: $H(j\omega) = jk\omega$

$$P(\omega) = 0, \quad Q(\omega) = k\omega$$

6. Nyquist plot:
for $k > 0$



Differentiator

7. Bode plot:

Differentiator

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Differentiator

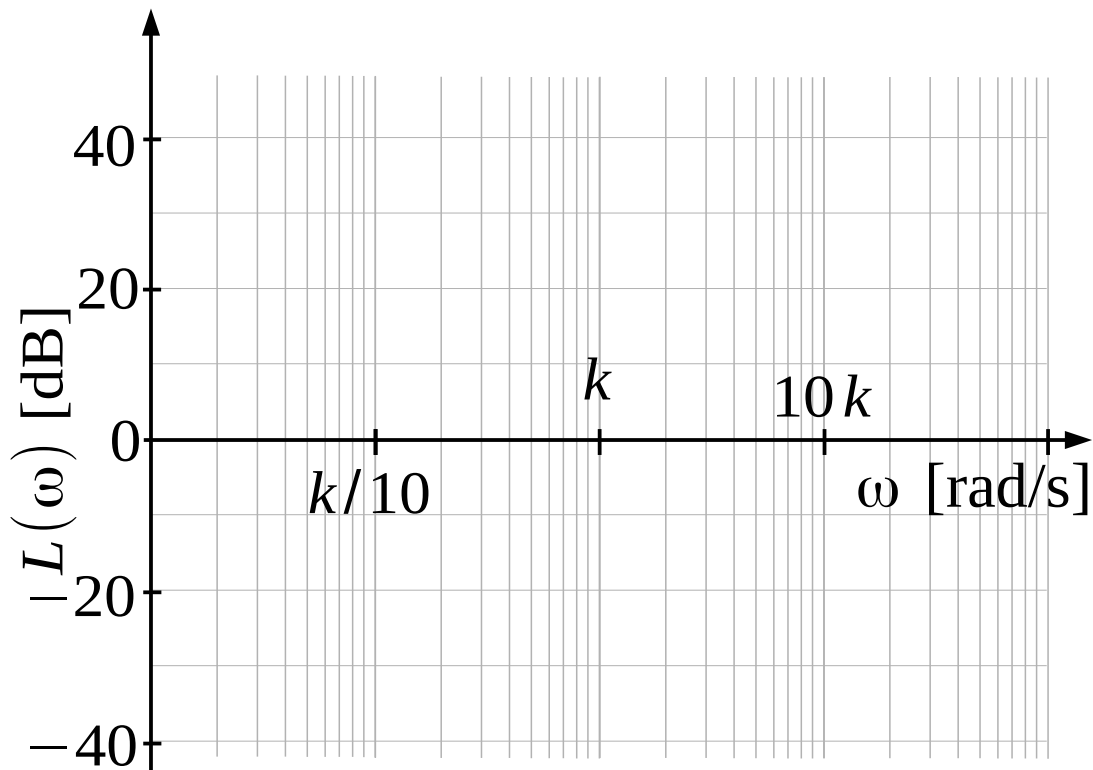
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$$L(\omega) = 20 \log A(\omega) = 20 \log |k \omega| \quad \varphi(\omega) = \arctan \frac{Q}{P} = \arctan(\infty)$$

Differentiator

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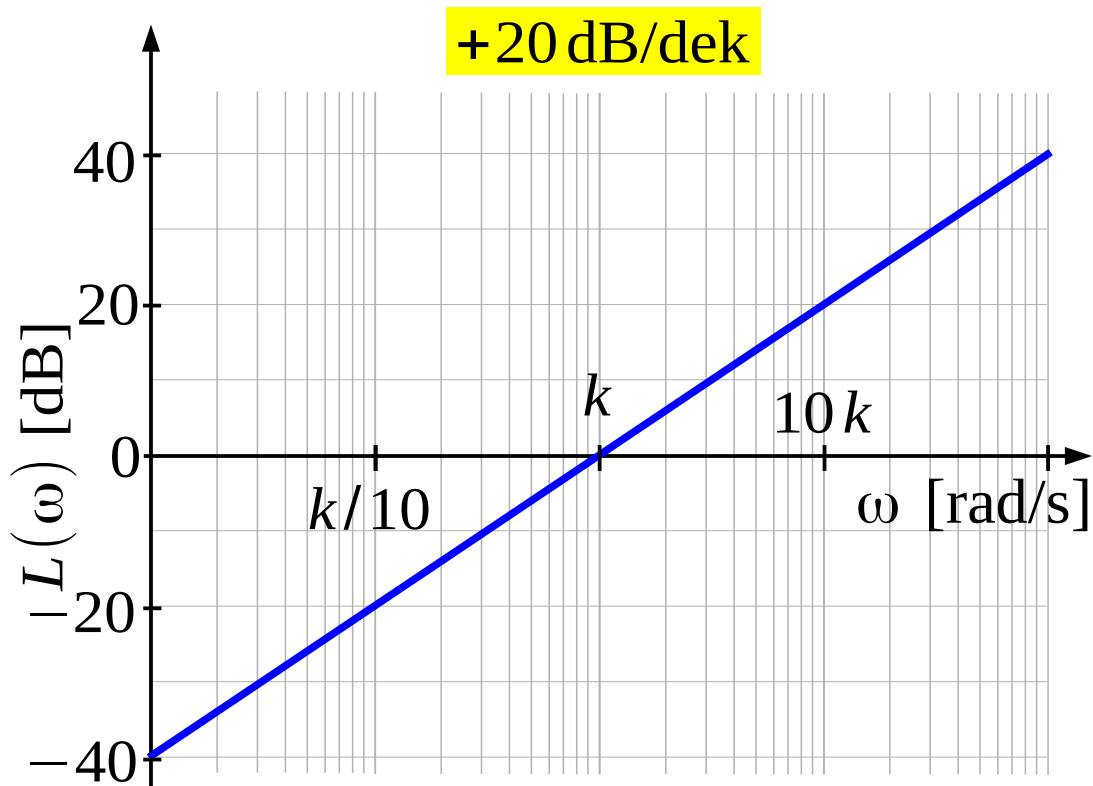
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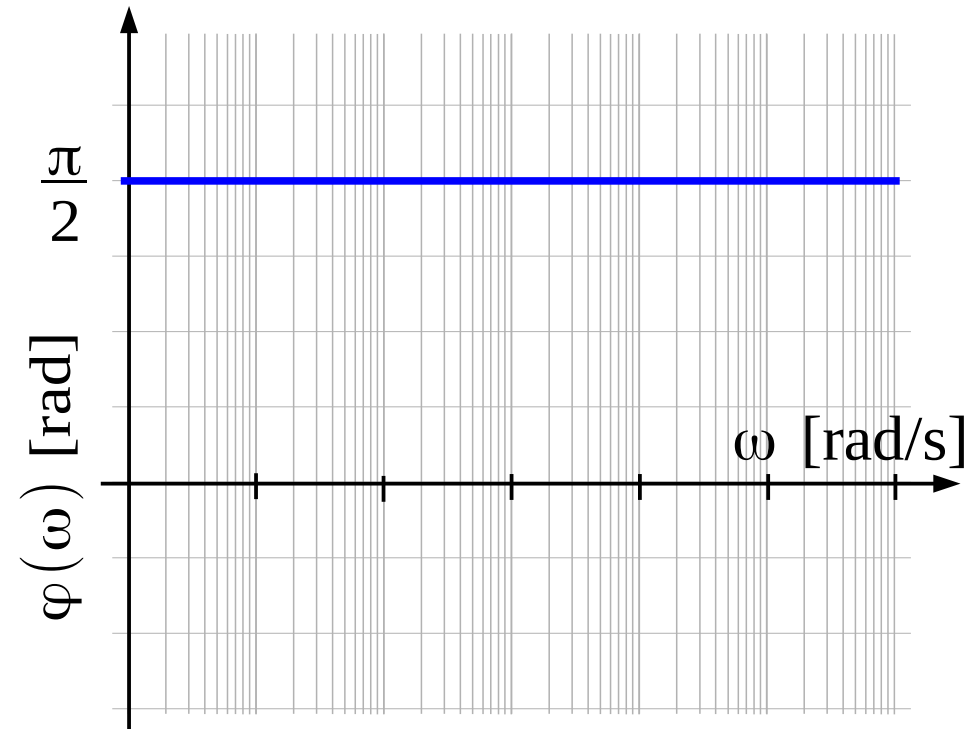
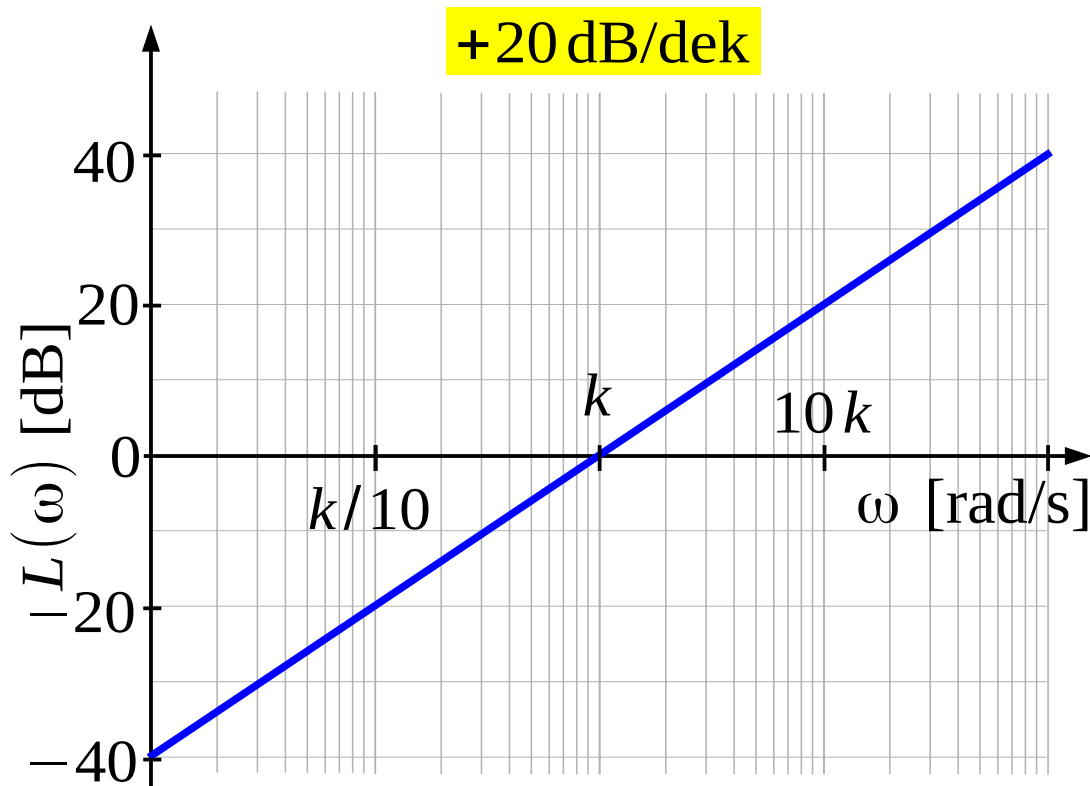
Differentiator

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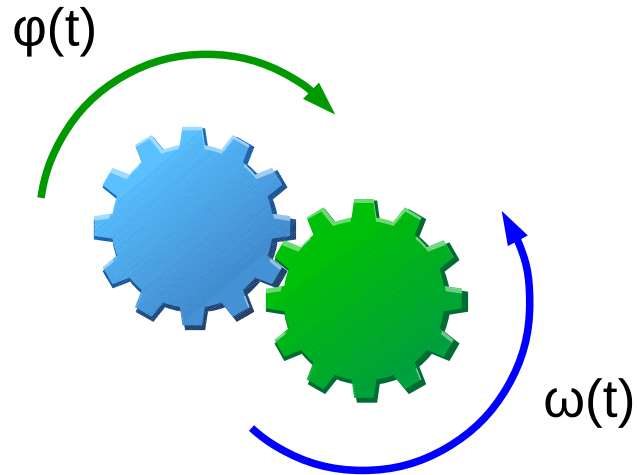
for $k > 0$



Differentiator

Examples

①



GEARBOX:

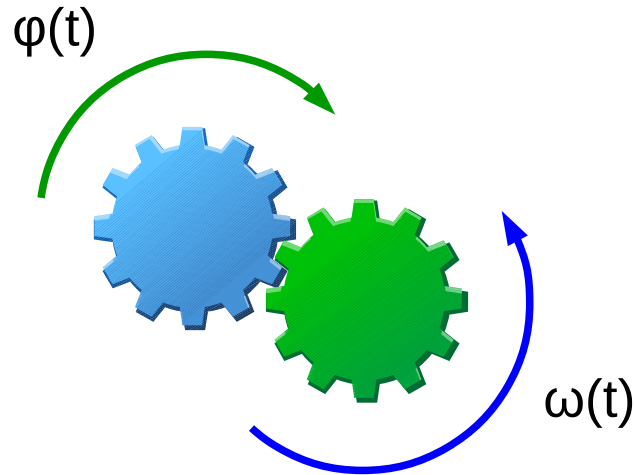
input – rotation angle $\varphi(t)$

output – angular velocity $\omega(t)$

Differentiator

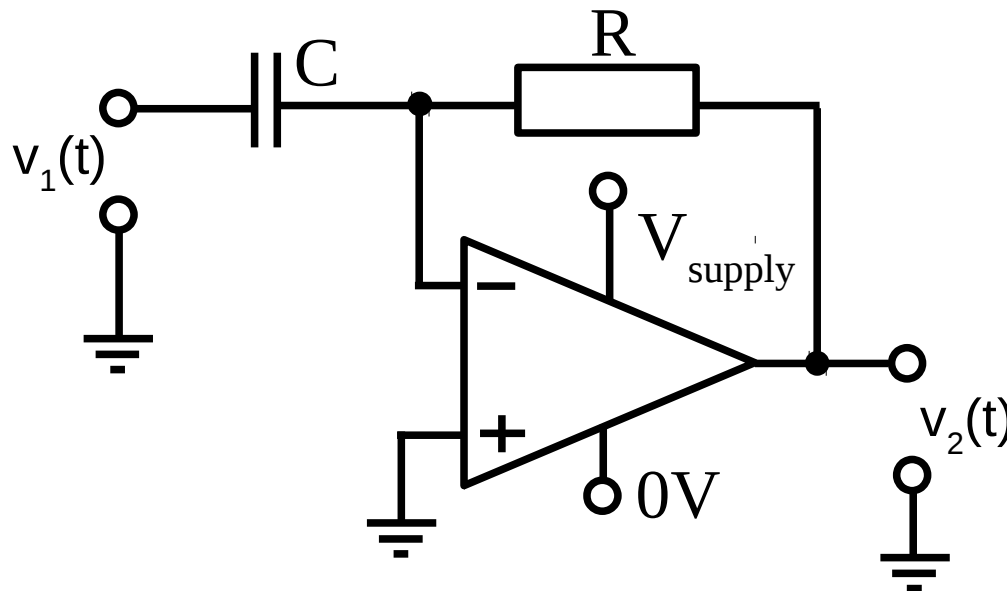
Examples

①



GEARBOX:
input – rotation angle $\varphi(t)$
output – angular velocity $\omega(t)$

②



OPERATIONAL AMPLIFIER:
input – voltage $v_1(t)$
output – voltage $v_2(t)$

$$v_2(t) = -RC \frac{dv_1(t)}{dt}$$

Real differentiator (derivative+1st order)

1. Element equation: $T \frac{dy(t)}{dt} + y(t) = k \frac{du(t)}{dt}$ $u(t)$ - input
 $y(t)$ - output

Real differentiator (derivative+1st order)

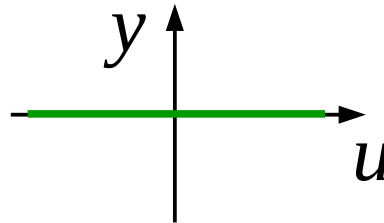
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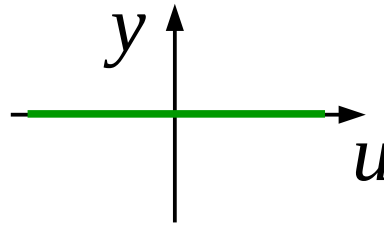
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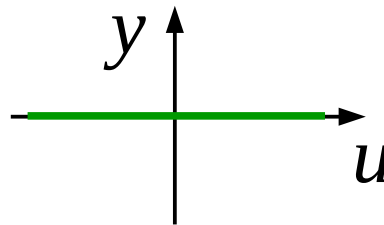


3. Transfer function:

Real differentiator (derivative+1st order)

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 $y(t)$ - output

2. Static characteristic (steady state): $y = 0$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$



3. Transfer function: $H(s) = \frac{k s}{T s + 1}$

Real differentiator (derivative+1st order)

4. Step response:

Real differentiator (derivative+1st order)

4. Step response:

$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

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Real differentiator (derivative+1st order)

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$$\text{output: } y(t) = L^{-1}\{Y(s)\} = k u_0 e^{-t/T}$$

Real differentiator (derivative+1st order)

4. Step response:

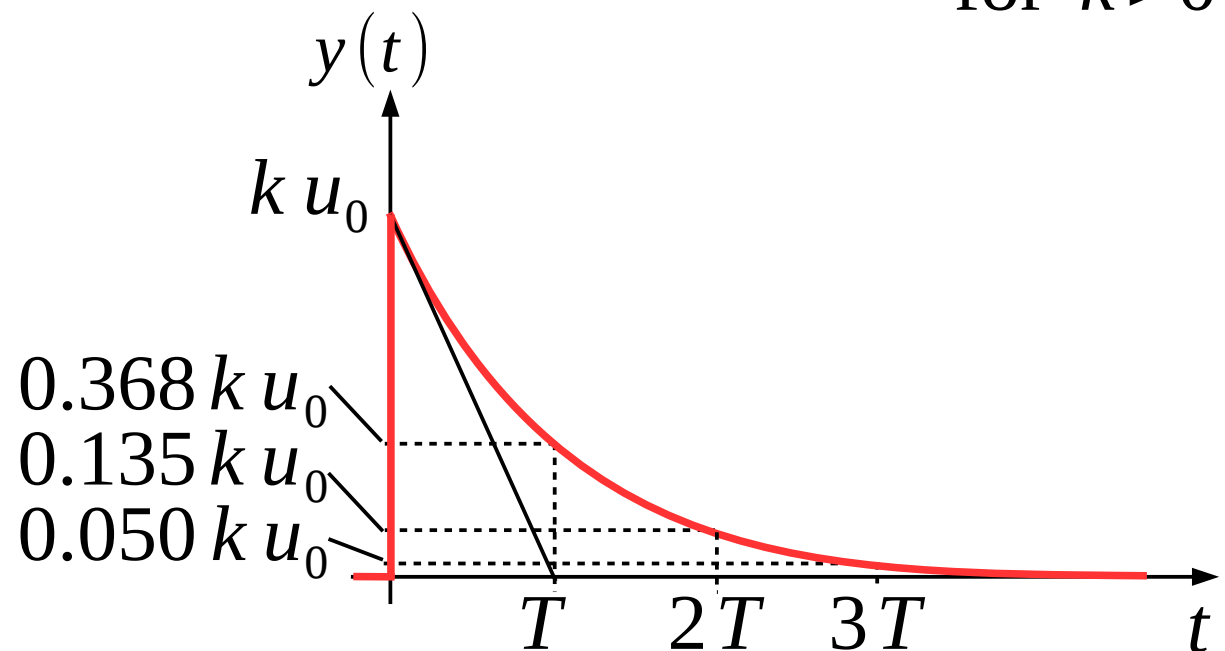
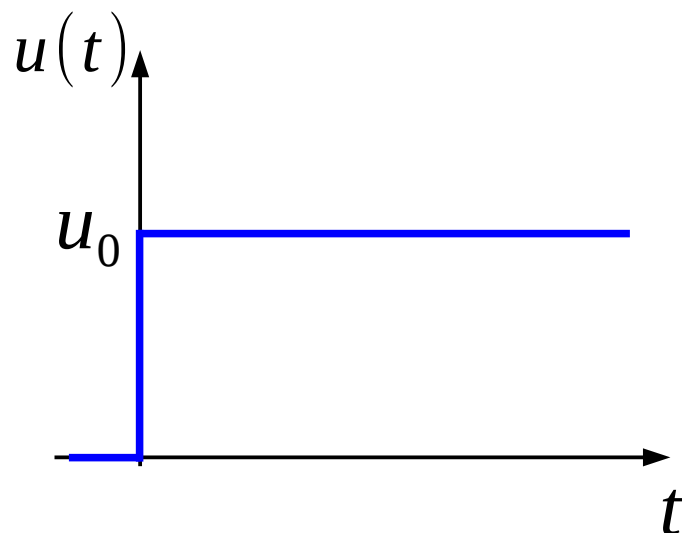
$$\text{input: } u(t) = u_0 1(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

$$\text{Laplace of output: } Y(s) = H(s) U(s) = \frac{k u_0}{Ts + 1}$$

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for $k > 0$



Real differentiator (derivative+1st order)

5. Frequency response:

Real differentiator (derivative+1st order)

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Real differentiator (derivative+1st order)

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$$P(\omega) = \frac{k T \omega^2}{T^2 \omega^2 + 1}, \quad Q(\omega) = \frac{k \omega}{T^2 \omega^2 + 1}$$

Real differentiator (derivative+1st order)

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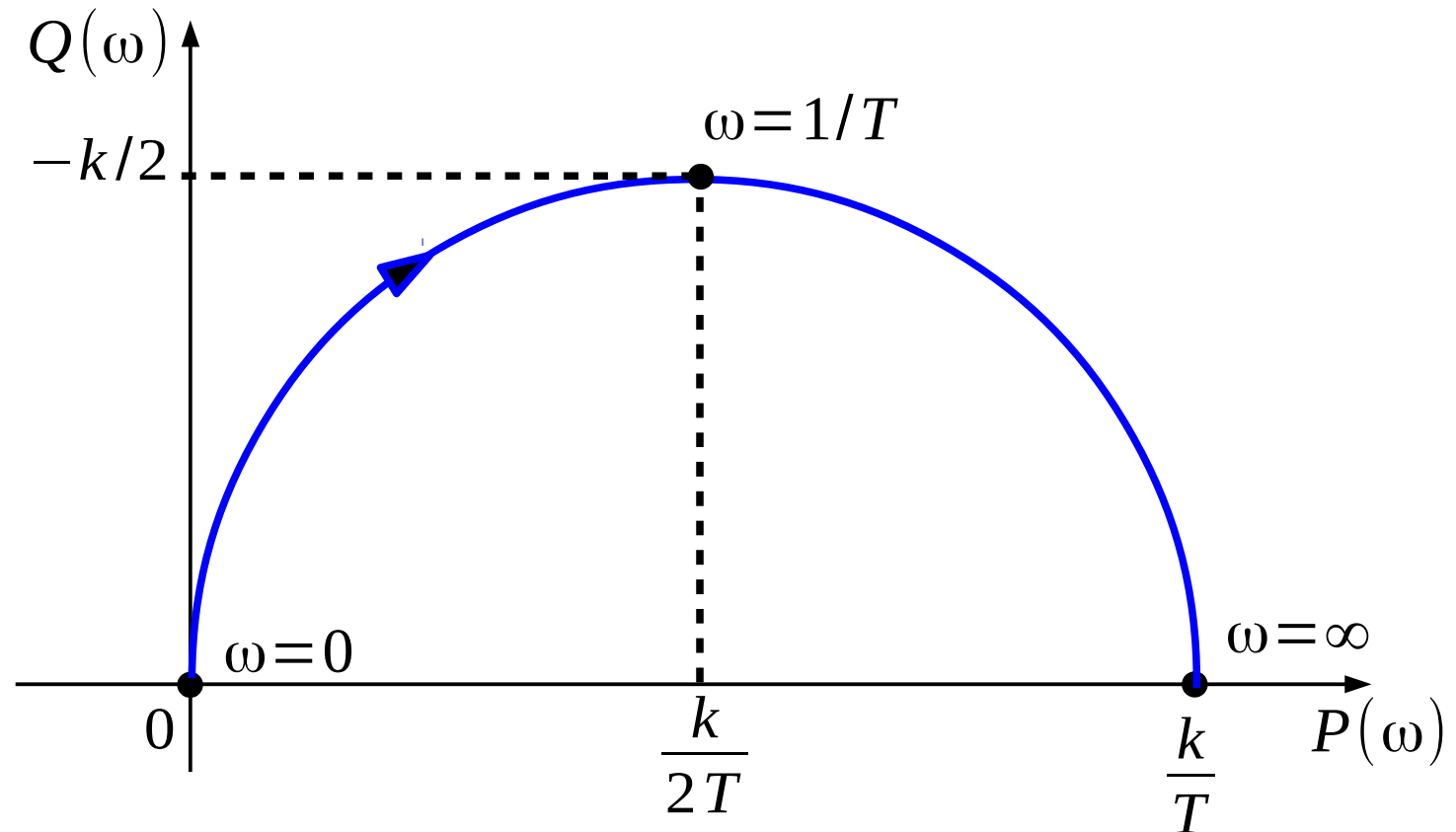
6. Nyquist plot:

Real differentiator (derivative+1st order)

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6. Nyquist plot:
for $k > 0$



Real differentiator (derivative+1st order)

7. Bode plot:

Real differentiator (derivative+1st order)

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = |k \omega| / \sqrt{T^2 \omega^2 + 1}$

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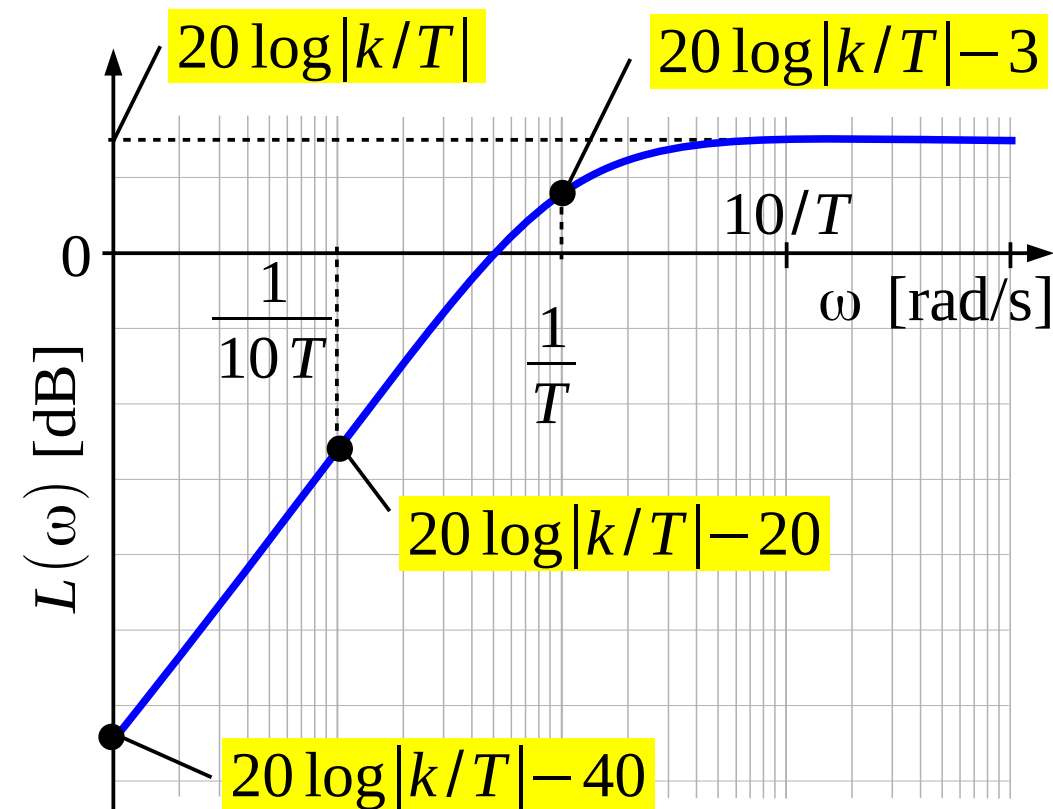
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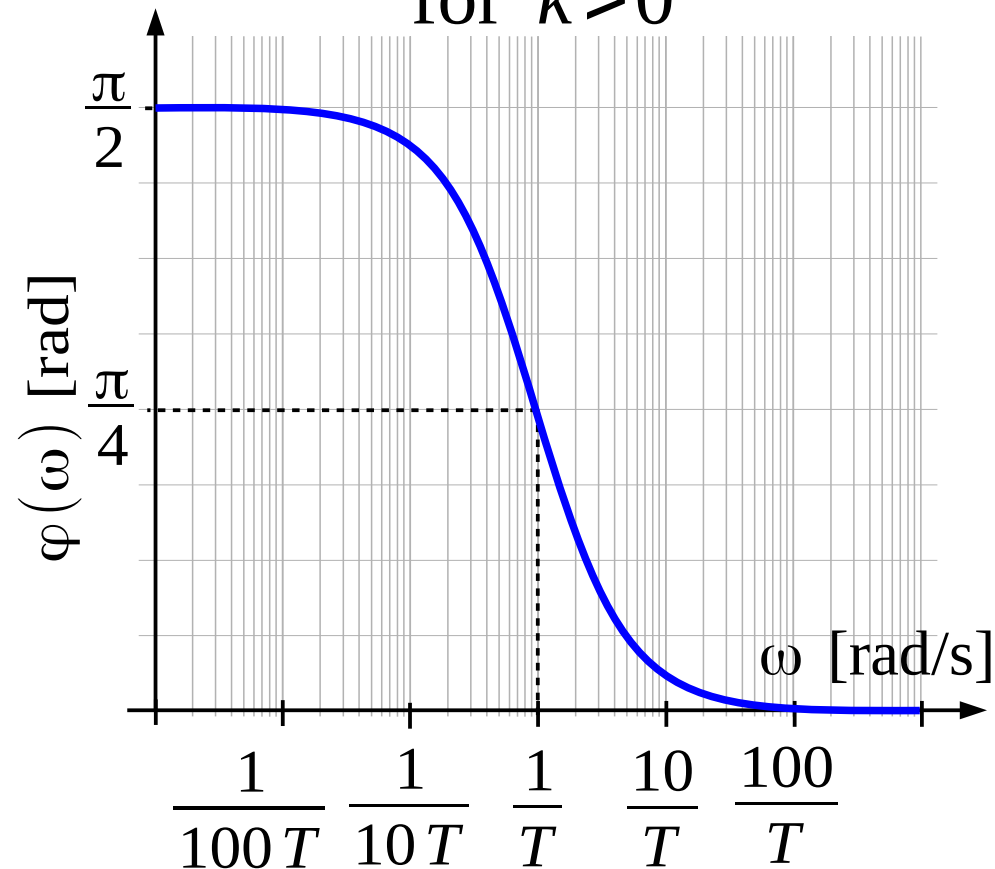
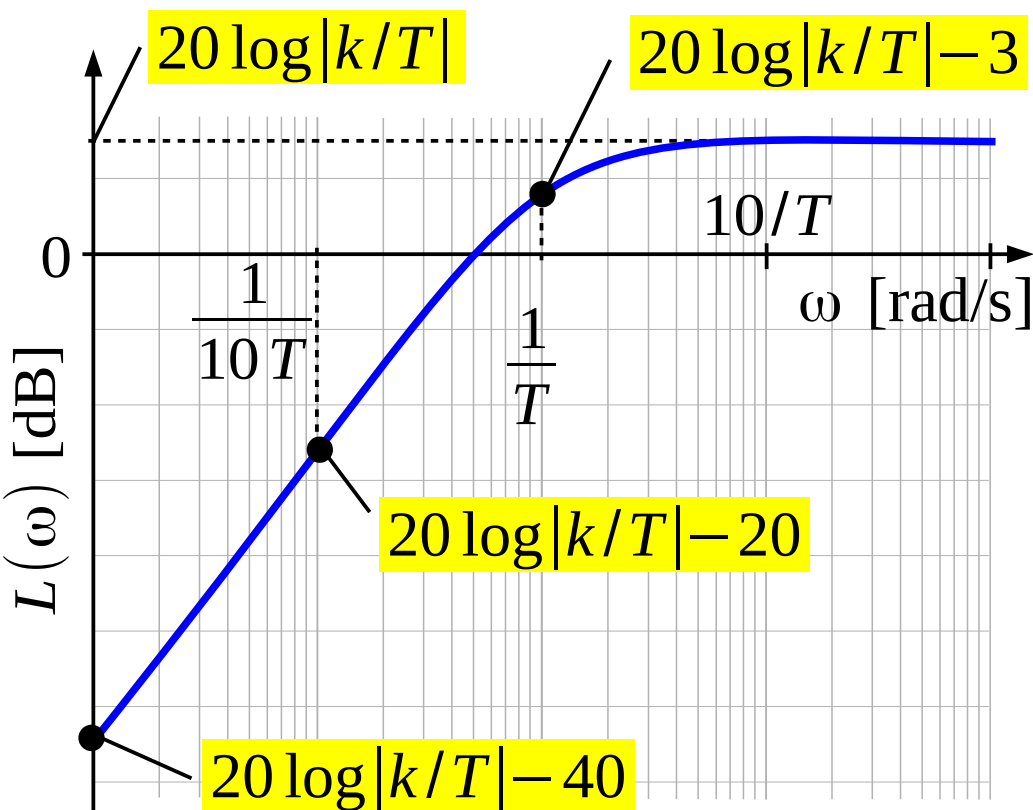
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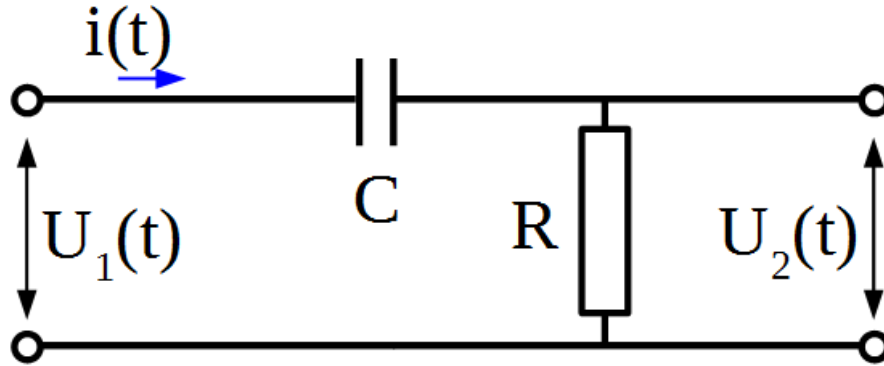
for $k > 0$



Real differentiator (derivative+1st order)

Examples

1



RC CIRCUIT:
input – voltage $u_1(t)$
output – voltage $u_2(t)$

Delay

1. Element equation: $y(t) = u(t - \tau)$

$u(t)$ - input
 $y(t)$ - output

Delay

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$u(t)$ - input
 $y(t)$ - output

2. Static characteristic (steady state):

for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

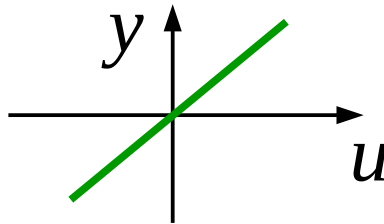
Delay

1. Element equation: $y(t) = u(t - \tau)$

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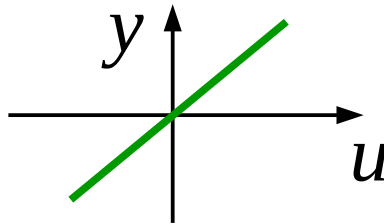
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3. Transfer function:

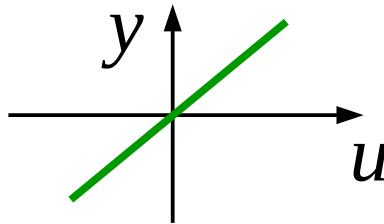
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 $y(t)$ - output

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3. Transfer function: $H(s) = e^{-\tau s}$

Delay

4. Step response:

Delay

4. Step response:

input: $u(t) = u_0 \mathbf{1}(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Delay

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$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

$$\text{Laplace of output: } Y(s) = H(s) U(s) = \frac{u_0}{s} e^{-\tau s}$$

Delay

4. Step response:

$$\text{input: } u(t) = u_0 \mathbf{1}(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

$$\text{Laplace of output: } Y(s) = H(s) U(s) = \frac{u_0}{s} e^{-\tau s}$$

$$\text{output: } y(t) = L^{-1}\{Y(s)\} = u_0 \mathbf{1}(t - \tau)$$

Delay

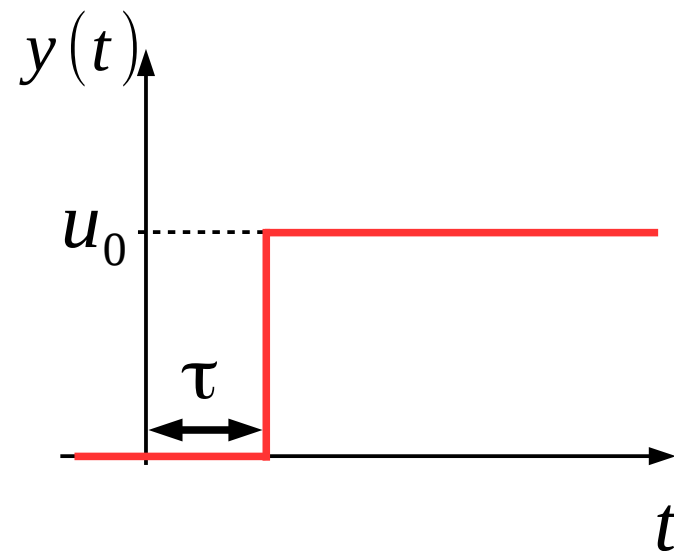
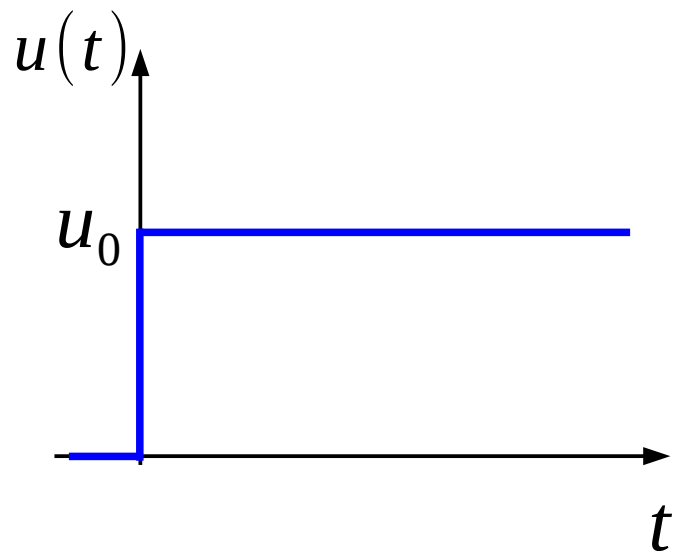
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Delay

5. Frequency response:

Delay

5. Frequency response: $H(j\omega) = e^{-\tau j\omega}$

Delay

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$$e^{-jx} = \cos x - j \sin x$$

$$P(\omega) = \cos(\tau\omega), \quad Q(\omega) = -\sin(\tau\omega)$$

Delay

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$$e^{-jx} = \cos x - j \sin x$$

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6. Nyquist plot:

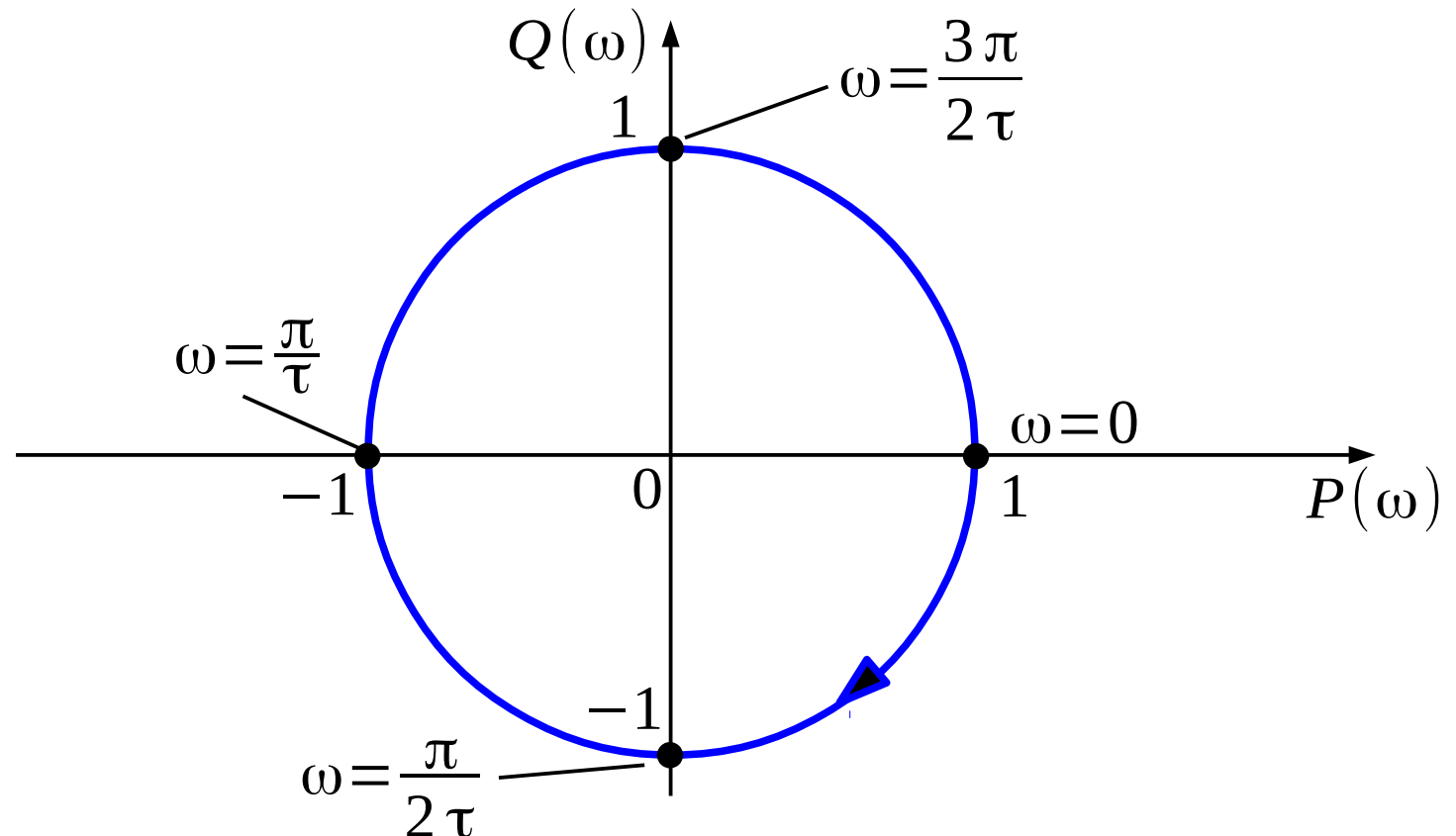
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6. Nyquist plot:
for $k > 0$



Delay

7. Bode plot:

Delay

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2} = 1$

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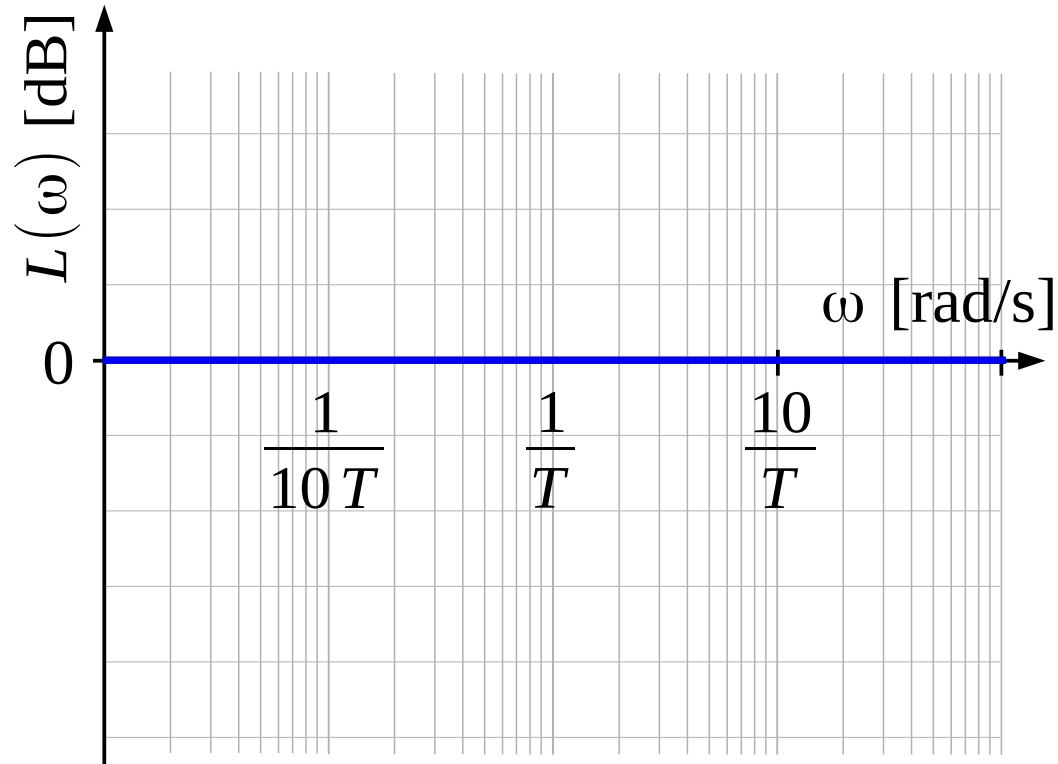
$$\varphi(\omega) = \arctan \frac{Q}{P} = \arctan(-\tan(\tau \omega)) = -\tau \omega$$

Delay

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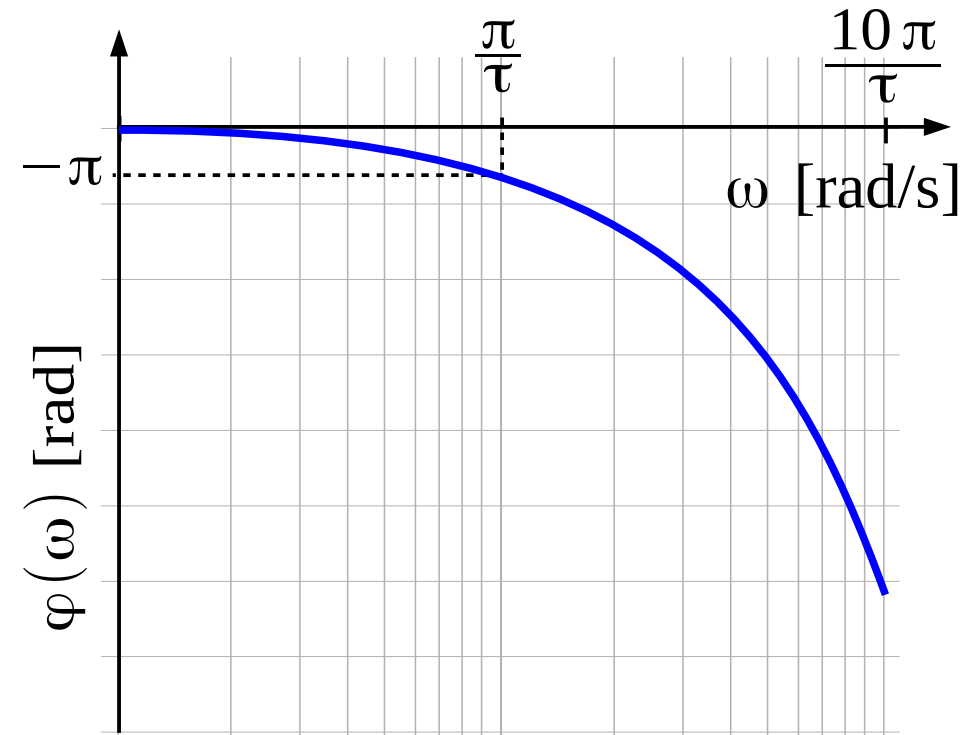
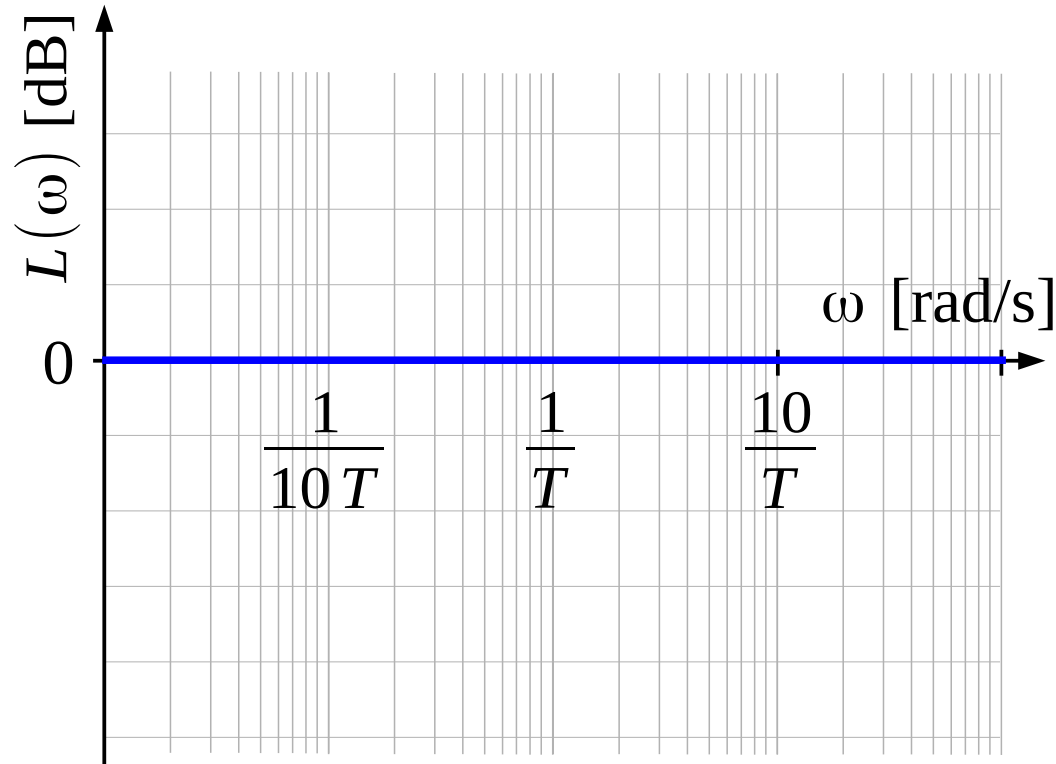


Delay

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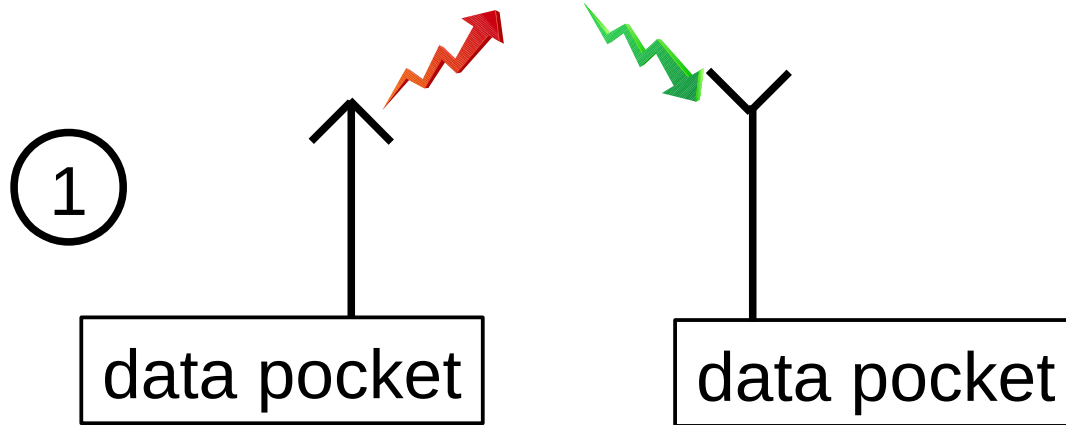
$$L(\omega) = 20 \log A(\omega) = 20 \log 1 = 0$$

$$\varphi(\omega) = \arctan \frac{Q}{P} = \arctan(-\tan(\tau\omega)) = -\tau\omega$$



Delay

Examples



WIRELESS TRANSMISSION:
input – sent data
output – received data

Second-order inertial element

1. Element equation:
$$T_1^2 \frac{d^2 y(t)}{dt^2} + T_2 \frac{dy(t)}{dt} + y(t) = k u(t)$$

Second-order inertial element

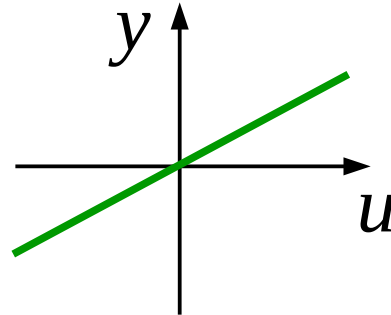
1. Element equation: $T_1^2 \frac{d^2 y(t)}{dt^2} + T_2 \frac{dy(t)}{dt} + y(t) = k u(t)$

2. Static characteristic (steady state): $\text{for } \frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$

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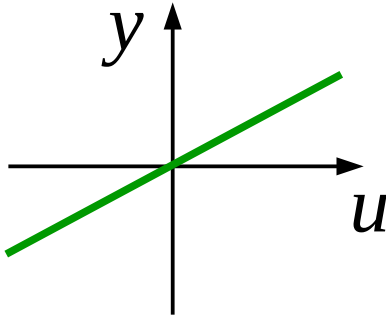
2. Static characteristic (steady state): $y = ku$ for $\frac{dy}{dt} = 0 \wedge \frac{du}{dt} = 0$



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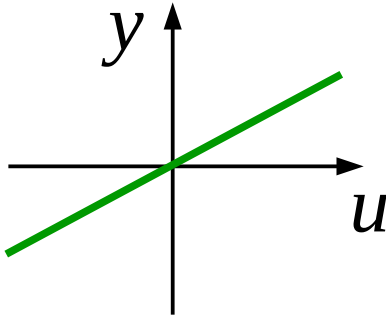


3. Transfer function:

Second-order inertial element

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3. Transfer function: $H(s) = \frac{k}{T_1^2 s^2 + T_2 s + 1}$

Second-order inertial element

4. Step response:

Second-order inertial element

4. Step response:

input: $u(t) = u_0 \mathbf{1}(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Second-order inertial element

4. Step response:

input: $u(t) = u_0 \mathbf{1}(t)$

Laplace of input: $U(s) = u_0 \frac{1}{s}$

Laplace of output: $Y(s) = H(s) U(s) = \frac{k u_0}{s(T_1^2 s^2 + T_2 s + 1)}$

Second-order inertial element

4. Step response:

$$\text{input: } u(t) = u_0 1(t)$$

$$\text{Laplace of input: } U(s) = u_0 \frac{1}{s}$$

$$\text{Laplace of output: } Y(s) = H(s) U(s) = \frac{k u_0}{s(T_1^2 s^2 + T_2 s + 1)}$$

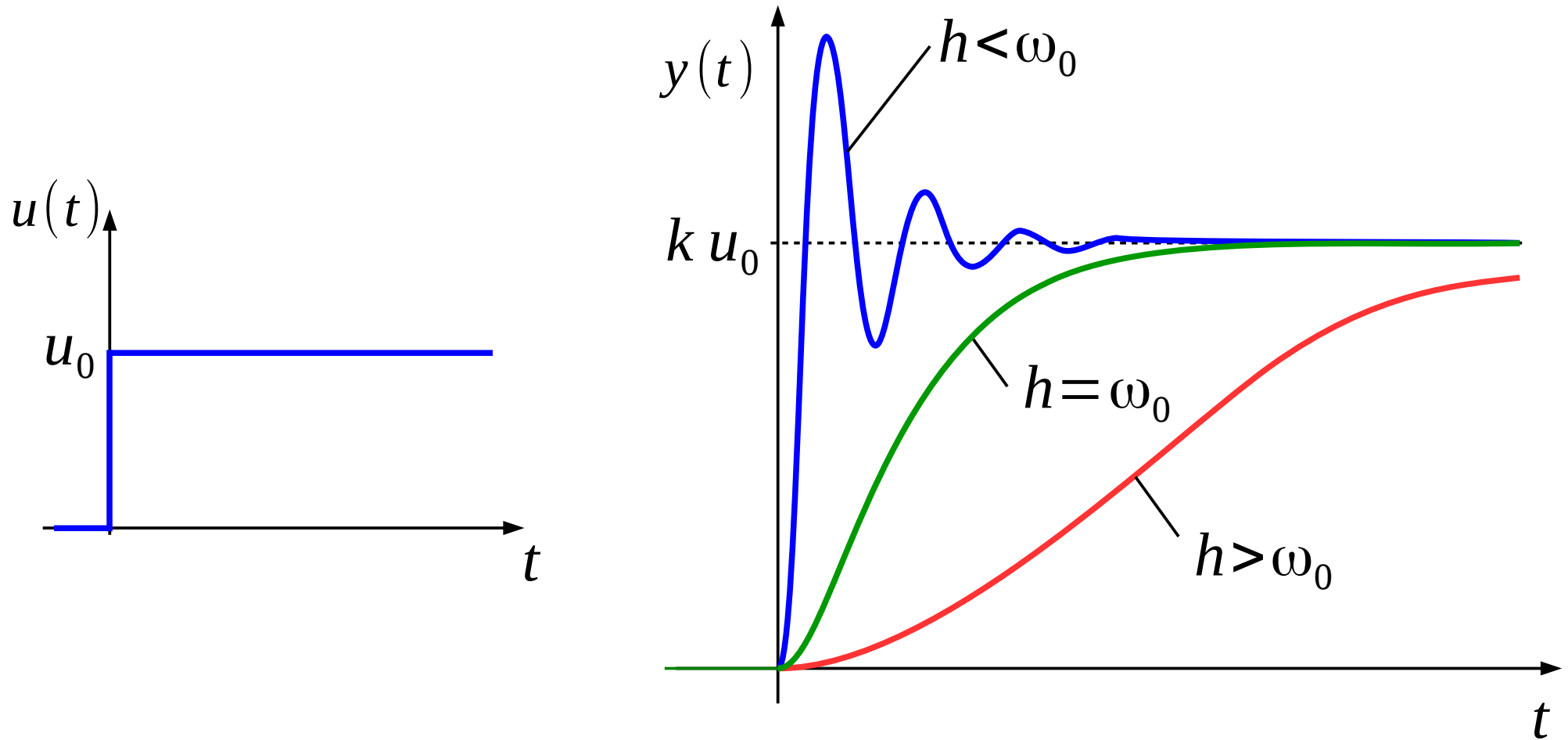
$$\text{output: } y(t) = L^{-1}\{Y(s)\} =$$

$$= \begin{cases} \frac{k u_0}{T_1^2} \left(1 - e^{-ht} \left(\cos \omega t + \frac{h}{\omega} \sin \omega t \right) \right), & \text{for } h \leq \omega_0 \\ \frac{k u_0}{T_1^2} \left(1 + e^{-ht} \left(\left(\frac{h+w}{2w} - 1 \right) e^{-wt} - \frac{h+w}{2w} e^{wt} \right) \right), & \text{for } h \geq \omega_0 \end{cases}$$

$$\text{where: } h = \frac{T_2}{2T_1^2}, \quad \omega_0 = \frac{1}{T_1}, \quad \omega = \sqrt{\omega_0^2 - h^2}, \quad w = \sqrt{h^2 - \omega_0^2}$$

Second-order inertial element

4. Step response:



Second-order inertial element

5. Frequency response:

Second-order inertial element

5. Frequency response:
$$H(j\omega) = \frac{k}{-T_1^2 \omega^2 + T_2 j\omega + 1}$$

Second-order inertial element

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$$P(\omega) = \frac{k(1 - T_1^2 \omega^2)}{(1 - T_1^2 \omega^2)^2 + T_2^2 \omega^2}, \quad Q(\omega) = \frac{-k T_2 \omega}{(1 - T_1^2 \omega^2)^2 + T_2^2 \omega^2}$$

Second-order inertial element

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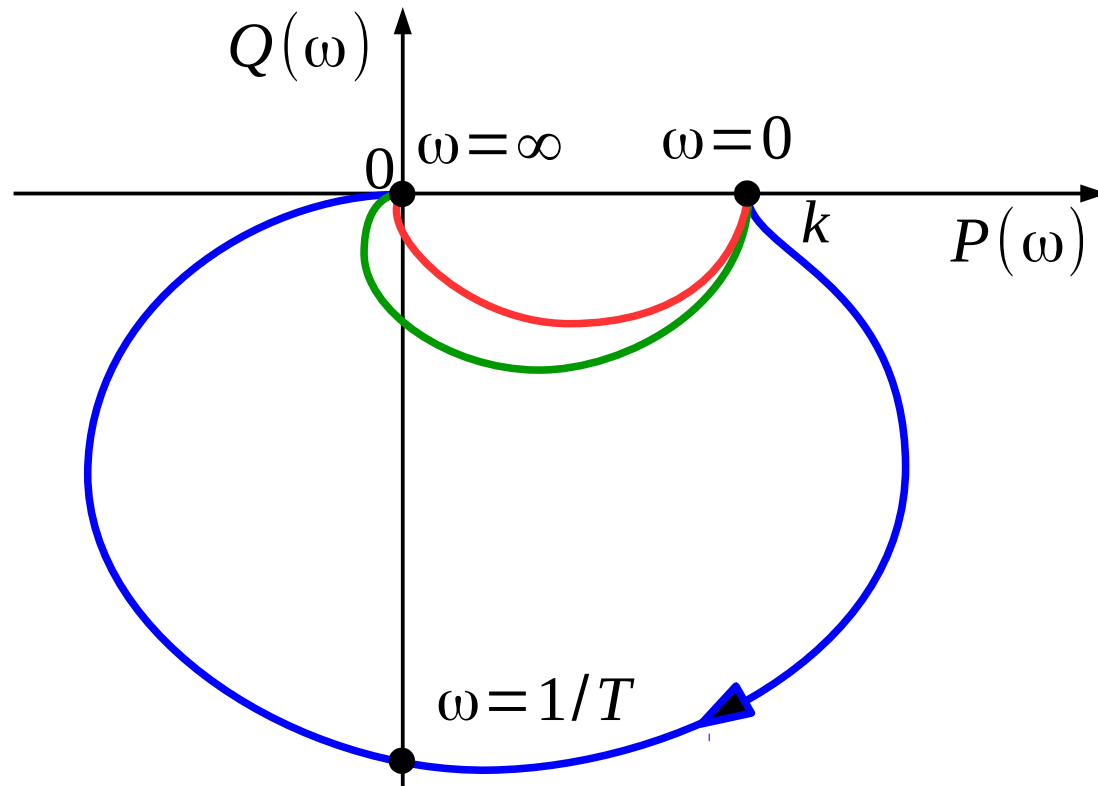
6. Nyquist plot:

Second-order inertial element

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6. Nyquist plot:
for $k > 0$



- for $h < \omega_0$
- for $h = \omega_0$
- for $h > \omega_0$

Second-order inertial element

7. Bode plot:

Second-order inertial element

7. Bode plot: $A(\omega) = \sqrt{P^2 + Q^2}$

$$L(\omega) = 20 \log A(\omega)$$

$$\varphi(\omega) = \arctan \frac{Q}{P}$$

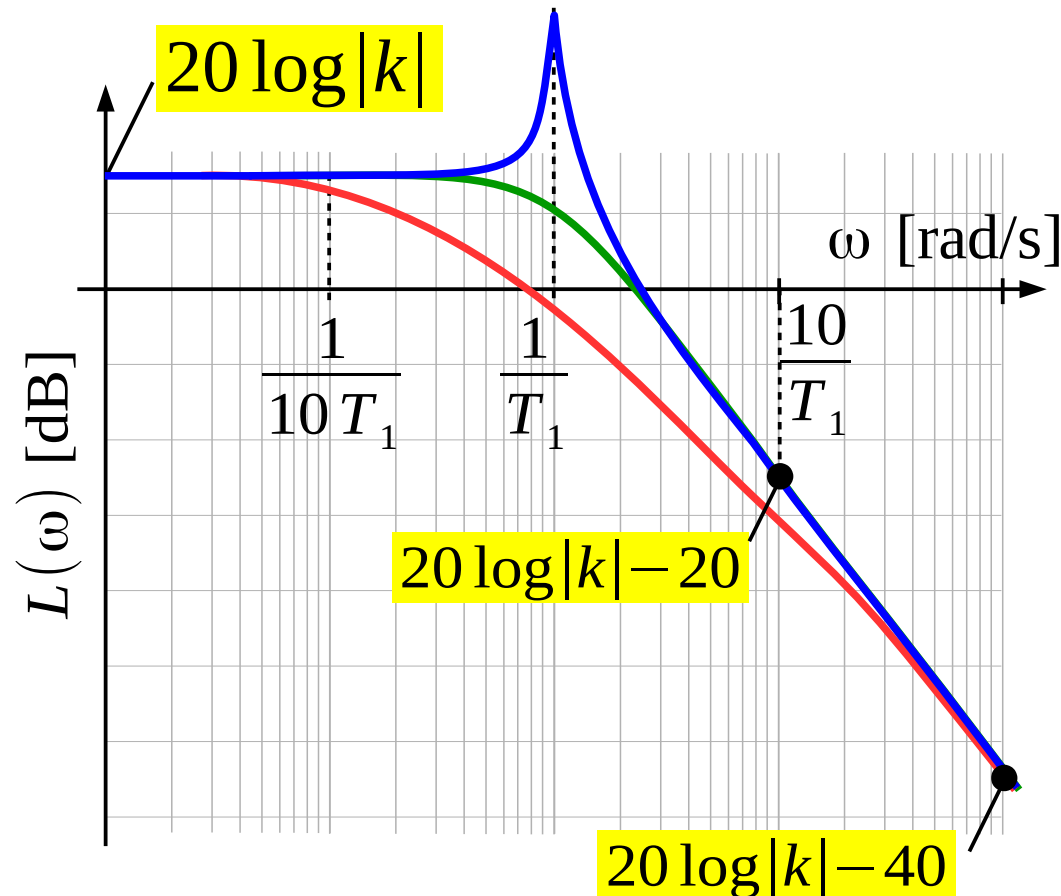
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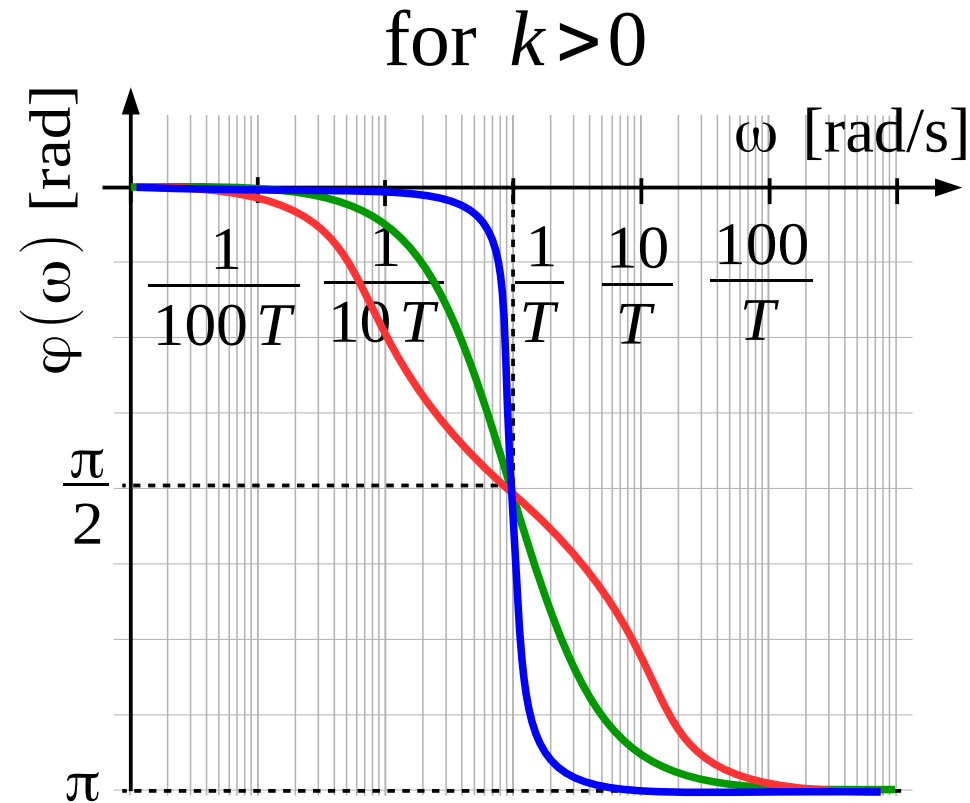
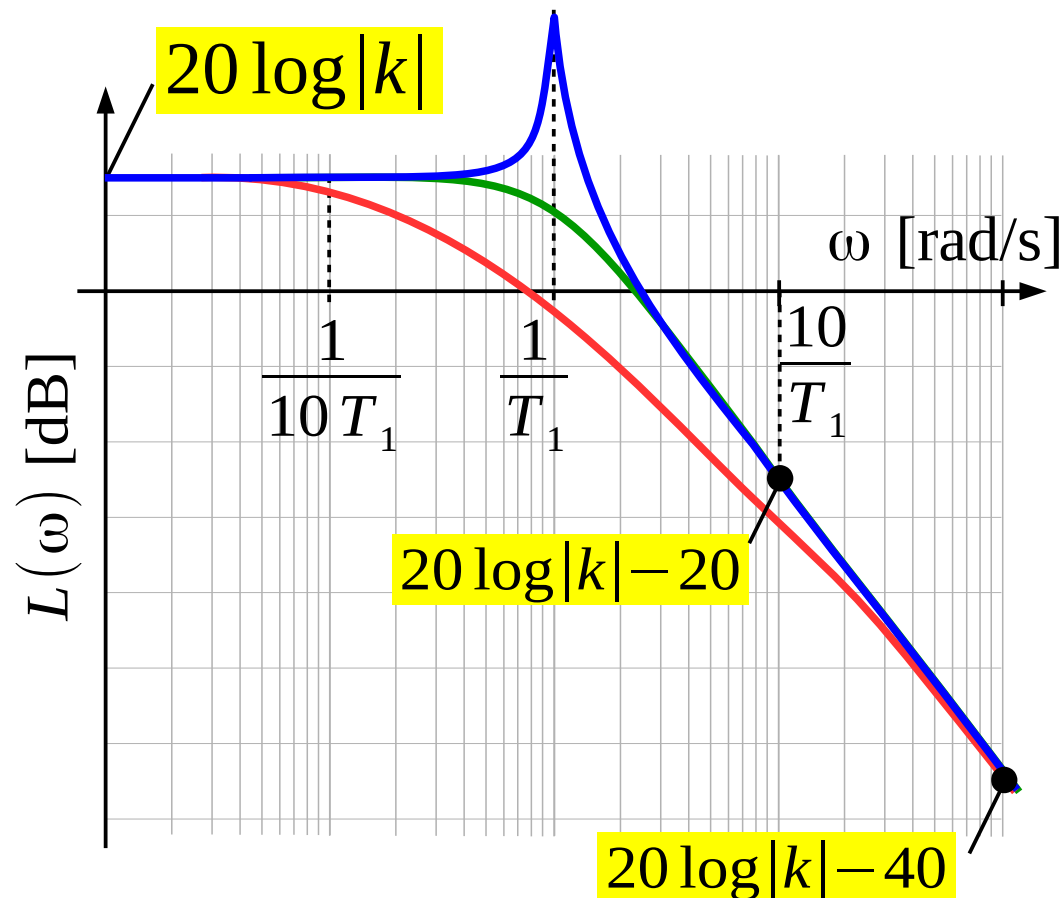
Second-order inertial element

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Second-order inertial element

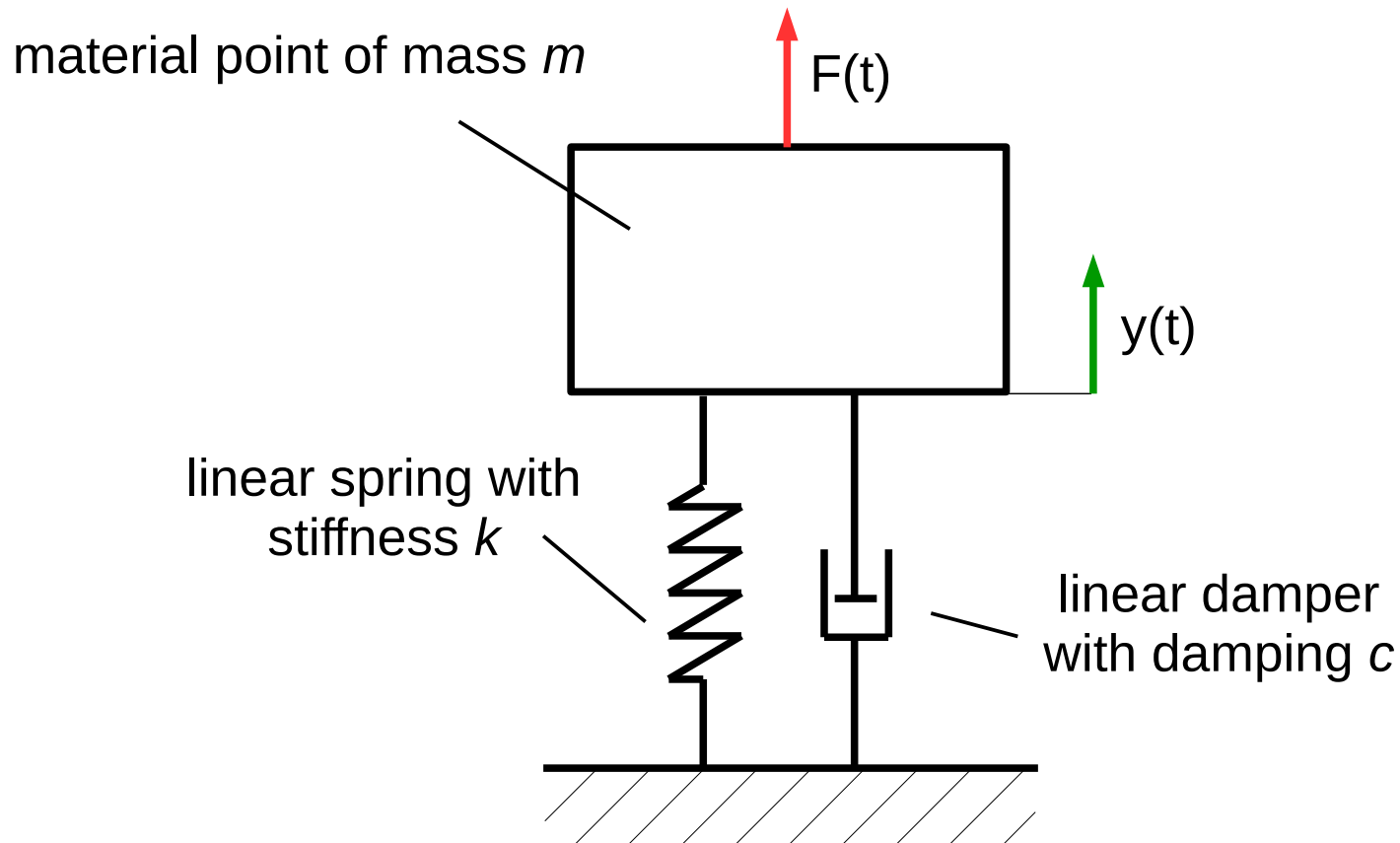
Examples

①

VIBRATING SYSTEM:

input – force $F(t)$

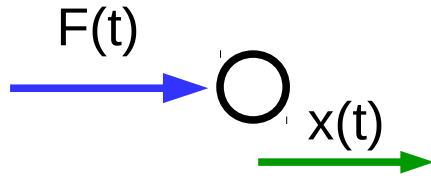
output – displacement $y(t)$



Second-order inertial element

Examples

②



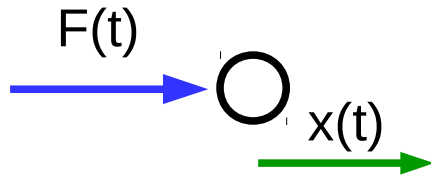
LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
input – force $F(t)$
output – displacement $x(t)$

example: car driving on a flat surface with air resistance proportional to velocity, described using machine equation of motion, with assumption of constant reduced mass.

Second-order inertial element

Examples

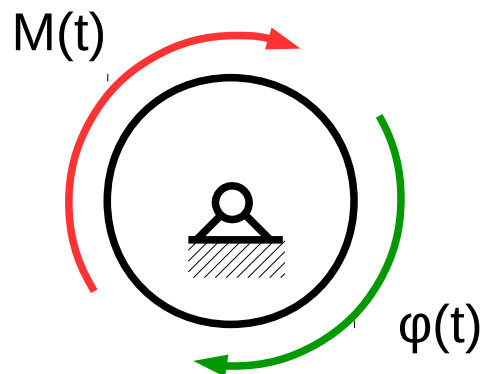
②



LINEAR MOTION OF A MATERIAL POINT WITH LINEAR DAMPING:
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example: car driving on a flat surface with air resistance proportional to velocity, described using machine equation of motion, with assumption of constant reduced mass.

③

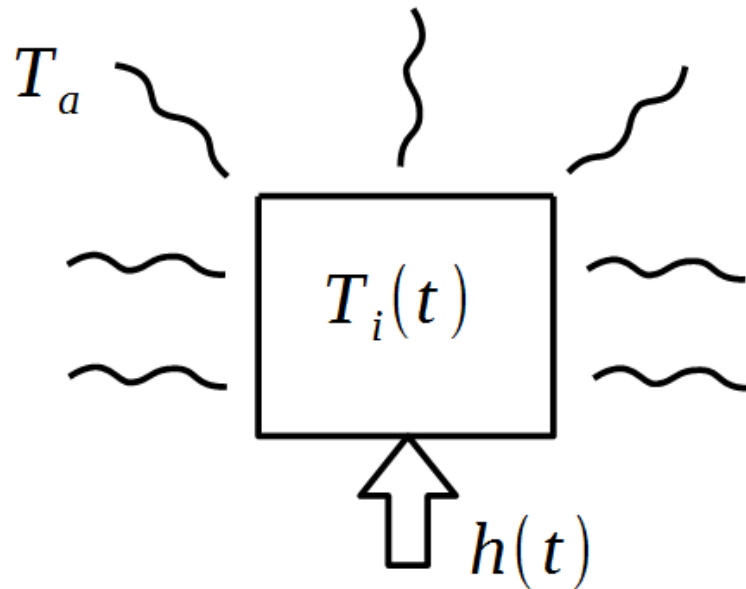


ANGULAR MOTION OF A RIGID BODY WITH LINEAR DAMPING:
input – torque $M(t)$
output – angle $\varphi(t)$

Second-order inertial element

Examples

4



HEATED OBJECT WITH HIGH
INERTIA:
input – heater power $h(t)$
output – object temperature $T_i(t)$

Classification of basic automatic systems

Element name	Transfer function
proportional	k
first order (inertial)	$\frac{k}{Ts + 1}$
integrator	$\frac{k}{s}$
differentiator	ks
differentiator with inertia	$\frac{ks}{Ts + 1}$
delay	$e^{-\tau s}$
second order (oscillator)	$\frac{k}{T_1^2 s^2 + T_2 s + 1}$